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A NEW PREHOMOGENEOUS VECTOR SPACE OF CHARACTERISTIC p

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Abstract

Let K be an algebraically closed field of characteristic 3. Let Λ_1, Λ_2, Λ_3 denote all the fundamental dominant weights of $GL(4)$. Then the K-dimension of the irreducible $GL(4)$-module V with the highest weight $\Lambda_1+\Lambda_2$ is equal to 16, and it is denoted by $V(16)$. In this paper, the following results are proved:

(1) $(GL(4), \Lambda_1+\Lambda_2, V(16))$ is a regular irreducible prehomogeneous vector space. The degree of its irreducible relative invariant is 8, the associated character is

$$\chi(g) = (\det g)^6.$$

(2) There exist only one 6-dimensional $GL(4)$-orbit and one 9-dimensional $GL(4)$-orbit in $V(16)$. When $m=7$, 8 or $1\leqslant m\leqslant 5$, there are no m-dimensional $GL(4)$-orbits.

Sato and Kimura [7] have completed the classification of irreducible prehomogeneous vector spaces in characteristic 0. The author investigated the irreducible prehomogeneous vector spaces in characteristic $p>0$ and found that almost every irreducible prehomogeneous vector space could be obtained by reduction mod p from the corresponding one in characteristc 0 (see [2]). One of the exceptions is $(GL(4), \Lambda_1+\Lambda_2, V(16))$ $(p=3)$. When the characteristic of the base field is 0, the triplet $(GL(4), \Lambda_1+\Lambda_2, V(20))$ is not a prehomogeneous vector space. The purpose of this paper is to investigate the triplet $(GL(4), \Lambda_1+\Lambda_2, V(16))$ $(p=3)$. This triplet can be proved to be a regular irreducible prehomogeneous vector space with the irreducible relative invariant of degree 8. Then we can prove that there are no other orbits of dimension less than 10 in this space except a 9-dimensional one, a 6-dimensional one and a 0-dimensional one $\{0\}$.

1. Preliminaries.

Let K be an algebraically closed field, V be a finite-dimensional vector space over K. Let G be a connected affine algebraic group over K and ρ: $G\to GL(V)$ be a rational representation of G. Then G acts on V by ρ. If there is a Zariski open G-orbit in V, then we call the triplet (G, ρ, V) a prehomogeneous vector space (abbrev. PV). All the triplets (G, ρ, V) in this paper are assumed to be a PV. If there

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exists a nontrivial rational function $P\in K(V)$ such that

$$P(\rho(g)x)=\chi(g)P(x), \quad \forall g\in G,\ x\in V,$$

then P is called a relative invariant. χ: $G\to K^*$ is a character of G called the character of P.

Now we denote the dual G-module of V by V^* with the corresponding contragredient representation ρ^*: $G\to GL(V^*)$. Taking a dual basis in V^*, we can define a morphism

$$\text{grad } P\colon \Omega\to V^*$$

$$: x\mapsto \begin{pmatrix} \frac{\partial P}{\partial x_1}(x) \\ \vdots \\ \frac{\partial P}{\partial x_n}(x) \end{pmatrix} = \text{grad } P(x).$$

Here Ω denote the Zariski open orbit. If the morphism (grad $P)/P$: $\Omega\to V^*$ is dominant, then (G, ρ, V) is considered to be regular. If the representation ρ is irreducible, then we call the (G, ρ, V) an irreducible PV.

Please refer to [1, 6] for the detail.

2. The construction of $(GL(4), \Lambda_1+\Lambda_2, V(16))$ $(p=3)$.

From now on, we assume that the characteristic of K is 3. Let $k=GF(3)$ be the prime field of K. We are going to construct the irreducible representation $\Lambda_1+\Lambda_2$ of $G=GL(4)$ $(=GL(4, K))$.

Take a 4-dimensional vector space over the complex field $\mathbb{C}$:

$$V_1(\mathbb{C})=\{{}^t(x_1, x_2, x_3, x_4)\,|\,x_i\in\mathbb{C},\ i=1, 2, 3, 4\}$$

with a canonical basis

$$f_1={}^t(1, 0, 0, 0),$$
$$f_2={}^t(0, 1, 0, 0),$$
$$f_3={}^t(0, 0, 1, 0),$$
$$f_4={}^t(0, 0, 0, 1).$$

Then the natural representation $\rho_1=\Lambda_1$ of $GL(4, \mathbb{C})$ on $V_1(\mathbb{C})$ is defined as

$$\rho_1(g).\,v=gv, \quad \forall g\in GL(4,\mathbb{C}),\ v\in V_1(\mathbb{C}).$$

Thus, $V_1(\mathbb{C})$ is an irreducible $GL(4,\mathbb{C})$-module with the highest weight Λ_1.

Put $V_2(\mathbb{C})=\{X\in M(4,\mathbb{C})\,|\,{}^tX=-X\}$ and define a representation $\rho_2=\Lambda_2$ of $GL(4,\mathbb{C})$ on $V_2(\mathbb{C})$ as follows:

$$\rho_2(g)X=gX{}^tg, \quad \forall g\in GL(4, \mathbb{C}),\ X\in V_2(\mathbb{C}).$$

Obviously, $\dim_{\mathbb{C}} V_2(\mathbb{C})=6$ and $V_2(\mathbb{C})$ is an irreducible $GL(4, \mathbb{C})$-module with the highest weight Λ_2. Denote

$$e_{ij}=\begin{pmatrix} 0 & & & 1 \\ & 0 & & \\ & & 0 & \\ -1 & & & 0 \end{pmatrix}\begin{matrix} i \\ \\ \\ j \end{matrix}, \quad i\neq j,\ i,j=1,2,3,4.$$
$$\qquad\qquad i \qquad j$$

Then $\{e_{ij}\mid 1\leqslant i<j\leqslant 4\}$ form a basis of $V_2(\mathbb{C})$ and $e_{ji}=-e_{ij}$.

The tensor representation $\Lambda_1\otimes\Lambda_2$ of $GL(4,\mathbb{C})$ on $V_1(\mathbb{C})\otimes V_2(\mathbb{C})$ is defined as

$$\rho_1\otimes\rho_2(g)(v\otimes X)=\rho_1(g)v\otimes\rho_2(g)X=gv\otimes gX^tg,$$
$$\forall g\in GL(4,\mathbb{C}),\ v\in V_1(\mathbb{C}),\ X\in V_2(\mathbb{C}).$$

As a $GL(4,\mathbb{C})$-module, $V_1(\mathbb{C})\otimes V_2(\mathbb{C})$ has a decomposition to direct summands

$$V_1(\mathbb{C})\otimes V_2(\mathbb{C})=W_1(\mathbb{C})+W_2(\mathbb{C}),$$

where

$$\begin{aligned} W_1(\mathbb{C})=&\,\mathbb{C}f_1\otimes e_{12}+\mathbb{C}f_1\otimes e_{13}+\mathbb{C}f_1\otimes e_{14}+\mathbb{C}f_2\otimes e_{21}+\mathbb{C}f_2\otimes e_{23}\\ &+\mathbb{C}f_2\otimes e_{24}+\mathbb{C}f_3\otimes e_{31}+\mathbb{C}f_3\otimes e_{32}+\mathbb{C}f_3\otimes e_{34}+\mathbb{C}f_4\otimes e_{41}\\ &+\mathbb{C}f_4\otimes e_{42}+\mathbb{C}f_4\otimes e_{43}+\mathbb{C}(f_2\otimes e_{13}+f_3\otimes e_{12})\\ &+\mathbb{C}(f_1\otimes e_{23}+2f_2\otimes e_{13}+f_3\otimes e_{12})+\mathbb{C}(f_2\otimes e_{14}+f_4\otimes e_{12})\\ &+\mathbb{C}(f_1\otimes e_{24}+2f_2\otimes e_{14}+f_4\otimes e_{12})+\mathbb{C}(f_3\otimes e_{14}+f_4\otimes e_{13})\\ &+\mathbb{C}(f_1\otimes e_{34}+2f_3\otimes e_{14}+f_4\otimes e_{13})+\mathbb{C}(f_3\otimes e_{24}+f_4\otimes e_{23})\\ &+\mathbb{C}(f_2\otimes e_{34}+2f_3\otimes e_{24}+f_4\otimes e_{23}),\\ W_2(\mathbb{C})=&\,\mathbb{C}(f_1\otimes e_{23}-f_2\otimes e_{13}+f_3\otimes e_{12})+\mathbb{C}(f_1\otimes e_{24}-f_2\otimes e_{14}\\ &+f_4\otimes e_{12})+\mathbb{C}(f_1\otimes e_{34}-f_3\otimes e_{14}+f_4\otimes e_{13})\\ &+\mathbb{C}(f_2\otimes e_{34}-f_3\otimes e_{24}+f_4\otimes e_{23}), \end{aligned}$$

$$\dim_{\mathbb{C}} W_1(\mathbb{C})=20,$$
$$\dim_{\mathbb{C}} W_2(\mathbb{C})=4.$$

In fact, $W_1(\mathbb{C})$ is the irreducible $GL(4,\mathbb{C})$-module with the highest weight $\Lambda_1+\Lambda_2$. Now we take a lattice $GL(4,\mathbb{Z})$ in $GL(4,\mathbb{C})$ and take an admissible $\mathbb{Z}$-form of $W_1(\mathbb{C})$ as follows:

$$\begin{aligned} W_{1,\mathbb{Z}}=&\,\mathbb{Z}f_1\otimes e_{12}+\mathbb{Z}f_1\otimes e_{13}+\mathbb{Z}f_1\otimes e_{14}+\mathbb{Z}f_2\otimes e_{21}+\mathbb{Z}f_2\otimes e_{23}\\ &+\mathbb{Z}f_2\otimes e_{24}+\mathbb{Z}f_3\otimes e_{31}+\mathbb{Z}f_3\otimes e_{32}+\mathbb{Z}f_3\otimes e_{34}+\mathbb{Z}f_4\otimes e_{41}\\ &+\mathbb{Z}f_4\otimes e_{42}+\mathbb{Z}f_4\otimes e_{43}+\mathbb{Z}(f_2\otimes e_{13}+f_3\otimes e_{12})\\ &+\mathbb{Z}(f_2\otimes e_{14}+f_4\otimes e_{12})+\mathbb{Z}(f_3\otimes e_{14}+f_4\otimes e_{13})\\ &+\mathbb{Z}(f_3\otimes e_{24}+f_4\otimes e_{23})+\mathbb{Z}(f_1\otimes e_{23}+2f_2\otimes e_{13}+f_3\otimes e_{12})\\ &+\mathbb{Z}(f_1\otimes e_{24}+2f_2\otimes e_{14}+f_4\otimes e_{12})+\mathbb{Z}(f_1\otimes e_{34}+2f_3\otimes e_{14}\\ &+f_4\otimes e_{13})+\mathbb{Z}(f_2\otimes e_{34}+2f_3\otimes e_{24}+f_4\otimes e_{23}). \end{aligned}$$

Let $GL(4,K)=GL(4,\mathbb{Z})\otimes_{\mathbb{Z}}K$ and $W_{1,K}=W_{1,\mathbb{Z}}\otimes_{\mathbb{Z}}K$, then $W_{1,K}$ is an indecomposable $GL(4)$-module. In fact, $W_{1,K}$ is a Weyl module of $GL(4)$ with the highest weight $\Lambda_1+\Lambda_2$. There exists a unique maximal $GL(4)$-submodule M in $W_{1,K}$ as follows:

$$M=(f_1\otimes e_{23}+2f_2\otimes e_{13}+f_3\otimes e_{12})\otimes K+(f_1\otimes e_{24}+2f_2\otimes e_{14}+f_4\otimes e_{12})\otimes K$$
$$+(f_1\otimes e_{34}+2f_3\otimes e_{14}+f_4\otimes e_{13})\otimes K+(f_2\otimes e_{34}+2f_3\otimes e_{24}+f_4\otimes e_{23})\otimes K.$$

The quotient module $W_{1,K}/M=V$ is an irreducible $GL(4)$-module with the highest weight $\Lambda_1+\Lambda_2$. We denote a basis of V by

$$e_{iij}=(f_i\otimes e_{ij})\otimes 1+M, \qquad i\neq j,\ i,j=1,\ 2,\ 3,\ 4.$$
$$e_{ijk}=(f_j\otimes e_{ik}+f_k\otimes e_{ij})\otimes 1+M, \quad 1\leqslant i<j<k\leqslant 4.$$

Then $\dim_K V=16$. Let ρ denote the representation of $GL(V)$ on $V(=V(16))$.

Let

$$g=(g_{ij})\in GL(4).$$

Then

$$\rho(g)e_{iij}=\sum_{p\neq q}(g_{pi}^2g_{qj}-g_{pi}g_{pj}g_{qi})e_{ppq}+\sum_{p<q<r}(-g_{pi}g_{qi}g_{rj}-g_{pi}g_{qj}g_{ri}-g_{pj}g_{qi}g_{ri})e_{pqr},$$
$$i\neq j,\ i,j=1,\ 2,\ 3,\ 4,$$
$$\rho(g)e_{ijk}=\sum_{p\neq q}(g_{pi}g_{pj}g_{qk}+g_{pi}g_{pk}g_{qj}+g_{pj}g_{pk}g_{qi})e_{ppq}+\sum_{p<q<r}(g_{pi}g_{qj}g_{rk}+g_{pi}g_{qk}g_{rj}$$
$$+g_{pj}g_{qi}g_{rk}+g_{pj}g_{qk}g_{ri}+g_{pk}g_{qi}g_{rj}+g_{pk}g_{qj}g_{ri})e_{pqr},$$
$$1\leqslant i<j<k\leqslant 4.$$

This representation ρ: $GL(4)\to GL(V)$ can induce a representation of the Lie algebra $\mathfrak{gl}(4)$ on the same vector space V as follows:

$$d\rho:\ \mathfrak{gl}(4)\to\mathfrak{gl}(V).$$

Let

$$A=(a_{ij})\in\mathfrak{gl}(4),$$

then

$$d\rho(A)e_{iij}=(a_{jj}-a_{ii})e_{iij}-a_{ji}e_{jji}+\sum_{q\neq i,j}a_{qj}e_{iiq}+\sum_{p\neq i,j}(-a_{pi})e_{(pij)},$$
$$i\neq j,\ i,j=1,\ 2,\ 3,\ 4,$$
$$d\rho(A)e_{ijk}=a_{ij}e_{ijk}+a_{ji}e_{jjk}+a_{ik}e_{iij}+a_{ki}e_{kkj}+a_{jk}e_{jji}+a_{kj}e_{kki}+(a_{ii}+a_{jj}+a_{kk})e_{ijk}$$
$$+a_{pi}e_{(pjk)}+a_{pj}e_{(pik)}+a_{pk}e_{(pij)}$$
$$1\leqslant i<j<k\leqslant 4.$$

In these formulas, $e_{(pij)}=e_{i_1i_2i_3}$, where $(i_1,\ i_2,\ i_3)$ is a rearrangement of $(p,\ i,\ j)$ from less to greater.

3. Lemma 1. *$(GL(4),\ \Lambda_1+\Lambda_2,\ V)$ is a PV.*

Proof Let

$$x_0=e_{123}+e_{124}+e_{134}+e_{234}\in V.$$

Then the stabilizer of $G=GL(4)$ at x_0 is

$$G_{x_0}=\{g\in G\,|\,\rho(g)\cdot x_0=x_0\}$$
$$=\{E_{1,i_1}+E_{2,i_2}+E_{3,i_3}+E_{4,i_4},-E_{i_1,1}-E_{i_1,2}-E_{i_1,3}$$
$$-E_{i_1,4}+E_{i_2,j_2}+E_{i_3,j_3}+E_{i_4,j_4}\,|\ (i_1,\ i_2,i_3,\ i_4)$$

is an arrangement of (1,2, 3, 4); $(j_2,\ j_3,\ j_4)$ is an arrangement of any 3 numbers taken from $\{1, 2, 3, 4\}$.}, where

$$E_{i,j}=\begin{pmatrix} & 1 \\ 0 & \end{pmatrix}^{i}_{j}.$$

Thus G_{x_0} is a finite subgroup of G, and $|G_{x_0}|=120$. Let $\rho(G).\ x_0$ denote the G-orbit of x_0, then

$$\dim \rho(G).\ x_0=\dim G-\dim G_{x_0}=16=\dim V.$$

Therefore $\rho(G).\ x_0$ must be a Zariski open subset in the affine space V. This proves that $(GL(4),\ \Lambda_1+\Lambda_2,\ V)$ is a PV.

4. Lemma 2. *There exist relative invariants in the PV $(GL(4),\ \Lambda_1+\Lambda_2,\ V)$, and all its relative invariants have the form cp^m, where $c\in K^*$ $m\in\mathbb{Z}$, and $P\in k[V]$ is irreducible.*

Proof(after [7]) Suppose that G is a reductive algebraic group, H is its closed subgroup, then G/H is an affine variety iff H is reductive (see [3]). Now G_{x_0} is a reductive closed subgroup of the reductive group $G=GL(4)$ (Lemma 2), so $G/G_{x_0}\cong \rho(G).\ x_0$ is an affine variety. Since $\rho(G).\ x_0$ is an open subset of V, $V-\rho(G).\ x_0$ is an algebraic set of pure codimension 1. Since $K[V]$ is a unique factorization domain, $V-\rho(G).\ x_0=Z(P)$, where $P\in K[V]$. If $Z(P)=Z(P_1)\cup\cdots\cup(P_r)$ is a decomposition to irreducible components, then $P_1, \cdots, P_r$ can be taken to be irreducible polynomials of $K[V]$ such that $P=P_1P_2\cdots P_r$. Hence the relative invariants of the $PV(G,\ \rho,\ V)$ have the form $cP_1^{m_1}\cdots P_r^{m_r}$, where $c\in K^*$ and $(m_1, \cdots, m_r)\in\mathbb{Z}^r$ (see [1] or [6]). Since ρ is irreducible, $r=1$(see [1] or [6]). So P can be taken to be an irreducible polynomial in $K[V]$ and all the relative invariants of this PV have the form $cP^m(c\in K^*,\ m\in\mathbb{Z})$. Further-more, since $GL(4)$ is defined over k, P can betaken to be an irreducible polynomial in $k[V]$[1].

5. Let $\mathfrak{g}_x$ denote the centralizer of the Lie algebra $\mathfrak{g}=\mathfrak{gl}(4)$ at $x\in V$. Then

$$\mathfrak{g}_x=\{A\in\mathfrak{g}\,|\,d\rho(A).\ x=0\}.$$

If $y=\rho(g).\ x$, then $\mathfrak{g}_y=Ad\ g.\ \mathfrak{g}_x$. Hence $\dim\mathfrak{g}_y=\dim\mathfrak{g}_x$. Let $L(G_x)$ denote the Lie algebra of G_x, then

$$L(G_x)\subseteq\mathfrak{g}_x.$$

So

$$\dim G_x=\dim L(G_x)\leqslant\dim\mathfrak{g}_x.$$

Now we define

$$V_r=\{x\in V\,|\dim d\rho(\mathfrak{g}).\ x\leqslant r\}=\{x\in V\,|\dim\mathfrak{g}_x\geqslant 16-r\},\quad 0\leqslant r\leqslant 16.$$

Then $\{E_{i,j}|0\leqslant i,\ j\leqslant 4\}$ is a basis of $\mathfrak{g}$. Hence $\{d\rho(E_{i,j})|\ 0\leqslant i,j\leqslant 4\}$ generates $d\rho(\mathfrak{g})$. Construct the 16×16 matrix

$$M_x=(d\rho(E_{1,1}).\ x,\ \cdots,\ d\rho(E_{4,4}).\ x).$$

Then $\dim d\rho(\mathfrak{g}).\ x\leqslant r$ iff all the minors of degree $r+1$ of M_x are null. Hence V_r is closed in V. It is not difficult to see that V_r is stable under the action of G. Since

$$\sum_{i=1}^{4} d\rho(E_{i,i})=0,$$

we have

$$V=V_{16}=V_{15}\supseteq V_{14}\supseteq\cdots\supseteq V_1\supseteq V_0=\{0\}.$$

Lemma 3. *The degree of the irreducible relative invariant of the PV* $(GL(4), \Lambda_1+\Lambda_2, V)$ *is* 8 *and the character associated with* P *is* $\chi(g)=(\det g)^6$.

Proof Since

$$\mathfrak{g}_{x_0}=\left\{\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \middle| \, a\in K\right\},$$

$x_0=e_{123}+e_{124}+e_{134}+e_{234}\in V_{15}$, $x_0\notin V_{14}$. Hence $V_{14}\subseteq V-\rho(G).x_0$, that is, $V_{14}\subseteq Z(P)$, where $P\in k[V]$ is the irreducible relative invariant of the PV.

If we take $x_1=e_{112}+e_{334}+e_{442}$, then

$$G_{x_1}=\left\{\begin{pmatrix} t^{-2} & & & \\ & t^4 & & \\ & & t & \\ & & & t^{-2} \end{pmatrix} \middle| \, t\in K^*\right\},$$

$$\mathfrak{g}_{x_1}=\left\{\begin{pmatrix} a & & & \\ & a & b & \\ & a & -b & \\ & & & a \end{pmatrix} \middle| \, a, b\in K\right\}.$$

As $\dim G_{x_1}=1$,

$$\dim \overline{\rho(G).\, x_1}=15.$$

Since $x_1\in V_{14}$, $\overline{\rho(G).\, x_1}\subseteq V_{14}\subseteq Z(P)$. Since $\overline{\rho(G).\, x_1}$ and $Z(P)$ are all irreducible closed subset, considering the fact that $\dim \overline{\rho(G).\, x_1}=\dim Z(P)$, we can conclude that

$$\overline{\rho(G).\, x_1}=V_{14}=Z(P).$$

As an orbit of a connected algebraic group, $\rho(G).x_1$ must be a locally closed subset, so $\rho(G).\, x_1$ is a relative open subset contained in the irreducible closed subset

$$Z(P)=V_{14}.$$

This implies that $\rho(G).\, x_1$ is the unique 15-dimensional orbit in V.

Now let Q be a nontrivial minor of degree 15 of the matrix M_{x_1}, then

$$Z(P)=V_{14}\subseteq Z(Q).$$

Hence $P\,|\,Q$, that is,

$$\deg P\leqslant \deg Q=15.$$

Let $\deg P=m$ and let χ denote the character associated with P. Then $\chi(g)=$

$(\deg g)^r$. If we take

$$g=t.1=\begin{pmatrix} t & & & \\ & t & & \\ & & t & \\ & & & t \end{pmatrix}\in GL(4),\quad t\in K^*,$$

then from the relation below

$$P(\rho(g).x)=\chi(g)P(x),\quad \forall g\in G,\ x\in V,$$

we can get

$$P(\rho(g).x)=P(t^3.x)=t^{3m}P(x),\quad \forall x\in V.$$

Hence

$$t^{3m}=\chi(g)=t^{4r}.$$

So $3m=4r$.

Supposing that $g\in G_{x_0}$, we have $\chi(g)=(\det g)^r=1$. But we know that there exists some $g\in G_{x_0}$ such that $\det g=-1$. Hence r must be an even integer. Therefore $m=8$, $r=6$.

6. Lemma 4. *Let (G, ρ, V) be a PV over an algebraically closed field K of characteristic $p>0$, such that $V-\Omega=Z(P)$, where P is a nontrivial relative invariant. If $\deg P\not\equiv 0 \pmod p$, then (G, ρ, V) is a regular PV.*

Proof See [6].

From this lemma we can obtain immediately the following

Lemma 5. *$(GL(4), \Lambda_1+\Lambda_2, V)$ is a regular PV.*

7. Using these lemmas, we can obtain the following

Theorem 1. *Let K be an algebraically closed field of characteristic 3, let $GL(4)=GL(4, K)$, then $(GL(4), \Lambda_1+\Lambda_2, V)$ is a regular irreducible prehomogeneous vector space. The degree of its irreducible relative invariant $P\in k[V]$ is 8, the associated character is $\chi(g)=(\det g)^6$.*

We investigated the possible irreducible PV over an algebraically closed field of characteristic $p>2$ in [2]. We found that almost all irreducible PV could be obtained by reduction mod p from the corresponding ones in characteristic 0. But there are 3 exceptions, that is: $(GL(n), (1+p^s)\Lambda_1, V(n^2))$ $(s>0, n\geqslant 2)$, $(GL(n), \Lambda_1+p^s\Lambda_{n-1}, V(n^2))$ $(s>0, n\geqslant 3)$ and $(GL(4), \Lambda_1+\Lambda_2, V(16))$ $(p=3)$. The first ones can be obtained by twisted modules. The last one is the most interesting. We know that the degree of its irreducible relative invariant is 8, but it need a lengthy calculation to write it explicitely. The determination of this relative invariant must be very interesting.

8. Let $\{e^*_{iij}, e^*_{ijk}\}$ denote the dual basis in V^* with respect to the basis $\{e_{iij}, e_{ijk}\}$ of V. Let

$$\det\begin{pmatrix} E_{\alpha_1,\beta_1}, & \cdots, & E_{\alpha_r,\beta_r} \\ e^*_{i_1,j_1,k_1}, & \cdots, & e^*_{i_r,j_r,k_r} \end{pmatrix}$$

$$= \det\begin{pmatrix} \langle d\rho(E_{\alpha_1,\beta_1}).\, x,\ e^*_{i_1,j_1,k_1}\rangle, & \cdots, & \langle d\rho(E_{\alpha_r,\beta_r}).\, x,\ e^*_{i_1,j_1,k_1}\rangle \\ & \cdots\cdots\cdots & \\ \langle d\rho(E_{\alpha_1,\beta_1}).\, x,\ e^*_{i_r,j_r,k_r}\rangle, & \cdots, & d\rho(E_{\alpha_r,\beta_r}).\, x, e^*_{i_r,j_r,k_r}\rangle \end{pmatrix},$$

where x is a certain fixed point in V, $i=j\neq k$ or $i<j<k$. Obviously, it is a minor of degree r of M_x.

For a permutation $\sigma\in S_4$, define a permutation matrix P_σ as follows:

$$P_\sigma=E_{1,\sigma(1)}+E_{2,\sigma(2)}+E_{3,\sigma(3)}+E_{4,\sigma(4)}\in GL(4).$$

Then

$$\rho(P_\sigma)\cdot e_{iij}=e_{\sigma(i),\sigma(i),\sigma(j)},\quad i\neq j,$$
$$\rho(P_\sigma).\, e_{ijk}=e_{(\sigma(i),\sigma(j),\sigma(k))},\quad i<j<k.$$

Lemma 6. *Let $x_7=e_{112}+e_{331}$, let $x_6=e_{112}$, then*

$$V_9=\rho(G).\, x_7\cup\rho(G).\, x_{10}\cup\{0\}.$$

Proof Suppose that $x\neq 0$, $x\in V_9$ and

$$x=\sum_{i\neq j}x(iij)e_{iij}+\sum_{i<j<k}x(ijk)e_{ijk}.$$

If $x(iij)=0$ for all $1\leqslant i\neq j\leqslant 4$, then there must exist some $x(ijk)\neq 0$, say, $x(123)\neq 0$. In this case

$$\det\begin{pmatrix} E_{11}, & E_{12}, & E_{13}, & E_{21}, & E_{23}, & E_{31}, & E_{32}, & E_{41}, & E_{42}, & E_{43} \\ e_{112}, & e_{113}, & e_{123}, & e_{124}, & e_{134}, & e_{221}, & e_{223}, & e_{234}, & e_{331}, & e_{332} \end{pmatrix}=\pm x(123)^{10}\neq 0,$$

so $x\notin V_9$. This implies that there exists some $x(iij)\neq 0$. Put

$$\sigma=\begin{pmatrix} i & j & k & l \\ 1 & 2 & 3 & 4 \end{pmatrix}\in S_4,$$

and denote $x'=\rho(P_\sigma).\, x$, then $x'(112)=x(iij)\neq 0$. Hence without loss of generality, we can assume that $x(112)\neq 0$.

Take

$$g_1=\begin{pmatrix} 1 & & & \\ & 1 & & \\ & -\dfrac{x(113)}{x(112)} & 1 & \\ & -\dfrac{x(114)}{x(112)} & & 1 \end{pmatrix}\in GL(4),$$

and put $y=\rho(g_1).\, x$. Then

$$y(112)=x(112),\ y(113)=y(114)=0.$$

We distinguish two cases:

(1) $y(331)$ or $y(441)$ is not equal to 0. We can assume $y(331)\neq 0$. Take

$$g_2=\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -\frac{y(334)}{y(331)} & 0 & -\frac{y(134)}{y(331)} & 1 \end{pmatrix}\in GL(4)$$

and let $z=\rho(g_2).y$. Then

$$z(112)=y(112),$$
$$z(113)=z(114)=z(334)=z(134)=0,$$
$$z(331)=y(331).$$

Since

$$\det\begin{pmatrix} E_{21}, & E_{22}, & E_{23}, & E_{31}, & E_{32}, & E_{33}, & E_{41}, & E_{42}, & E_{43}, & E_{44} \\ e_{112}, & e_{113}, & e_{114}, & e_{123}, & e_{124}, & e_{134}, & e_{221}, & e_{331}, & e_{441}, & e_{443} \end{pmatrix}$$
$$=\pm z(112)^5 z(331)^3 z(441)^2,$$

this implies $z(441)=0$. Then

$$\det\begin{pmatrix} E_{21}, & E_{22}, & E_{24}, & E_{31}, & E_{32}, & E_{33}, & E_{41}, & E_{42}, & E_{43}, & E_{44} \\ e_{112}, & e_{113}, & e_{114}, & e_{123}, & e_{124}, & e_{134}, & e_{221}, & e_{234}, & e_{331}, & e_{443} \end{pmatrix}$$
$$=\pm z(112)^6 z(331)^2 z(443)^2,$$

so $z(443)=0$. In this case

$$\det\begin{pmatrix} E_{21}, & E_{22}, & E_{23}, & E_{31}, & E_{32}, & E_{33}, & E_{41}, & E_{42}, & E_{43}, & E_{44} \\ e_{112}, & e_{113}, & e_{114}, & e_{123}, & e_{124}, & e_{134}, & e_{221}, & e_{234}, & e_{331}, & e_{334} \end{pmatrix}$$
$$=\pm z(112)^4 z(331)^4 z(124)^2,$$

so $z(124)=0$. Similarly, from

$$\det\begin{pmatrix} E_{21}, & E_{22}, & E_{24}, & E_{31}, & E_{32}, & E_{33}, & E_{41}, & E_{42}, & E_{43}, & E_{44} \\ e_{112}, & e_{113}, & e_{114}, & e_{123}, & e_{124}, & e_{134}, & e_{221}, & e_{224}, & e_{331}. & e_{441} \end{pmatrix}$$
$$=\pm z(112)^6 z(331)^2 z(442)^2,$$

we can conclude $z(124)=0$. Since

$$\det\begin{pmatrix} E_{21}, & E_{22}, & E_{31}, & E_{32}, & E_{33}, & E_{34}, & E_{41}, & E_{42}, & E_{43}, & E_{44} \\ e_{112}, & e_{113}, & e_{114}, & e_{123}, & e_{124}, & e_{134}, & e_{221}. & e_{223}, & e_{224}, & e_{331} \end{pmatrix}$$
$$=\pm z(112)^6 z(331)^2 z(224)^2,$$

$z(224)=0$. Furthermore

$$\det\begin{pmatrix} E_{21}, & E_{22}, & E_{23}, & E_{24}, & E_{31}, & E_{32}. & E_{33}, & E_{41}, & E_{42}, & E_{43} \\ e_{112}, & e_{113}, & e_{114}, & e_{123}, & e_{124}, & e_{134}, & e_{221}, & e_{223}, & e_{224}, & e_{331} \end{pmatrix}$$
$$=\pm z(112)^6 z(331)^2 z(234)^2,$$

implies $z(234)=0$. At last, we have

$$\det\begin{pmatrix} E_{11}, & E_{12}, & E_{21}, & E_{22}, & E_{23}, & E_{31}, & E_{32}, & E_{41}, & E_{42}, & E_{43} \\ e_{112}, & e_{113}, & e_{114}, & e_{123}, & e_{124}, & e_{134}, & e_{221}, & e_{223}, & e_{331}, & e_{332} \end{pmatrix}$$
$$=\pm z(112)^3 z(331)[z(112)z(332)+z(221)z(331)-z(123)^2]^3,$$

this implies that

$$z(112)z(332)+z(221)z(331)-z(123)^2=0.$$

Now we put

$$\begin{aligned}
&v=z(112),\\
&w_2^2=z(331),\\
&w_1^3=-[z(221)z(123)+z(112)z(223)]/w_2,\\
&u_1^3=\frac{[z(221)z(123)+z(112)z(223)]^2}{z(112)^3z(331)}-\frac{z(221)^3}{z(112)^3},\\
&u_2^3=\frac{z(331)[z(221)z(123)+z(112)z(223)]}{z(112)^3}-\frac{z(123)^3}{z(112)^3},
\end{aligned}$$

then

$$\begin{aligned}
&z(221)z(123)+z(112)z(223)=-w_1^3w_2,\\
&z(221)=w_1^2-u_1v,\\
&z(123)=-w_1w_2-u_2v,\\
&z(223)=w_1^2u_2-w_1w_2u_1-u_1u_2v,\\
&z(112)z(332)=z(123)^2-z(221)z(331)=w_2^2u_1v-w_1w_2u_2v+u_2^2v^2.
\end{aligned}$$

So

$$z(332)=w_2^2u_1-w_1w_2u_2+u_2^2v.$$

Put

$$g_3=\begin{pmatrix}1&0&0&0\\u_1&v&w_1&0\\u_2&0&w_2&0\\0&0&0&1\end{pmatrix}\in GL(4).$$

Then

$$z=\rho(g_3).\,x_7.$$

This proves that

$$x\in\rho(G).\,x_7.$$

(2) Both $y(331)$ and $y(441)$ are equal to zero. Note that $y(113)=y(114)=0$. Since

$$\det\begin{pmatrix}E_{11}, & E_{21}, & E_{23}, & E_{31}, & E_{32}, & E_{33}, & E_{34}, & E_{41}, & E_{42}, & E_{43}\\ e_{112}, & e_{113}, & e_{114}, & e_{123}, & e_{124}, & e_{134}, & e_{221}, & e_{331}. & e_{441}, & e_{443}\end{pmatrix}$$
$$=\pm y(112)^5y(134)^5,$$

$y(134)=0$. In this case

$$\det\begin{pmatrix}E_{11}, & E_{21}, & E_{23}, & E_{24}, & E_{31}, & E_{32}, & E_{33}, & E_{41}, & E_{42}, & E_{43}\\ e_{112}. & e_{113}, & e_{114}, & e_{123}, & e_{124}, & e_{221}, & e_{234}, & e_{332}, & e_{334}, & e_{443}\end{pmatrix}$$
$$=\pm y(112)^6y(334)^4,$$

so $y(334)=0$. Similarly, $y(443)=0$.

Let

$$g_4=\begin{pmatrix} 1 & & & \\ \frac{y(221)}{y(112)} & 1 & & \\ \frac{y(123)}{y(112)} & & 1 & \\ \frac{y(124)}{y(112)} & & & 1 \end{pmatrix}\in GL(4),$$

$$z=\rho(g_4).y.$$

Then

$$z(113)=z(114)=z(221)=z(331)=z(334)=z(441)$$
$$=z(443)=z(123)=z(124)=z(134)=0.$$

In this case

$$\det\begin{pmatrix} E_{11}, & E_{12}, & E_{21}, & E_{22}, & E_{24}, & E_{31}, & E_{32}, & E_{34}, & E_{41}, & E_{42} \\ e_{112}, & e_{113}, & e_{114}, & e_{123}, & e_{124}, & e_{221}, & e_{224}, & e_{234}. & e_{441}, & e_{442} \end{pmatrix}$$
$$=\pm z(112)^6 z(442)^4.$$

It follows that $z(442)=0$. By symmetry, we can prove that $z(332)=0$. Furthermore

$$\det\begin{pmatrix} E_{11}, & E_{12}, & E_{21}, & E_{23}, & E_{31}, & E_{32}, & E_{34}, & E_{41}, & E_{42}, & E_{43} \\ e_{112}, & e_{113}, & e_{114}, & e_{123}, & e_{124}, & e_{134}, & e_{221}, & e_{224}, & e_{332}, & e_{442} \end{pmatrix}$$
$$=\pm z(112)^6 z(234)^4,$$

implies $z(234)=0$. Finally, if $z(223)=z(224)=0$, then

$$z=z(112)e_{112}.$$

So

$$x\in\rho(G).\ x_{10}.$$

Otherwise, we may suppose that $z(223)\neq 0$. Let

$$g_5=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{z(224)}{z(223)} & 1 \end{pmatrix}\in GL(4),$$

then

$$\rho(g_5).\ z=x_7.$$

Hence

$$x\in\rho(G).\ x_7.$$

9. By calculation, we know that

$$\mathfrak{g}_{x_7}=\left\{\begin{pmatrix} a_1 & b_4 & b_5 & b_1 \\ 0 & a_1 & b_6 & b_2 \\ -b_6 & b_5 & a_1 & b_3 \\ 0 & 0 & 0 & a_2 \end{pmatrix}\middle|\ a_i,\ b_i\in K\right\},$$

$$G_{x_7}=\left\{\left.\begin{pmatrix} t_1^{-2} & u_4 & u_5 & u_1 \\ 0 & t_1^4 & 0 & u_2 \\ 0 & t_1^3u_5 & t_1 & u_3 \\ 0 & 0 & 0 & t_2 \end{pmatrix}\right| t_1,\ t_2\in K^*,\ u_i\in K\right\}.$$

So $\dim \mathfrak{g}_{x_7}=8$ and $\dim G_{x_7}=7$. That implies $x_7\in V_8\subseteq V_9$ and $\dim \rho(G).\,x_7=9$.

Moreover.

$$\mathfrak{g}_{x_{10}}=\left\{\left.\begin{pmatrix} a_1 & b_1 & b_2 & b_3 \\ 0 & a_1 & b_4 & b_5 \\ 0 & 0 & a_2 & b_6 \\ 0 & 0 & b_7 & a_3 \end{pmatrix}\right| a_i,\ b_i\in K\right\},$$

$$G_{x_{10}}=\left\{\left.\begin{pmatrix} t_1 & u_1 & u_2 & u_3 \\ 0 & t_1^{-1} & u_4 & u_5 \\ 0 & 0 & t_2 & u_6 \\ 0 & 0 & u_7 & t_3 \end{pmatrix}\right| t_i\in K^*,\ u_i\in K\right\}.$$

So $\dim \mathfrak{g}_{x_{10}}=\dim G_{x_{10}}=10$, $x_{10}\in V_6\subseteq V_7$ and $\dim \rho(G).\,x_{10}=6$.

Theorem 2. *There exist only one 6-dimensional G-orbit and one 9-dimensional G-orbit in V. They are $\rho(G).\,x_{10}$ and $\rho(G).\,x_7$ respectively. In addition, when $m=7$, 8 or $1\leqslant m\leqslant 5$, there are no m-dimensional G-orbits.*

Proof We have proved that

$$V_9=V_8=\overline{\rho(G).x_7},$$
$$V_7=V_6=\overline{\rho(G).\,x_{10}},$$
$$V_5=V_4=\cdots=V_0=\{0\}.$$

Supposing that $m\leqslant 9$, if x is a point in a m-dimensional orbit, then

$$\dim \mathfrak{g}_x=\dim G_x=n-\dim \rho(G).\,x=n-m\geqslant n-9.$$

That is

$$x\in V_9.$$

Hence $m=9$, 6 or 0 by lemma 6.

10. Other orbits.

By calculation, we can know that there exist following orbits in V:

$$x_2=e_{112}+e_{334}+e_{441},$$

$$G_{x_2}=\left\{\left.\begin{pmatrix} t^4 & u & 0 & 0 \\ 0 & t^{-8} & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t^{-2} \end{pmatrix}\right| \begin{matrix} t\in K^* \\ u\in K \end{matrix}\right\}$$

$$\mathfrak{g}_{x_2}=\left\{\left.\begin{pmatrix} a & b & -c & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & 0 & a \end{pmatrix}\right| a,\ b,\ c\in K\right\},$$

$$x_2\in V_{13}-V_{12}.$$

12-dimensional orbit:

$$x_4=e_{112}+e_{334},$$

$$G_{x_4}=\left\{\begin{pmatrix} t_1 & u_1 & 0 & 0\\ 0 & t_1^{-1} & 0 & 0\\ 0 & 0 & t_2 & u_2\\ 0 & 0 & 0 & t_2^{-1}\end{pmatrix}\middle| \begin{matrix} t_1,\ t_2\in K^*\\ u_1, u_2\in K\end{matrix}\right\},$$

$$\mathfrak{g}_{x_4}=\left\{\begin{pmatrix} a_1 & b_1 & 0 & 0\\ 0 & a_1 & 0 & 0\\ 0 & 0 & a_2 & b_2\\ 0 & 0 & 0 & a_2\end{pmatrix}\middle| a_1,\ a_2,\ b_1,\ b_2\in K\right\},$$

$$x_4\in V_{12}-V_{11}.$$

11-dimensional orbit:

$$x_5=e_{114}+e_{123},$$

$$G_{x_5}=\left\{\begin{pmatrix} t_1 & -t_1^2t_2u_3 & -t_1u_2 & u_1\\ 0 & t_2 & 0 & t_2u_2\\ 0 & 0 & t_1^{-1}t_2^{-1} & u_3\\ 0 & 0 & 0 & t_1^{-1}\end{pmatrix}\middle| \begin{matrix} t_1,\ t_2\in K^*\\ u_1,u_2,u_3\in K\end{matrix}\right\},$$

$$\mathfrak{g}_{x_5}=\left\{\begin{pmatrix} a_1 & -b_3 & -b_2 & b_1\\ 0 & a_2 & 0 & b_2\\ 0 & 0 & -a_1-a_2 & b_3\\ 0 & 0 & 0 & a_1\end{pmatrix}\middle| \begin{matrix} a_1,\ a_2,\\ b_1,\ b_2,b_3\in K\end{matrix}\right\}.$$

$$x_5\in V_{11}-V_{10}.$$

10-dimensional orbit:

$$x_6=e_{123},$$

$$G_{x_6}=\left\{\begin{pmatrix} t_1 & 0 & 0 & u_1\\ 0 & t_2 & 0 & u_2\\ 0 & 0 & t_1^{-1}t_2^{-1} & u_3\\ 0 & 0 & 0 & t_3\end{pmatrix}\middle| \begin{matrix} t_1,t_2,\ t_3\in K^*\\ u_1,\ u_2,\ u_3\in K\end{matrix}\right\},$$

$$\mathfrak{g}_{x_6}=\left\{\begin{pmatrix} a_1 & 0 & 0 & b_1\\ 0 & a_2 & 0 & b_2\\ 0 & 0 & -a_1-a_2 & b_3\\ 0 & 0 & 0 & a_3\end{pmatrix}\middle| \begin{matrix} a_1,\ a_2,\ a_3\\ b_1,\ b_2,\ b_3\in K\end{matrix}\right\},$$

$$x_6\in V_{10}-V_9.$$

Our conjecture is:

There are no other G-orbits. Equivalently, we have a chain of closed subvarieties as follows:

$$V=V_{16}=V_{15}\supsetneqq V_{14}\supsetneqq V_{13}\supsetneqq V_{12}\supsetneqq V_{11}\supsetneqq V_{10}\supsetneqq V_9=V_8\supsetneqq V_7$$

$$=V_6 \supseteq V_5 = V_4 = \cdots = V_0 = \{0\}.$$

All these V_i are irreducible.

It remains to prove that V_{10}, V_{11}, V_{12} and V_{13} are irreducible.

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