Chin. Ann. of Math.
8B (1) 1987

GRADED PI-RINGS AND GENERALIZED CROSSED PRODUCT AZUMAYA ALGEBRAS

F. Van Oystaeyen*

Abstract

A generalized crossed product of a ring R and a group G is a strongly graded ring $A=\bigoplus_{\sigma\in G}A_\sigma$ and $A_e=R$, where e is the neutral element of G. This paper investigates the conditions on G and on the gradation of A, which will ensure that A is an Azumaya algebra whenever A_e is one. And the author extends Proposition 3.8 in [9] to arbitrary finite group and some results of [10] concerning certain PI-rings to the case of not necessarily finitely generated grading groups.

Introduction

A generalized crossed product of a ring R and a group G is a strongly graded ring $A=\bigoplus_{\sigma\in G}A_\sigma$, i.e. $A_\sigma A_\tau=A_{\sigma\tau}$ for all σ, $\tau\in G$, and $A_e=R$, where e is the neutral element of G. We aim to investigate conditions on G and on the gradation of A, which will ensure that A is an Azumaya algebra whenever A_e is one. This problem has been studied for group rings by F. De Meyer, G. Janusz in [1] and by the author in [9, 10]. The proof of Proposition 3.8 in [9] is correct only for abelian groups, here we present an elementary proof for arbitrary finite groups. Also we extend some results of [10] concerning certain graded PI-rings to the case of not necessarily finitely generated grading groups, in fact also providing proofs for some claims of [10], e. g. in Propositions 4 and 5, which do not follow as easily rs pretended in loc. cit.

§1. Preliminaries

A ring R graded by G satisfies conditions (E) if for each $r_\sigma\neq 0$ in R_σ we have that $R_{\sigma^{-1}}r_\sigma\neq 0$. When R_e is a semiprime ring, condition (E) is left-right symmetri o. Recall Proposition 14 of [2].

Manuscript received February 19, 1986.

* Institute of Mathematics, University of Antwerp, U. I. A., Belgium.

Lemma 1.1. *Let R be graded by G such that (E) holds and assume that R_e is a semiprime left Goldie ring.*

1. *The set $S=\{s\in R,\ s$ regular and homogeneous in $R\}$ is a left Ore set of R.*
2. *The set $S_e=\{s\in R_e,\ s$ regular in $R_e\}$ is a left Ore set of R and $S^{-1}R=S_e^{-1}R$.*
3. *The ring $S^{-1}R$ is graded by G, it satisfies (E) and $(S^{-1}R)_e=S_e^{-1}R_e$. Moreover $S^{-1}R$ is a gr- semisimple gr-Artinian ring.*

In this paper we will be considering PI-rings and in particular semiprime PI-rings. By Amitsur's theorem such a PI-ring has a central extension of the form $\bigoplus_{i=1}^{d} M_i(C_i)$ where d is determined by the degree of the proper identity assumed on R and each C_i is areduced commutative ring. If the reader looks up Theorem 5.1 in C. Procesi's book [7], he will note that the theorem states that $\bigoplus_{i=1}^{d} M_i\ (C_i)$satisfies all stable identities (in particular multilinear ones) satisfied by R. From this, or by checking each step of the construction of the rings C_i, it follows that

$$R\cdot\Big(\bigoplus_{i=1}^{d} C_i\Big)=\bigoplus_{i=1}^{d} M_i(C_i).$$

If R is such a ring then $T^{-1}R$ is also such a ring for any left Ore set T of R. This follows from the fact that every localization (e.g. at the kernel functor associated to T) of the Azumaya algebra $\bigoplus_{i=1}^{d} M\ (C_i)$ is a central localization, or else by the following easy argument. If $qT^{-1}R$ is a nil right ideal of $T^{-1}R$ (i.e. contained in the prime radical of $T^{-1}R$) then, for some $t\in T$, tqR is a nil right ideal of the semiprime PI-ring R, hence $tq=0$ or $q=0$. Let us also mention the following special case of the Burnside problem (Corollary 2.8 p. 129 of [7]); a torsion group of units in a PI-algebra is locally finite! Hence if k is a field and the groupring kG is a PI-algebra then every torsion quotient group of G is locally finite (because if $\bar{G}$ is such a group then the canonical epimorphism $kG\to k\bar{G}$ makes $k\bar{G}$ into a PI-algebra too). As a final observation we point out that kG is always semiprime if char $k=0$; the same is true for kG^t and the proof runs along the lines of D. Passman's treatment of the groupring case using the trace function $\operatorname{tr}(\Sigma a_g u_g)=a_e$(cf. Theorem 3.4 using Lemma 3.3, 3.2 and 3.1 of [5], note that the proof is identically the same if one first extends k so that the cocycle describing kG^t is normalized i.e. $c(\sigma,\sigma^{-1})=1$ for all $\sigma\in G$, as in [10] p. 13, then $t(\sigma,\tau)^{-1}=t(\sigma^{-1},\tau^{-1})$ holds too). For general crossed products $k*G$, I do not know whether char $k=0$ makes $k*G$ semiprime, it is true in case $[k:k^G]<\infty$, where k^G is the fixed field under the action of G on k, because then one can sum over the finitely many conjugates and arrange the coefficients of homogeneous components of some nilpotent element such that they are in k.

§ 2. Graded PI-Rings Satisfying Condition E

Our aim is to extend the following result of D. Passman (of [4]): if K is a field and G is a group such that KG satisfies a proper polynomial identity then $[G:\Delta(G)] < \infty$ and $|\Delta(G)'| < \infty$, when $\Delta(G)$ is the finite conjugation subgroup of G. In particular when char $K=0$, or else when G is finitely generated, G contains a normal abelian subgroup of finite index. In Proposition 4 of [10] I considered the case of PI-rings divisorially graded by a finitely generated group G, and in Theorem 5 of loc. cit. the situation was generalized to gradations satisfying condition (E) but G still finitely generated.

We now provide the most general statement concerning graded PI-rings extending D. Passman's result. Next to obtaining now a complete similarity with the group ring case, we also provide here some proofs for some claims in Proposition 4 of [10], which were not so easy to verify as first thought. A graded ring R such that condition (E) holds and R_e is a semiprime (left) Goldie ring is said to have invariant cocycle if $Q^g=S^{-1}R$ (as in Lemma 1.1.3) is a direct sum of gr-simple gr-Artinian rings Q_i strongly graded by some subgroup G_i of G such that the action of G on $Z((Q_i)_e)$ leaves each $c_i(\sigma, \tau)$ invariant, where c_i is the cocycle describing the crossed product $Q_i=(Q_i)_e{*}G$. Note that this condition is not invariant under taking an equivalent cocycle!

Examples of such graded rings with invariant cocycles include both skew and twisted group rings and crossed products (for example of Galois-type) with factor sets in $H^2_{sym}(G, K^G)$ or $H^2(G, (K^G)^*)$.

Lemma 2.1. *Let R be graded by G over the semiprime Goldie ring R_e such that condition (E) holds, then*

1. *Put $G_1=\sup_G(R)=\{\sigma\in G, R_\sigma\neq 0\}$, then G_1 is a normal subgroup of G.*
2. *If $Q^g=Q_1\oplus\cdots\oplus Q_e$ is the decomposition of the gr-semisimple gr-Artinian ring Q^g into gr-simple components, then for every $\sigma\in G_1$, $(Q_i)_\sigma\neq 0$ for $i=1, \cdots, t$.*

Proof 1. By definition of (E), G_1 is closed under taking inverses. Since R_e is semiprime, (E) is left-right symmetric. Hence if $\sigma, \tau\in G_1$ then both $R_\sigma R_\tau$ and $R_\tau R_\sigma$ are nonzero, i.e. G_1 is closed under products and obviously, if $\sigma\in G$ then $R_\gamma K_\sigma R_{\gamma^{-1}}\neq 0$ yields normality.

2. For every $\sigma\in G_1$ and every left ideal L of R_e, $L\neq 0$, we have that $R_\sigma R_{\sigma^{-1}} L \subset L\cap R_\sigma R_{\sigma^{-1}}$ and it is clear from (E) that $R_\sigma R_{\sigma^{-1}} L\neq 0$. Hence $R_\sigma R_{\sigma^{-1}}$ is essential as a left ideal of R_e, similar on the right since (E) is left-right symmetric). Consequently $1\in Q_eR_\sigma R_{\sigma^{-1}}$ or $Q^g=Q^gR_\sigma R_{\sigma^{-1}}$, if $R_\sigma\cap Q_n=0$ for some $1\leqslant n\leqslant t$ then $Q^gR_\sigma Q^g\subset \bigoplus_{i\neq n} Q_i$ but this contradicts

$$Q^g R_\sigma Q^g \supset Q^g R_\sigma R_{\sigma^{-1}} = Q^g.$$

Theorem 2.2. *Let R be G–graded over the semiprime Goldie ring R such that property (E) holds and assume that R satisfies a proper polynomial identity. In each of the following cases:*

Case 1. *Char $R_e=0$ and R is semiprime,*

Case 2. *Char $R_e=0$ and R has invariant cocycle,*

Case 3. *G is finitely generated and R has invariant cocycle,*

G_1 contains a normal abelian subgroup of finite index.

Proof Reduce to $G=G_1$ by restricting attention to Sup_G (R). Write $Q^g=Q_1\oplus\cdots\oplus Q_t$ when each Q_i is gr– simple gr–Artinian as before, and

$$(Q^g)_\sigma=(Q_1)_\sigma\oplus\cdots\oplus(Q_t)_\sigma$$

for every $\sigma\in G$ because the Q_i are graded ideals. By the graded version of the Artin-Wedderburn theorem (cf. [3]), $Q_i=M_{n_i}(D_i)(\sigma^i)$, where D_i is a g–division algebra and $\sigma^i\in G^t$ determines the gradation of $M_{n_i}(D_i)$as follows $(d_{kj})\in M_{n_i}(D_i)(\sigma^i)_\gamma$ if and only if $d_{kj}\in D_{\sigma_k\gamma\sigma_j^{-1}}$. Let G_i be the subgroup(!) of G consisting of all $\sigma\in G$ such that $(D_i)_\sigma\neq 0$. (Note that $G_i\neq G$ is indeed possible, look at $M_2(k[T, T^{-1}])$ with deg $T=2$ and define the $\mathbb{Z}$–gradation according to $(0, 1)\in\mathbb{Z}\times\mathbb{Z}$ as described above, then there are nonzero elements of degree one in the matrix ring but not in the gr–field $k[T, T^{-1}]$). If $\delta\in G$ then $(Q_i)_\delta\neq 0$ implies that some entry in $(M_{n_i}(D_i)(\sigma^i))_\delta$ is nonzero, i.e. $\sigma^i_\lambda\delta(\sigma^i_\mu)^{-1}\in G_i$ for certain $\lambda, \mu\in G$. Now look at the finite set of subgroups of G, $\mathscr{S}=\{G_i, (\sigma^i_\lambda)^{-1}G_i\sigma^i_\lambda, \lambda=1, \cdots, n_i\}$. For every $\delta\in G$, $(Q_i)_\delta\neq 0$ yields that G is a finite union of right cosets of the forementioned groups i.e.

$$G=G_i\cup(\bigcup_{\lambda,\mu}\{(\sigma^i_\lambda)^{-1}G_i\sigma^i_\lambda\cdot((\sigma^i_\lambda)^{-1}\sigma^i_\mu)\}).$$

A result of B. Neumann (cf. [4]) yields that one of the subgroups in $\mathscr{S}$ has finite index in G, hence $[G: G_i]<\infty$. If we establish that G_i has an abelian normal subgroup of finite index then the same will hold for G,so we may assume from here on that $R=D$ a gr–division ring i.e. $R=R_e*G$, R_e a skewfield, and G acts on R_e via the morphism φ: $G\to Pic(\mathrm{Re})\to Aut(Z(R_e))$. We now proceed to prove that $Z(R_e)$ is a field of finite dimension over $Z(R_e)^G$.

Case 1. Assume that $Z(R_e)$has infinite $Z(R_e)^G$–dimension. Consider independent elements $x_1, \cdots, x_m\in Z(R_e)$ and suppose

$$(*)\qquad x_1y_1+\cdots+x_my_m=0 \text{ with } y_i\in Z(R).$$

Write $y_i=\sum_{\sigma\in G}(y_i)_\sigma$, $i=1, \cdots, m$. Since the x_j have degree e, we obtain:

$$(**)\qquad x_1(y_1)_\sigma+\cdots+x_m(y_m)_\sigma=0 \text{ foreach } \sigma.$$

Since $y_i\in Z(R)$ we have $z_\tau y_i=(y_i)_\sigma z_\tau$ for all $z_\tau\in R_\tau$, $\tau\in G$, yielding

$$z_\tau(y_i)_\sigma=(y_i)_\sigma z_\tau,$$

where $y=\tau\sigma\tau^{-1}$. So, if $(y_i)_\sigma\neq 0$ then $z_\tau(y_i)_\sigma\neq 0$ and $(y_i)_{\tau\sigma\tau^{-1}}\neq 0$. Consequently, in

$(*)$ we may assume that the y_i are such that the degrees appearing in their homogeneous decomposition are exactly the finitely many conjugates $\{\tau\sigma\tau^{-1}, \tau\in G\}$. From $(**)$ we derive, for all σ, $x_1(y_1)_\sigma(y_m)_\sigma^{-1}+\cdots+x_m=0$, and after summation over the conjugation class $C(\sigma)$ of a fixed σ we get

$$({}_*{}^*{}_*)\qquad x_1\sum_{\gamma\in C(\sigma)}(y_1)_\gamma(y_m)_\gamma^{-1}+x_2\sum_{\gamma\in C(\sigma)}(y_2)_\gamma(y_m)_\gamma^{-1}+\cdots+dx_m=0,$$

where $d=|C(\sigma)|=[G: C_G(\sigma)]$. Since $d\neq 0$ (char $R_e=0$), the relation$({}_*{}^*{}_*)$ is not trivial. Moreover, for any $\tau\in G$ and $z_\tau\in R_\tau$ we calculate

$$z_\tau(y_i)_\sigma(y_m)_\sigma^{-1}=(y_i)_{\tau\sigma\tau^{-1}}z_\tau(y_m)_\sigma^{-1},$$

$$z_\tau(y_m)_\sigma=(y_m)_{\tau\sigma\tau^{-1}}\,z_\tau$$

and

$$(y_m)^{-1}_{\tau\sigma\tau^{-1}}\,z_\tau=z_\tau(y_m)_\sigma^{-1}.$$

Consequently we obtain the following relations

$$z_\tau(y_i)_\sigma(y_m)_\sigma^{-1}=(y_i)_{\tau\sigma\tau^{-1}}(y_m)^{-1}_{\tau\sigma\tau^{-1}}z_\tau,$$

$$z_\tau\Big(\sum_{y\in C(\sigma)}(y_i)_y(y_m)_y^{-1}\Big)=\sum_{y\in C(\sigma)}(y_i)_{\tau y\tau^{-1}}(y_m{}^{-1}_{\tau y\tau^{-1}})\,z_\tau.$$

The coefficients in$({}_*{}^*{}_*)$ are therefore in $Z(R)\cap R_e=Z(R_e)^G$ and this contradicts the assumption on the $x_1, \cdots, x_m$. It follows that R contains a free $Z(R)$-module of infinite rank but then the central extension $\bigoplus_{i=1}^{d} M_i(C_i)$ (cf. Section 1; this applies here becauce R is semiprime i.e. also R^g and D_i at the beginning of this proof) also contain a free module of infinite rank over $\bigoplus_i C_i$, which is impossible (e.g. Localize to semisimple Artinian ring or reduce modelo a prime or maximal ideal of the centre). We have established that $Z(R_e)$ is a Galois extension of $Z(R_e)^G$ with Galois group Imφ, a finite group. Crossed products of $Z(\mathrm{Re})$ and G with action given by φ and cocycle $c:G\times G\to U(Z(R_e))$ will be denoted by $(Z(R_e), G, \varphi, c)$. Since

$$R=R_e*G$$

is a PI-algebra, $Z(R_e)*G$ is a PI-algebra too. For $A=(Z(R_e), G, \varphi, c)$ and $B=(Z(R_e), G, \varphi, \alpha)$ we claim that $A\otimes_{Z(R_e)^G}B$ contains the $Z(R_e)^G$—algbra $(Z(R_e), G, \varphi, cd)$ as a subalgebra. The proof of this is identical to the proof of the product theorem for 2-cocycles in Galois cohomology (cf. R. Pierce [6] p. 258), up to noting that the finiteness of G used in that proof may be replaced by the finiteness of Imφ: Gal$(Z(R_e)/Z(R_e)^G$. Applying this argument to the $Z(R_e)^G$-algebra S, $S=(Z(R_e), G, \varphi, c)\otimes_{Z(R_s)^G}(Z(R_e), G, \varphi, c)^0$ we obtain that S is a PI-algebra by A. Regev's theorems and S contains $(Z(R_e),G,\varphi,1)$. It follows that the latter ring as well as the subring $Z(R_e)^G G$ is a PI-ring. Now D. Passman's result in characteristic may be invoked to conclude that there is a normal abelian A in G of finite index.

Case 2 and 3. As before we reduce the situation to $Z(R_e)*G$ but now we know that the cocycle is invariant underthe action of G. Consider the set map:

$$\psi:\ G\to(Z(R_e)*G)\otimes_{Z(R_e)^G}(Z(R_e)\otimes G)^0;\ \sigma\to u_\sigma\otimes u_{\sigma^{-1}}.$$

we calculate that

$$\psi(\sigma)\psi(\tau)=c(\sigma,\tau)c(\tau^{-1},\sigma^{-1})\psi(\sigma\tau).$$

Since $c(\sigma,\tau)\in Z(R_e)^G$ we may find a field extension L of $Z(R_e)^G$ containing all $\sqrt{c(\sigma,\sigma^{-1})}$ such that c is equivalent to a normalized cocycle c': $G\times G\to L^*$, i.e. satisfying $c'(y,y^{-1})=1$ for all $y\in G$. So in fact we may define ψ: $G\to LG^{c'}\otimes_L(LG^{c'})^0$, $\sigma\mapsto\psi(\sigma)\sqrt{c(\sigma,\sigma^{-1})}$ and since $LG^{c'}\cong LG^c$ is a subring of $L\otimes_{Z(R_e)^G}Z$, we have presented G as a subgroup of the units of some PI-algebra, so we may invoke the special case of the Burnside problem mentioned in Section 1 to conclude that $\operatorname{Im}\varphi$ is locally finite. Indeed if $\sigma\in G$ is a torsion element then φ_σ is torsion; if $\sigma\in G$ is not torsion, then $Z(R_e)*\langle\sigma\rangle$ is a PI-ring of the form $Z(R_e)[X,X^{-1},\varphi_\sigma]$, hence φ_σ^p is inner for some $p\in\mathbb{N}$ (well known) i.e. φ_σ is torsion, hence $\operatorname{Im}\varphi$ is always a torsion group. For Case 3, it follows that $\operatorname{Im}\varphi$ is finite and one proceeds as in Case 1.

For Case 2, it suffices now to observe that G will contain a normal abelian subgroup of finite index if and only if every finitely generated subgroup of G has the same property. If $H\subset G$ is finitely generated then $Z(R_e)*H$ is again PI and now $Z(R_e)$ has finite dimension over $Z(R_e)^H$, so, as in Case 1, it follows that H contains a normal abelian subgroup of finite index and Case 2 follows by the foregoing observation.

Remark 2.3. Does it follow from char $R_e=0$ that R_e*G is semiprime? (cf. Section 1). The fact that this case be established when $Z(R_e)$ is finite dimensional over $Z(R_e)^G$ yields another proof for Case 2 by reduction to Case 1.

Corollary 2.4 (cf. the proof). *If R is as in the theorem but have invariant cocycle then without restriction on G or char R_e one deduces from the proof given that G is a pi-group, i.e. for some field k (in fact char $k=$ char R_e) kG is a PI-algebra. This then entails Case 2 and Case 3 directly.*

2.5 Added in proof: Possman and Montgomery have a result that enables us to give a positive answer to 2.3.

§ 3. Generalized Crossed Product Azumaya Algebras

Throughout this section A_e is a (left) Goldie ring. We say that a gradation on A is quasi-inner if $Z(A_e)\subset Z(A)$ and A is strongly graded. Note that a quasi-inner gradation always has invariant cocycle. we first present a proof for Proposition 3.8 in [9], which is also valid for nonabelian groups.

Theorem 3.1. *Let A be strongly graded by a finite group G such that A_e is an Azumaya algebra, the gradation is quasi-inner, and $|G|^{-1}\in A_e$, then A is an Azumaya algebra.*

Proof Since A is an A_e-bimodule centralizing $Z(A_e)$, it follows that $A=$

$A_e \otimes_{Z(A_e)} C_A(A_e)$. Obviously $C_A(A_e)$ is G-graded and from $A_\sigma \otimes_{A_e} A_{\sigma-1} \cong A_e$ it follows that $(A_\sigma)^{(A_e)} \otimes (A_{\sigma-1})^{(A_e)}$ is isomorphic to $Z(A_e)$ (where $(-)^{(A_e)}$ denotes the commutation functor yielding the equivalence of the category of A_e-bimodules over $Z(A_e)$ and the category of $z(A_e)$-modules). Clearly $C_A(A_e)$ is strongly graded by G over $Z(A_e)$. If $C_A(A_e)$ is separable over $Z(A_e)$, then A will be an Azumaya algebra. Consider a maximal ideal m in $Z(A_e)$ and look at the strongly graded ring B, $B = C_A(A_e)/mC_A(A_e)$, over $Z(A_e)/m$. Since the latter is a field k central in B, it follows that $B \cong kG^t$ and from $|G|^{-1} \in k$ we conclude that B is k-separable. The local-global property for separability yields that $C_A(A_e)$ is $Z(A_e)$-separable.

We now provide an extension of Corollary 10 in [10], that includes the case of twisted and common group rings.

Proposition 3.2. *Let A be strongly graded by G such that A satisfies the identities of $n \times n$-matricos* (cf. C. Procesi [7]). *If A is gr-semisimple gr-Artinian and $|G'|^{-1} \in A_e$ then A is an Azumaya algebra.*

Proof Obviously A_e is a semisimple Artinian ring and $A^{(G')}$ is gr-semisimple gr-Artinian. But since each A_σ, $\sigma \in G$ is a finitely generated projective A_e-module, it is clear that $A^{(G')} = \bigoplus_{\sigma \in G'} A_\sigma$ is finite dimensional over $Z(A_e)$, hence Artinian. On the other hand, since $|G'|^{-1} \in A_e$, we have

$$J(A^{(G')}) = J^g(A^{(G')}) = A^{(G')},$$

$$(J^g(A^{(G')}) \cap A_e = 0$$

(cf. [9] for properties of the graded Jacobson radical J^g), hence $A^{(G')}$ is semisimple Artinian and A is strongly graded by G/G' over $A^{(G')}$.

Put $A^{(G')} = A_{\bar{0}}$ with $\bar{0} \in G/G'$; let $S_{\bar{0}}$ be the set of regular elemente in $A_{\bar{0}}$. By Lemma 1.1, it follows that A is gr-semisimple gr-Artinian in the G/G'-gradation because $S_{\bar{0}}$ is invertible in $A_{\bar{0}}$, i.e. $S_{\bar{0}}^{-1}A = A$. Write $A = A_1 \oplus \cdots \oplus A_t$, each A_i being gr-simple in the G/G'-gradation. Since G/G' is abelian, $Z(A_i)$ is a gr-field (the centre is graded here!) and a non-trivial multilinear central polynormial f_i for A_i cannot vanish at all homogeneous substitutions, hence it must take at least one nonzero homogeneous central value. The Formanek centre of A_i must then equal $Z(A_i)$, i.e. A is an Azumaya algebra.

Corollary 3.3 (cf. the proof). *The result in Proposition* 3.2 *is also valid if the gradation on A satisfies* (E).

Proof Since A is gr-semisimple gr-Artinian, A_e is semiprime and $A = Q^g(A)$. So from the proof of Lemma 2.1 (2) we retain $Q^g A_\sigma A_{\sigma^{-1}} = Q^g$, hence $Q^g A_{\sigma^{-1}} = Q^g$ for all $\sigma \in G$. This yields $AA_{\sigma^{-1}} = A$ for all $\sigma \in G$, or $A_\sigma A_{\sigma^{-1}} = A_e$, i.e. A is strongly graded by G.

Proposition 3.4. *Let A be graded by a group G.*

1. *If A is an Azumaya algebra, then there is a finitely generated subgroup H of G such that $A^{(H)}$ is an Azumaya algebra too.*

2. *If some central localization $S^{-1}A$ containing A is a finite module over its centre, then there is a finitely generated subgroup H of G such that A is Azumaya whenever $A^{(H)}$ is an Azumaya algebra.*

Proof 1. There is a finitely generated ring C_0 in C (over the base ring generated by 1) and an Azumaya algebra A_0 over C_0 such that $A=A_0C=A_0\otimes_{C_0}C$. Let H be the subgroup of G generated by all degrees of homogeneous elements appearing in the decomposition of the ring generators for C_0 and the generators for A_0 as a C_0-module. Obviously $A_0C\subset A^{(H)}$. If $z\in Z(A^{(H)})$ then z commutes with A_0 hence with A, thus $Z(A^{(H)})=Z(A)\cap A^{(H)}$. Also,

$$Z(A_0)\subset A^{(H)}\cap Z(A)=Z(A^{(H)}),$$

thus we obtain: $A^{(H)}=A_0\otimes_{C_0}Z(A^{(H)})$, i.e. $A^{(H)}$ is an Azumaya algebra.

2. Let $x_1\cdots, x_m$ generate $S^{-1}A$ as a $Z(S^{-1}A)$-module and let $s\in S\subset Z(A)$ be such that $sx_i\in A$ for all i. Let B be the subring of A generated by the homogeneous components of all the sx_i, $i=1, \cdots, m$, and let H be the subgroup of G generated by the degrees of these components. Obviously $B\subset A^{(H)}$. If $Z\in A^{(H)}$ commutes with B then it commutes with all the x_i, hence we get

$$Z(A^{(H)})=Z(A)\cap A^{(H)}=Z(S^{-1}A)\cap A^{(H)}$$

and also that $Z(A^{(H)}Z(A))=Z(A)$. If $A^{(H)}$ is an Azumaya algebra then so is $A^{(H)}\otimes_{Z(A^{(H)})}Z(A)$ and also $A^{(H)}Z(A)$. From

$$A=A^{(H)}Z(A)\otimes_{Z(A)}C_A(A^{(H)})=A^{(H)}Z(A(\otimes_{Z(A)}Z(A),$$

it then follows that A is an Azumaya algebra.

We now extend Theorem 17 of [10] as follows.

Theorem 3.5. *Let A be a PI-ring satisfying the identities of $n\times n$-matrices and let A be quasi-inner graded by a group G such that G' is finite and $|G'|^{-1}\in A$. If A_e is an Azumaya algebra and a Q-algebra then A is an Azumaya algebra.*

Proof By the commutator method we reduce the problem to the case

$$A_e=Z(A_e).$$

It is sufficient to prove for every maximal ideal M of $Z(A)$ that A/MA is separable over $Z(A)/M$. Put $P=M\cap Z(A_e)$. Then A/MA is an epimorphic image of the strongly graded A/PA over A_e/PA_e. Let $\bar{a}$ be the image of $a\in A/PA$ in A/MA. Since $Z(A)/M$ embeds into A/MA, it follows that the field of fractions K of A_e/P also embeds in A/MA. Therefore we may define a map $A_P\to A/MA$ as the composition of the canonical map $A_P\to K\otimes_{A_e/P}A/PA$ and the map $K\otimes_{A_e/P}A/PA\to A/MA$, $x\otimes a\mapsto x\bar{a}$, which is obviously a ring epimorphism. Since $(A_e)_P$ is local, we may write $A_P=(A_e)_PG^t$ and in fact $K\otimes_{A_e/P}A/PA=KG^t$ where we have used the image of the cocycle describing the structure of A_P. If we prove that A_P is an Azumaya algebra, then $Z((A_e)_P\, G^t)$ maps to $Z(KG^t)$ and $Z(A)$ maps to a subring R of $Z(KG^t)$ such that $KR = K \otimes_{A_e/P} R = Z(KG^t)$. The centre of KG^t consists of linear combinations of ray-class sums(cf. [10]) and such a sum in $Z(KG^t)$ may be written as $\sum_j \lambda_j \otimes y_j$ with $\lambda_j \in K$, $y_j \in Z(A/PA)$, because KG^t is obtained by central localization of A/PA; hence $\sum_j \lambda_j \otimes y_j$ maps to $\sum_j \lambda_j \otimes \bar{y}_j \in Z(A)/M$ and so $Z(A)/M$ is the image of $Z(KG^t)$. Then $Z(KG^t)$ separability of KG^t entails that A/MA is $Z(A)/M$-separable and by the globalization property of separability: A is separable over $Z(A)$. So we have to establish that $(A_e)_P G^t$ is Azumaya. Since the localization is central and A satisfies the identities of $n \times n$-matrices, we use Proposition 3.4 to reduce to the case where G is finitely generated. Since G is a PI-group by the result of Section 2, $[G: Z(G)] < \infty$. Since ray- class sums are central in $(A_e)_P\, G^t$ (even if A_e is not a domain this holds but then one cannot easily show that the centre is freely generated by such ray-class sums), we may reproduce Proposition 14 of [10], modified in a very trivial way in order to deal with the fact that $Q(A_e)$ is only semisimple here, and conclude that the C-regular elements, G_{reg} say, in $Z(G)$ have finite index in G. So $(A_e)_P\, G_{reg}$ is central in $(A_e)_P G$ and then we invoke Theorem 3.1 to finish the proof.

References

[1] De Meyer, F. and Janusz,G., Group Rings which are Azumaya Algebras (to appear).

[2] Nastasescu, C. Nauwelaerts, E. and Van Oystaeyen, F. Arithmatically Graded Rings Revisited (to appear).

[3] Nastasescu, C. and Van Oystaeyen, R., Graded Ring Theory, *Library of Math.*, Vol 28, North Holland, Amsterdam 1982.

[4] Neumann, B., Groups Covered by Permutable Subsets, *J. Londen Math. Soc.*, **29** (1958), 236—248.

[5] Passmann, D. S., The Algebraic Structure of Group Rings, J. Wiley, 1977.

[6] Pierce, Associative Algebras, GTM 88, Springer Verlag, Berlin 1980.

[7] Procesi, C., Rings with Polynomial Identities, M. Dekker, *Pure and Applied Math.*, Vol. 17, New York 1973.

[8] Van Oystaeyen, F., Note on Central Class Groups of Orders over Krull Domains, Methods in Ring theory, Reidel, Dordrecnt (1984), 523—540.

[9] Van Oystaeyen, F., On Clifford Systems and Generalized Crossed Products, *J. of Algebra*, **87**. 2, (1984), 396—415.

[10] Van Oystaeyen, F., Azumaya Strongly Graded Rings and Ray Classes, *J. of Algebra*, (1986).