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NORMAL EXTENSIONS OF OPERATORS TO KREIN SPACES

WU JINGBO (伍镜波)*

Abstract

In this paper, it is proved that every bounded linear operator on a Hilbert space has a normal extension to a Krein space. Two criteria for J-subnormality are given. In particular, in order that T be subnormal, it suffices that exp $(-\bar{\lambda}T^*)$exp (λT) be a positive definite operator function on a bounded infinite subset of complex plane. This improves the condition of Bram [4]. Also it is proved that the local spectral subspaces are closed for J-subnormal operators.

In [1] we introduced the concept of J-subnormality. A bounded linear operator T on a Hilbert space H is called a J-subnormal operator of order n if on some Π_n-Pontrjagin space Π containing H, there exists a buonded J-normal operator $\tilde{T}$ such that $\tilde{T}f=Tf$ for every f in H and Π is spanned by the vectors of the form $\tilde{T}^{*k}f$, where $f\in H$ and $k=0, 1, 2, \cdots$. Here the Π_n-Pontrjagin space is a Krein space with negative rank n[2]. The J-subnormality of order 0 is equivalent to subnormality. In the first part of the paper, we prove that every bounded linear operator on a Hilbert space has a normal extension to a Krein space. As an application, we obtain two criteria for J-subnormality. In the second part of the paper, we investigate the local spectral theory for J-subnormal operators.

Throughout the paper, by an operator we mean a bounded linear transformation acting on a space. The inner product on a Hilbert space is denoted by $(. , .)$ and the indefinite inner product on a Krein space is denoted by $\langle . , . \rangle$.

Theorem 1. *If T is an operator on a Hilbert space H, ,then there is a Krein space K containing H such that $\langle f,g\rangle=(f,g)$ for all f and g in H and a normal operator $\tilde{T}$ on K such that $\tilde{T}f=Tf$ for all f in H and K is spanned by the vectors of the form $\tilde{T}^{*k}f$, where $f\in H$ and $k=0, 1, 2, \cdots$. In brief, every operator has a minimal normal extension to a Krein space.*

Proof Without loss of generality, we may assume that $0<\|T\|<1$.

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* Department of mathematics, Nankai University, Tianjin, China.

Let $\overline{H}=\bigoplus_{0}^{\infty} H$. Define $\overline{T}$ on $\overline{H}$ by $(\overline{T}\bar{x})_j=\sum_{i=0}^{\infty} T^{*i}T^j x_i$, where $\bar{x}=\{x_i\}_{i=0}^{\infty}\in\overline{H}$. We have

$$\|\overline{T}\bar{x}\|^2=\sum_{j=0}^{\infty}\|\sum_{i=0}^{\infty}T^{*i}T^j x_i\|^2\leqslant\sum_{j=0}^{\infty}(\sum_{i=0}^{\infty}\|T\|^{i+j}\|x_i\|)^2\leqslant\|\bar{x}\|^2/(1-\|T^2\|^2)$$

and

$$(\overline{T}\bar{x},\bar{x})=\sum_{i,j=0}^{\infty}(T^{*i}T^j x_i, x_j)=(\bar{x},\overline{T}\bar{x}).$$

Hence $\overline{T}$ is a bounded symmetric operator on $\overline{H}$.

Let K_0 be the set of all sequences of the form $f=\{f_k\}_{k=0}^{\infty}$, where

$$f_k=\sum_{j=0}^{\infty}T^{*j}T^k x_j,\ \bar{x}=\{x_j\}_{j=0}^{\infty}\in\overline{H}.$$

Define

$$\langle f, g\rangle=(\overline{T}\bar{x},\bar{y}),\ [f, g]=(|\overline{T}|\bar{x},\bar{y}) \tag{1}$$

for $f=\{f_k\}_{k=0}^{\infty}$ and $g=\{g_k\}_{i=0}^{\infty}$ in K_0, where $g_k=\sum_{j=0}^{\infty}T^{*j}T^k y_j$, $\bar{y}=\{y_j\}_{j=0}^{\infty}\in\overline{H}$. Hence

$$\langle f, g\rangle=\sum_{j=0}^{\infty}(f_j, y_j)=\sum_{i,j=0}^{\infty}(T^j x_i, T^i y_j). \tag{2}$$

If $f=0$, then $|\overline{T}|\bar{x}=\overline{T}\bar{x}=0$. It is easy to see that the values $\langle f, g\rangle$ and $[f, g]$ are uniquely determined by f and g. If $[f, f]=0$, then $\overline{T}\bar{x}=|\overline{T}|\bar{x}=0$, and so $f=0$. Thus K_0 is a pre-Hilbert space with inner product $[.,.]$. Let K be the completion of K_0. Define $\overline{H}_-=\ker\overline{T}^+$ and $\overline{H}_+=\overline{H}\ominus\overline{H}_-$. Let K_-^0 and K_+^0 be sets of all sequences of forms $f^-=\{f_k^-\}_{k=0}^{\infty}$ and $f^+=\{f_k^+\}_{k=0}^{\infty}$, respectively, where

$$f_k^-=\sum_{j=0}^{\infty}T^{*j}T^k x_j^-,\ x^-=\{x_j^-\}_{j=0}^{\infty}\in\overline{H}_-$$

and

$$f_k^+=\sum_{j=0}^{\infty}T^{*j}T^k x_j^+,\ \bar{x}^+=\{x_j^+\}_{j=0}^{\infty}\in\overline{H}_+.$$

Let K_- and K_+ be the completions of K_-^0 and K_+^0, repectively. For

$$f^-=\{f_k^-\}_{k=0}^{\infty}\in K_-^0\quad\text{and}\quad f^+=\{f_k^+\}_{k=0}^{\infty}\in K_+^0,$$

it follows from (1) that

$$[f^-, f^+]=(|\overline{T}|\bar{x}^-,\bar{x}^+)=(\overline{T}^-\bar{x}^-,\bar{x}^+)=(\bar{x}^-,\overline{T}^-\bar{x}^+)=0.$$

Thus $K_+\perp K_-$, and so $K=K_+\oplus K_-$ (with respect to $[.,.]$). Let P_+ and P_- be the orthogonal projections (with respect to $[.,.]$) of K onto K_+ and K_-, respectively. Denote $J=P_+-P_-$. For $f, g\in K_0$, it follows from (1) that

$$\langle f,g\rangle=[Jf, g].$$

Then K may be considered to be a Krein space with indefinite inner product $\langle .,.\rangle$ It is clear that $K_+\oplus K_-$ be a fundamental decomposition of K.

The space H may be embedded as a subspace in K. In fact, we define

$$f_x=\{T^k x\}_{k=0}^{\infty}$$

for all $x\in H$ and note that $\langle f_x, f_y\rangle=(x, y)$ for all $x, y\in H$. For $f=\{f_k\}_{k=0}^{\infty}\in K_0$, define $\tilde{T}f=\{f_{k+1}\}_{k=0}^{\infty}$. It follows that $f_{Tx}=\tilde{T}f_x$ for $x\in H$.

We are going now to show that the operator T is bounded in K_0. Define $\bar{S}$ on $\bar{H}$ by $(\bar{S}\bar{x})_j=Tx_j$, where $\bar{x}=\{x_j\}_{j=0}^{\infty}\in\bar{H}$. We have

$$\|\bar{T}\bar{S}\bar{x}\|^2=\sum_{j=0}^{\infty}\|\sum_{i=0}^{\infty}T^{*i}T^{j+1}x_i\|^2=\sum_{j=1}^{\infty}\|\sum_{i=0}^{\infty}T^{*i}T^{j}x_i\|^2\leqslant\|\bar{T}\bar{x}\|^2.$$

It follows that

$$\|\bar{S}^*|\bar{T}|\bar{S}\bar{x}\|\leqslant\|T\|\cdot\||\bar{T}|\bar{S}\bar{x}\|=\|T\|\cdot\|\bar{T}\bar{S}\bar{x}\|\leqslant\|T\|\cdot\|\bar{T}\bar{x}\|=\|T\|\cdot\||\bar{T}|\bar{x}\|.$$

By Heinz's inequality [3], we have

$$(\bar{S}^*|\bar{T}|\bar{S}\bar{x},\bar{x})\leqslant\|T\|(|\bar{T}|\bar{x},\bar{x}).$$

For $f\in K_0$, this implies that

$$[\tilde{T}f,\tilde{T}f]=(|\bar{T}|\bar{S}\bar{x},\bar{S}\bar{x})\leqslant\|T\|(|\bar{T}|\bar{x},\bar{x})=\|T\|[f,f]. \tag{3}$$

Thus $\tilde{T}$ may be considered to be a bounded operator on K.

Let $\tilde{T}^*$ be the adjoint of $\tilde{T}$ with respect to $\langle.,.\rangle$. For $f\in K_0$, we have

$$\tilde{T}^*f=\{f_k'\}_{k=0}^{\infty},$$

where

$$f_k'=\sum_{j=1}^{\infty}T^{*j}T^{k}x_{j-1}.$$

It follows that

$$\tilde{T}^*\tilde{T}f=\tilde{T}^*\{f_{k+1}\}_{k=0}^{\infty}=\left\{\sum_{j=1}^{\infty}T^{*j}T^{k+1}x_{j-1}\right\}_{k=0}^{\infty}=\tilde{T}\{f_k'\}_{k=0}^{\infty}=\tilde{T}\tilde{T}^*f.$$

Thus $\tilde{T}$ is normal with respec to $\langle.,.\rangle$.

For every $f=\{f_k\}_{k=0}^{\infty}\in K_0$, let $f^m=\{f_k^m\}_{k=0}^{\infty}$, where $f_k^m=\sum_{j=0}^{m}T^{*j}T^kx_j$. Define $\bar{x}^m=\{x_k^m\}_{k=0}^{\infty}$, where $x_k^m=0$ for $k\leqslant m$ and $x_k^m=x_k$ for $k>$m. Since $\lim_{m\to\infty}\|x^m\|=0$, we have

$$\lim_{m\to\infty}[f-f^m,f-f^m]=\lim_{m\to\infty}(|\bar{T}|\bar{x}^m,\bar{x}^m)=0.$$

It is easy to see that $f^m=\sum_{j=0}^{m}\tilde{T}^{*j}f_{x_j}$. Then K is spanned by the vectors of the form $\tilde{T}^{*k}f_x$, where $x\in H$ and $k=0,1,2,\cdots$.

This completes the proof of Theorem 1.

Theorem 2. *Let T be an operator on a Hilbert space H and let Ω be a bounded infinite subset of complex plane. A necessary and sufficient condition for T to be a J-subnormal operator of order n is that the Hermitian form*

$$\sum_{i,j=1}^{\infty}(\exp(\lambda_jT)x_i,\exp(\lambda_iT)x_j)\alpha_j\bar{\alpha}_i$$

has at most n negative squares for every choice of a finite number of $x_1,\cdots,x_m\in H$ and $\lambda_1,\cdots,\lambda_m\in\Omega$ and has exactly n negative squares for the least one choice of $x_1,\cdots,x_m$ and $\lambda_1,\cdots,\lambda_m$. In particular, a necessary and sufficient condition for T to be subnormal is that

$$\sum_{i,j=1}^{u}(\exp(\lambda_jT)x_i,\exp(\lambda_iT)x_j)\geqslant 0$$

for every choice of a finite number of $x_1, \cdots, x_m \in H$ and $\lambda_1, \cdots, \lambda_m \in \Omega$.

Proof Let λ_0 be a limit point of Ω. By Theorem 1, operator T has a minimal normal extension $\widetilde{T}$ to a Krein space K. For $x_1, \cdots, x_m \in H$ and $\lambda_1, \cdots, \lambda_m \in \Omega$, we have

$$\sum_{i,j=1}^{m}(\exp(\lambda_j T)x_i, \exp(\lambda_i T)x_j)\alpha_i\bar{\alpha}_j = \sum_{i,j=1}^{m}\langle \exp(\lambda_j \widetilde{T})x_i, \exp(\lambda_i \widetilde{T})x_j\rangle \alpha_i\bar{\alpha}_j$$
$$= \langle \sum_{i=1}^{m}\alpha_i \exp(\bar{\lambda}_i \widetilde{T}^*)x_i, \sum_{j=1}^{m}\alpha_j \exp(\bar{\lambda}_j \widetilde{T}^*)x_j\rangle. \tag{4}$$

Let K' be the linear closed set spanned by the vectors of the form $\exp(\bar{\lambda}\widetilde{T}^*)x$, where $\lambda \in \Omega$ and $x \in H$. It is clear that $\exp(\bar{\lambda}_0\widetilde{T}^*)x \in K'$ for $x \in H$. Assume that $\exp(\bar{\lambda}_0\widetilde{T}^*)\widetilde{T}^{*i}x \in K'$ for $x \in H$ and $i = 0, 1, \cdots, k-1$. It follows that for $x \in H$

$$\exp(\bar{\lambda}_0\widetilde{T}^*)\widetilde{T}^{*k}x = \lim_{\lambda\in\Omega,\lambda\to\lambda_0}\frac{k!}{(\bar{\lambda}-\bar{\lambda}_0)^k}\Big[\exp(\bar{\lambda}\widetilde{T}^*)x$$
$$-\sum_{i=1}^{k-1}\frac{1}{i!}(\bar{\lambda}-\bar{\lambda}_0)^i\exp(\bar{\lambda}_0\widetilde{T}^*)\widetilde{T}^{*i}x\Big]$$

and so $\exp(\bar{\lambda}_0\widetilde{T}^*)\widetilde{T}^{*k}x \in K'$. This means that $\exp(\bar{\lambda}_0\widetilde{T}^*)\widetilde{T}^{*k}x \in K'$ for all $x \in H$ and $k = 0, 1, 2, \cdots$. Since the extension is minimal, this implies that $K' = K$. The desired conclusion follows from (4).

Remark. Setting $x_i = \exp(-\lambda_i T)y_i$, Theorem 2 gives an improvement of Theorem 2 of [1]. More precisely, in order that T be a J-subnormal operator of order n, it suffices that $\exp(-\bar{\lambda}T^*)\exp(\lambda T)$ be a quasi-positive definite operator function of order n on a bounded infinite subset of complex plane. In particular, in order that T be subnormal, it suffices that $\exp(-\bar{\lambda}T^*)\exp(\lambda T)$ be a positive definite operator function on a bounded infinite subest of complex plane. The last proposition is an improvement of a theorem of Bram[4].

Corollary. *Let T be a J-subnormal operator of order n and let S be a normal operator which commutes with T. Then $T+S$ is a J-subnormal operator of order n and TS is a J-subnormal operator of order $\leqslant n$.*

Proof By the Fuglede Theorem, we have $S^*T = TS^*$. For complex numbers $\lambda_1, \cdots, \lambda_m$ and for $x_1, \cdots, x_m \in H$,

$$\sum_{i,j=1}^{m}(\exp(\lambda_j(T+S))x_i, \exp(\lambda_i(T+S)x_j)\alpha_i\bar{\alpha}_j$$
$$= \sum_{i,j=1}^{m}(\exp(\lambda_j T)\exp(\bar{\lambda}_i S^*)x_i, \exp(\lambda_i T)\exp(\bar{\lambda}_j S^*)x_j)\alpha_i\bar{\alpha}_j.$$

By Theorem 2, $T+S$ is a J-subnormal operator of order n. For $x_{ij} \in H$, $i, j = 1, \cdots, m$,

$$\sum_{i,j,k,l=0}^{m}((TS)^j x_{ik}, (TS)^i x_{jl})\alpha_{ik}\bar{\alpha}_{jl} = \sum_{i,j,k,l=0}^{m}(T^j S^{*i} x_{ik}, T^i S^{*j} x_{jl})\alpha_{ik}\bar{\alpha}_{jl}.$$

By Theorem 1 of [1], TS is a J-subnormal operator of order $\leqslant n$.

Theorem 3. *Let T be an operator on a Hilbert space H and let Ω be an unbounded* *subset of complex plane with* $|\lambda|>\|T\|$ *for every* $\lambda\in\Omega$. *A necessary and sufficient condition for* T *to be a* J*-subnormal operator of order* n *is that the Hermitian form*

$$\sum_{i,j=1}^{m}((\lambda_j-T)^{-1}x_i,\ (\lambda_i-T)^{-1}x_j)\alpha_i\bar{\alpha}_j$$

has at most n *negative squares for every choice of a finite number of* $x_1, \cdots, x_m\in H$ *and* $\lambda_1, \cdots, \lambda_m\in\Omega$ *and has exactlyin negative squares for the least one choice of* $x_1, \cdots, x_m$ *and* $\lambda_1, \cdots, \lambda_m$. *In particular, a necessary and sufficient condition for* T *to be subnormal is that*

$$\sum_{i,j=1}^{m}((\lambda_j-T)^{-1}x_i,\ (\lambda_i-T)^{-1}x_j)\geqslant 0$$

for every choice of a finite number of $x_1, \cdots, x_m\in H$ *and* $\lambda_1, \cdots, \lambda_m\in\Omega$.

Proof By Theorem 1, operator T has a minimal normal extension $\tilde{T}$ to a Krein space K. By the inequality (3), we have $\lambda\in\rho(\tilde{T})$ and $\bar{\lambda}\in\rho(\tilde{T}^*)$ for every $\lambda\in\Omega$. For $x_1, \cdots, x_m\in H$ and $\lambda_1, \cdots, \lambda_m\in\Omega$, we have

$$\sum_{i,j=1}^{m}((\lambda_j-T^{-1})x_i,\ (\lambda_i-T)^{-1}x_i)\alpha_i\bar{\alpha}_j=\sum_{i,j=1}^{m}\langle(\lambda_j-\tilde{T})^{-1}x_i,\ (\lambda_i-\tilde{T})^{-1}x_j\rangle\alpha_i\bar{\alpha}_j$$
$$=\langle\sum_{i=1}^{m}\alpha_i(\bar{\lambda}_i-\tilde{T}^*)^{-1}x_i,\ \sum_{j=1}^{m}\alpha_j(\lambda_j-\tilde{T}^*)^{-1}x_j\rangle. \tag{5}$$

Let $K\alpha$ be the linear closed set spanned by the vectors of the form $(\bar{\lambda}-\tilde{T}^*)^{-1}x$, where $\lambda\in\Omega$ and $x\in H$. Since $\operatorname*{s-lim}_{\lambda\in\Omega,\lambda\to\infty}\bar{\lambda}(\bar{\lambda}-\tilde{T}^*)^{-1}=I$, we have $H\subset K'$. Assume that $\tilde{T}^{*i}x\in K'$ for $x\in H$ and $i=0, 1, \cdots, k-1$. It follows that for $x\in H$,

$$\tilde{T}^{*k}x=\lim_{\lambda\in\Omega,\lambda\to\infty}\bar{\lambda}^{k+1}[(\bar{\lambda}-\tilde{T}^*)^{-1}-\sum_{j=0}^{k-1}\bar{\lambda}^{-j-1}\tilde{T}^{*j}]x$$

and so $\tilde{T}^{*k}x\in K'$. Since the extension is minimal, this means that $K'=K$. Thus the desired conclusion follows from (5).

Theorem 4. *Suppose that* T *is a* J*-subnormal operator of order* n. *Then* T *has the single-valued extension property.*

Proof Let T be a J-subnormal operator of order n on H. By [5], there exists a polynomial $Q(\lambda)=\prod_{k=1}^{n}(\lambda-\mu_k)$ such that $T|H_0$ is a subnormal operator on $H_0=[\operatorname{ran} Q(T)]^-$. Suppose that $f(\lambda)$ is an H-valued function analytic on an open subset Ω of complex plane such that

$$(T-\lambda)f(\lambda)=0,\quad \lambda\in\Omega. \tag{6}$$

Since a subnormal operator has the single-valued extension property, it follows from (6) that

$$\prod_{k=1}^{n}(T-\mu_k)f(\lambda)=0,\quad \lambda\in\Omega. \tag{7}$$

From (6) and (7), we have $(\mu_n-\lambda)\prod_{k=1}^{n-1}(T-\mu_k)f(\lambda)=0$,and so

$$\prod_{k=1}^{n-1}(T-\mu_k)f(\lambda)=0,\quad \lambda\in\Omega.$$

Along this way, we conclude that $f(\lambda)=0$, $\lambda\in\Omega$. This completes the proof.

Theorem 5. *Let T be a J-subnormal operator of order n on H and let x be in H. Suppose that the open subset Ω of complex plane contains no eigenvalues of T and that there is a bounded H-valued function $x(\lambda)$ anablytic on Ω, such that $(T-\lambda)x(\lambda)=x$ for all $\lambda\in\Omega$. Then $x(\lambda)$ is analytic on Ω.*

Proof By [5], there is a polynomial $Q(\lambda)=\prod_{k=1}^{n}(\lambda-\mu_k)$ such that $T|H_0$ is a subnormal operator on $H_0=[\text{ran } Q(T)]^-$. The identity $(T-\lambda)x(\lambda)=x(\lambda\in\Omega)$ implies that $(T-\lambda)Q(T)x(\lambda)=Q(T)x(\lambda\in\Omega)$. Directly from a theorem of Putnam [6], it follows that $\prod_{k=1}^{n}(T-\mu_k)x(\lambda)$ is analytic on Ω. By the assumption, $x(\lambda)$ is weakly continuous on Ω. Since $(T-\lambda)\prod_{k=1}^{n-1}(T-\mu_k)x(\lambda)$ is constant on Ω, $(\lambda-\mu_n)$, $\prod_{k=1}^{n-1}(T-\mu_k)x(\lambda)$ is analytic on Ω and so is $\prod_{k-1}^{n-1}(T-\mu_k)x(\lambda)$. Along this way, we conclude that $x(\lambda)$ is analytic on Ω. This completes the proof.

Let δ be a closed subset of complex plane and let T be an operator on H. We define the local spectral subspace as $\mathfrak{M}_T(\delta)=\{x\in H: \sigma(x, T)\subset\delta\}$, where $\sigma(x, T)$ is the local spectrum of T at x.

Theorem 6. *Let T be a J-subnormal operator of order n on H. Suppose that δ is a closed subset of complex plane. Then $\mathfrak{M}_T(\delta)$ is a closed invariant subspace for T.*

Proof It is clear that $\mathfrak{M}_T(\delta)$ is invariant under T. By [5], T has an invariant subspace H_0 with finite codimension in H such that $V|H_0$ is subnormal. By Proposition 1.10 of [7], we have $\mathfrak{M}_{T|H_0}(\delta)\subset\mathfrak{M}_T(\delta)$. Let $x\in H_0$. By Proposition 2.9 and Theorem 2.11 of [7], we have $\sigma(x, T|H_0)=\sigma(x, T)$. This implies that

$$[\mathfrak{M}_T(\delta)\ominus\mathfrak{M}_{T|H_0}(\delta)]\cap H_0=\{0\}.$$

By Corollary I. 3.4 of [2], we have $\dim\,[\mathfrak{M}_T(\delta)\ominus\mathfrak{M}_{T|H_0}(\delta)]<\infty$. Since $\mathfrak{M}_{T|H_0}(\delta)$ is closed, so is $\mathfrak{M}_T(\delta)$. This completes the proof.

Theorem 7. *Let T be a J-subnormal operator of order n and $n>0$. Then T^* is not dominant.*

Proof By [5], T has an invariant subspace H_0 with $0<\text{codim } H_0\leqslant n$ such that $T|H_0$ is subnormal. We may assume that there does not exist another invariant subspace of T containing H_0 on which T is subnormal. Since $H_1=H_0^{\perp}$ is invariant under T^*, there is an eigenvector x corresponding to the eigenvalue μ of $T^*|H_1$. If T^* is dominant, thenthere is an M_μ such that $\|(T-\bar{\mu})x\|\leqslant M_\mu\|(T^*-\mu)x\|$. This implies that $Tx=\bar{\mu}x$. Let $H_2=H_0\oplus\{\alpha x: \alpha\in C\}$. Thus H_2 is invariant under T and $T|H_2$ is subnormal. This gives a contradiction and completes the proof.

Theorem 8. *Let T be a J-subnormal operator of order n. If $[T]=T^*T-TT^*$, then* rank $[T]_-\leqslant n$, *where $[T]_-$ is the negative part of $[T]$.*

Proof Since $\text{ran}[T]_-\subset\text{ran}[T]_-^{1/2}$, it is sufficient to show that dim ran $[T]_-^{1/2}\leqslant$

n. If dim ran $[T]_-^{1/2} > n$, then there exist $x_1, \cdots, x_{n+1} \in H$ such that

$$([T]_-^{1/2} x_i, [T]_-^{1/2} x_j) = \delta_{ij}, \quad i, j = 1, \cdots, n+1.$$

Define $H_- = \ker [T]_+$. Let P_- be the orthogonal projection of H onto H_-. Therefore

$$\sum_{i,j=1}^{n+1} ([T] P_- x_i, P_- x_j) \alpha_i \bar{\alpha}_j = -\sum_{i,j=1}^{n+1} ([T]_- x_i, x_j) \alpha_i \bar{\alpha}_j = -\sum_{j=1}^{n+1} |\alpha_j|^2.$$

This is in contradiction with Theorem 4 of [1]. The proof is complete.

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