Chin. Ann. of Math.
8B (1) 1987

AN ANALYSIS OF A FINITE ELEMENT METHOD OF LOW DEGREE FOR THE NAVIER-STOKES PROBLEMS**

Li Likang (李立康)*

Abstract

There are many papers in which approximate solution of Navier-Stokes problem is discussed by finite element method. Their error estimates are optimal, but degree of piecewise polynomials for pressure p or degree of piecewise polynomials for velocity $\boldsymbol{u}$ are not the lowest. In this papre a new element is given. Its degre for p and degree for $\boldsymbol{u}$ are the lowest and error estimates are optimal.

§ 1. Introduction

Let Ω be a bounded open domain in R^2 and its boundary be Γ. In this paper we consider the stationary Stokes problem and Navier-Stokes problem

$$\begin{cases} -\nu\Delta\boldsymbol{u}+\mathbf{grad}\,p=\boldsymbol{f}, & \text{in } \Omega, \quad (1.1)\\ \operatorname{div}\boldsymbol{u}=0, & \text{in } \Omega, \quad (1.2)\\ \boldsymbol{u}=0, & \text{on } \Gamma, \quad (1.3)\end{cases}$$

$$\begin{cases} -\nu\Delta\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\mathbf{grad}\,p=\boldsymbol{f}, & \text{in } \Omega, \quad (1.4)\\ \operatorname{div}\boldsymbol{u}=0, & \text{in } \Omega, \quad (1.5)\\ \boldsymbol{u}=0, & \text{on } \Gamma. \quad (1.6)\end{cases}$$

Let Ω be a convex polygon. There are many papers in which Navier-Stokes problem is discussed by finite element method. In [1], an element of lower order in $\boldsymbol{u}$ is used, namely: a linear approximation for $\boldsymbol{u}$ and a linear approximation for p, where each triangle of triangulation for p is actually a macroelement made of four triangles of triangulation for $\boldsymbol{u}$ (see Fig. 1 in [1]). The approximations have H^1 error of h and L^2 error of h^2 for $\boldsymbol{u}$. They did not point out that the approximation has L^2 error of h for p which we can easily obtain. In [2], the finite element spaces for $\boldsymbol{u}$ and p are $(W_{h0})^2$ and M_h respectively, where W_{h0} is the set of piecewise polynomials of degree 2, each element of W_{h0} is equal to zero on Γ, M_h is the set of piecewise constant. Each triangle of triangulation for p coincides with each triangle

Manuscript received September 3, 1984.

* Department of Mathematics, Fudan University, Shanghai, China.

** Project Supported by Science Fund of the Chinese Academy of Science.

of triangulation for u. The approximations have H^1 error of h and L^2 error of h^2 for u. The approximation has L^2 error of h for p. In [1] and [2], these approximations are optimal, but the degree of piecewise polynomials for p in [1] and the degree of piecewise polynomials for u in [2] are not the lowest. We propose the following question: Let X_h^0 and Q_h denote the set of piecewise polynomials of degree 1 whose element is equal to zero on Γ and the set of piecewise constants respectively. We take $(X_h^0)^2$ and $Q_{h0}=Q_h\backslash R$ as the finite element spaces for u and p respectively. Can we conclude that the approximations have H^1 error of h and L^2 error of h^2 for u, the approximation has L^2 error of h for p? If the conclusion can be reached, then the approximations are optimal, the degree of piecewise polynomials for u and the degree of piecewise polynomials (piecewise constants) for p are the lowest. Obviously, this question has practial meaning. It is well known (see [2] § 2.2, Chapter 2) that the above conclusion does not hold as the triangulations for u and p are the same. In this paper we point out that the above conclusion is right as the triangulations for u and p are different. The other object of this paper is to extend the above conclusion to problem (1.4)$\sim$(1.6) with sufficiently smooth domain Ω.

The results obtained are organized as follows: In § 2 we give the triangulations for u and p and construct finite element spaces. In § 3 we prove Bpezzi inequality. In § 4 we give the error estimate for stokes problem. In § 5 we give the error estimate for Navier-Stokes problem.

§ 2. Triangulations and Finite Element Spaces

Let $\{K_i\}$ be a triangulation of Ω (see Fig. 1). We have $\overline{\Omega}=\bigcap_{i=1}^{m_0+m_1} K_i$, where K_1, $\cdots$, K_{m_0} are all boundary elements, K_{m_0+1}, $\cdots$, $K_{m_0+m_1}$ are all interiore lements. Triangulation satisfies the regular condition in [3]. Define

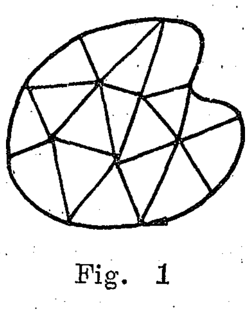

Fig. 1

$$Q_h=\{q_h \mid q_h\in L^2(\Omega),\ q_h \text{ on each } K_i \text{ is a constant } (i=1,\ \cdots,\ m_0+m_1)\}.$$

Let K_i be an interior element. K_{i_1}, $\cdots$, K_{i_4} is a triangulation of K_i by dividing K_i into 4 equal subtriangles (by joining the mid-sides, see Fig. 2). Let K_i be a boundary element. K_{i1}, $\cdots$, K_{i4} is a triangulation of K_i by dividing K_i into 4 subtriangles (see Fig 3). In Fig. 3 D and E are midpoints of sides OB and OA respectively. For simplicity, we assume $EC \parallel OB$. Define

$$X_h=\{v_h \mid v_h\in C^\circ(\Omega),\ v_h \text{ on each } K_{ij}\ (i=1,\ \cdots,\ m_0+m_1;\ j=1,\ \cdots,\ 4) \text{ is a linear function}\},$$

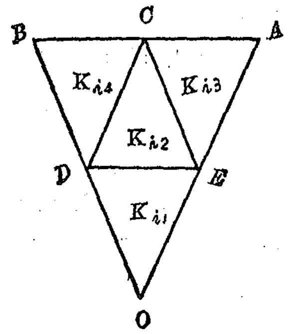

Fig. 2

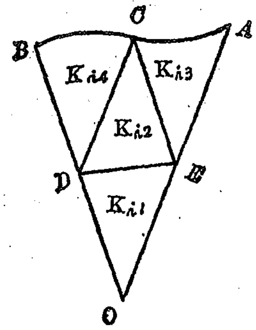

Fig. 3

Let $A_1, \cdots, A_{2m_0}$ be all boundary nodes of triangulation $\{K_{ij}\}$. Define

$$X_h^0=\{v_h \mid v_h\in X_h,\ v_h(A_i)=0\ (i=1, \cdots, 2m_0)\},$$

$$\boldsymbol{X}_h^0=X_h^0\times X_h^0,\quad Q_{ho}=Q_h\backslash R,$$

$$\boldsymbol{V}_h^0=\{\boldsymbol{v}_h \mid \boldsymbol{v}_h\in \boldsymbol{X}_h^0,\quad (\operatorname{div}\boldsymbol{v}_h,\ \mu_h)=0,\ \forall \mu_h\in Q_{ho}\}.$$

We shall not give the definition of Sobolev space $H^m(\Omega)$, $H_0^1(\Omega)$, $\cdots$, which are well known. We denote the norms and the seminorms of Sobolev spaces $H^m(\Omega)$, $H_0^1(\Omega)$, $\cdots$, by $\|\cdot\|_m$, $|\cdot|_m$, $\cdots$. We denote $H^0(\Omega)$ by $L^2(\Omega)$ and

$$L_0^2(\Omega)=\{v \mid v\in L^2(\Omega),\ \int_\Omega v\,dx=0\}(=L^2(\Omega)\backslash R),$$

where $dx=dx_1dx_2$. Let $v\in H^m(\Omega)\cap L_0^2(\Omega)$. We denote the norm and the seminorm of $H^m(\Omega)\cap L_0^2(\Omega)$ by $\|\cdot\|_{m\backslash 1}$ and $|\cdot|_{m\backslash 1}$ respectively.

Define

$$a(\boldsymbol{u},\ \boldsymbol{v})=\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,dx,\quad b(\boldsymbol{u},\ p)=(\operatorname{div}\boldsymbol{u},\ p),$$

$$\boldsymbol{f}(\boldsymbol{v})=\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}dx=(\boldsymbol{f},\boldsymbol{v}),\ (u,\ v)=\int_\Omega uv\,dx.$$

Variational problem P_{h1} of $(1.1)\sim(1.3)$: Find $(\boldsymbol{u}_h,\ p_h)\in \boldsymbol{X}_h^0\times Q_{ho}$ satisfying

$$\begin{cases} \nu a(\boldsymbol{u}_h,\ \boldsymbol{v}_h)-b(\boldsymbol{v}_h,\ p_h)=\boldsymbol{f}(\boldsymbol{v}_h), & \forall \boldsymbol{v}_h\in \boldsymbol{X}_h^0,\\ b(\boldsymbol{u}_h,\ \mu_h)=0, & \forall \mu_h\in Q_{ho}. \end{cases}$$

Variational problem P_{h2} of $(1.1)\sim(1.3)$: Find $\boldsymbol{u}_h\in \boldsymbol{v}_h^0$ satisfying

$$\nu a(\boldsymbol{u}_h,\ \boldsymbol{v}_h)=\boldsymbol{f}(\boldsymbol{v}_h),\quad \forall \boldsymbol{v}_h\in \boldsymbol{V}_h^0. \tag{2.2}$$

Lemma 2.1. *$\boldsymbol{X}_h^0$ and $\boldsymbol{V}_h^0$ are Hilbert space. The mapping $\boldsymbol{u}_h\in \boldsymbol{X}_h^0$(or $\boldsymbol{V}_h^0$)$\to|\boldsymbol{u}_h|_1=a(\boldsymbol{u}_h,\ \boldsymbol{u}_h)$ is a norm over space $\boldsymbol{X}_h^0$(or $\boldsymbol{V}_h^0$). Inaddition, we have $\boldsymbol{V}_h^0\neq\{\boldsymbol{0}\}$.*

We skip the proof since it is easy.

Theorem2.1. *Variational problem P_{h2} has a unique solution.*

Proof Sine the linear functional $\boldsymbol{v}_h\to\boldsymbol{f}(\boldsymbol{v}_h)$ and the bilinear form $(\boldsymbol{u}_h,\ \boldsymbol{v}_h)\to \nu a(\boldsymbol{u}_h,\ \boldsymbol{v}_h)$ are continuous over $\boldsymbol{V}_h^0$ and $\boldsymbol{V}_h^0\times \boldsymbol{V}_h^0$ respectively. Theorem 2.1 holds from Lemma 2.1 and Lax–Milgram Lemma.

Theorem 2.2. *Variational problem P_{h1} has a unique solution.*

Proof To prove the existence and uniqueness for (2.1) we need only show the only solution of (2.1) when $\boldsymbol{f}=0$ is $(\boldsymbol{u}_h, p_h)=(\boldsymbol{0}, 0)$.

Taking $\boldsymbol{v}_h=\boldsymbol{u}_h$, $\mu_h=p_h$ in (2.1), we obtain $a(\boldsymbol{u}_h, \boldsymbol{u}_h)=0$. Hence $\boldsymbol{u}_h=0$ follows from Lemma 2.1. By (2.1) we obtain

$$b(\boldsymbol{v}_h, p_h)=0, \quad \forall \boldsymbol{v}_h\in \boldsymbol{X}_h^0.$$

Then by the following Brezzi inequality (3.1) we have $\|p_h\|_{0\backslash 1}=0$, i. e. $p_h=0$.

Both Variational problems P_{h1} and P_{h2} can be characterized by the following Theorem 2.3.

Theorem 2.3. *Let $\boldsymbol{u}_h$ be a solution of (2.2). Then there exists a unique p_h such that $(\boldsymbol{u}_h, p_h)$ is a solution of (2.1). Conversely, let $(\boldsymbol{u}_h, p_h)$ be a solution of (2.1), then $\boldsymbol{u}_h$ is a solution of (2.2).*

We skip the proof since it is easy.

§ 3. Brezzi Inequality

Lemma 3.1. *Let K_i be a boundary element, $\boldsymbol{v}_h\in \boldsymbol{X}_h^0$. We have*

$$|\boldsymbol{v}_h|_{0,\gamma_{ji}}\leqslant Ch_i^{\frac{3}{2}}|\boldsymbol{v}_h|_{1,K_{ij}}, \quad i=1, \cdots, m_0;\ j=3, 4,$$

where $\gamma_{ij}=K_{ij}\cap\Gamma$, C is a positive constant independent of h_i, $h_i=\operatorname{diam}(K_i)$.

The proof may be found in [4].

In this paragraph we would like to show

$$\sup_{\boldsymbol{v}_h\in \boldsymbol{X}_h^0}\frac{b(\boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_1}\geqslant\alpha\|q_h\|_{0\backslash 1}, \quad \forall q_h\in Q_{h0}, \tag{3.1}$$

where α is a positive constant.

In this paper, C denotes a generic constant with possibly different values in different contexts.

Let $h_i=\operatorname{diam}(K_i)$, $h=\max\limits_{1\leqslant i\leqslant m_0+m_1} h_i$. Assume $\dfrac{h}{h_i}\leqslant C$. Let $\hat{K}$ be a reference eelment. Let K_i be any interior element. Then there exists an affine mapping

$$F_i(x=B_i\hat{x}+b_i):\ \hat{K}\to K_i.$$

Let K_i be any boundary element. We construct a triangle $\widetilde{K}_i$ such that three vertexes of K_i coincide with three vertexes of $\widetilde{K}_i$. Then there exists an affine mapping $F_i:\hat{K}\to\widetilde{K}_i$. Each F_i is an invertible mapping. Let $\widehat{A_iB_i}$ in K_i denote an arc on Γ and $\partial\hat{K}_i=F_i^{-1}(\widehat{A_iB_i})$. We construct $\Delta_1\hat{O}\hat{A}\hat{B}$ = a curved triangle $\hat{O}\hat{A}\hat{B}$ (as shown in Fig. 4), it consists of segments $\hat{O}\hat{A}$ and $\hat{O}\hat{B}$ and arc $\hat{A}\hat{C}\hat{B}$ such that

$$\partial\hat{K}_i\cap(\Delta_1\hat{O}\hat{A}\hat{B})^\circ=\phi \quad (i=1, \cdots, m_0),$$

where A° denotes the set of interior points of set A. Since Γ is sufficiently smooth, there exists such $\Delta_1\hat{O}\hat{A}\hat{B}$.

Let $q_h\in Q_{h0}$. According to Lemma 3.2 (see [2], Chapter 1), there exists one function $\boldsymbol{v}\in(H_0^1)^2$ such that

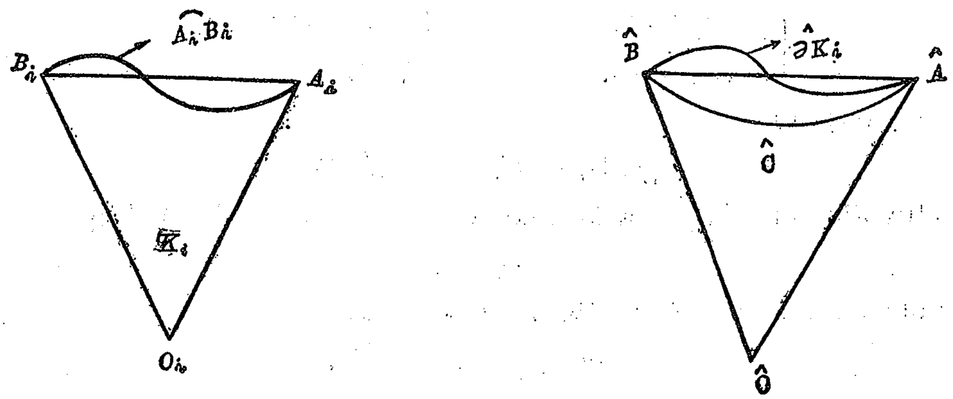

Fig. 4

$$\operatorname{div} \boldsymbol{v}=q_h, \quad |\boldsymbol{v}|_1 \leqslant C\|q_h\|_{0\setminus 1}. \tag{3.2}$$

According to Lemma 2.1 and Lax-Milgram lemma, there exists a unique $\boldsymbol{w}_h \in \boldsymbol{X}_h^0$ satisfying

$$a(\boldsymbol{w}_h-\boldsymbol{v}, \boldsymbol{y}_h)=0, \quad \forall \boldsymbol{y}_h \in \boldsymbol{X}_h^0. \tag{3.3}$$

Moreover, we have

$$|\boldsymbol{w}_h|_1 \leqslant |\boldsymbol{v}|_1. \tag{3.4}$$

Then we define $\boldsymbol{v}_h \in \boldsymbol{X}_h^0$ by:

1. $K_i (i=m_0+1, \cdots, m_0+m_1)$ is an interior element. Let $A_1=0$, $A_2=A$, $A_3=B$, $A_{23}=C$, $A_{13}=D$, $A_{12}=E$ (see Fig. 2). $\boldsymbol{v}_h$ on K_i is defined by

$$\begin{cases} \boldsymbol{v}_h(A_j)=\boldsymbol{w}_h(A_j), & j=1, 2, 3, \\ \int_{\overline{A_j A_k}} (\boldsymbol{v}_h-\boldsymbol{v})\, ds=0, & 1 \leqslant j<k \leqslant 3. \end{cases} \tag{3.5}$$

2. $K_i (i=1, \cdots, m_0)$ is a bounary element. Let $A_1=0$, $A_2=A$, $A_3=B$, $A_{23}=C$, $A_{13}=D$, $A_{12}=E$ (see Fig. 3). $\boldsymbol{v}_h$ on K_i is defined by

$$\begin{cases} \boldsymbol{v}_h(A_j)=\boldsymbol{w}_h(A_j), & j=1, 2, 3, \\ \boldsymbol{v}_h(A_{23})=0, \\ \int_{\overline{A_j A_k}} (\boldsymbol{v}_h-\boldsymbol{v})\, ds=0, & (j,k)=(1, 2), (1, 3). \end{cases} \tag{3.6}$$

We can easily prove that there exists a unique $\boldsymbol{v}_h$ satisfying (3.5)and (3.6).

Lemma 3.2. *The function $\boldsymbol{v}_h \in \boldsymbol{X}_h^0$ defined by (3.5) and (3.6) satisfies the following inequality.*

$$|\boldsymbol{v}_h|_1 \leqslant C\|q_h\|_{0\setminus R}. \tag{3.7}$$

Proof Let $e_h=\boldsymbol{v}_h-\boldsymbol{w}_h (\in \boldsymbol{X}_h^0)$, $\boldsymbol{e}=\boldsymbol{v}-\boldsymbol{w}_h$. By(3.2) and (3.4) we have

$$|\boldsymbol{v}_h|_1 \leqslant |\boldsymbol{w}_h|_1+|\boldsymbol{e}_h|_1 \leqslant C\|q_h\|_{0\setminus 1}+|\boldsymbol{e}_h|_1. \tag{3.8}$$

On each K_i $(i=1, \cdots, m_0+m_1)$, $\boldsymbol{e}_h$ is the form of

$$\boldsymbol{e}_h=\sum_{1 \leqslant j<k \leqslant 3} \boldsymbol{e}_h(A_{jk}) p_{jk}, \tag{3.9}$$

where p_{jk} is a linear function on each K_{il} $(l=1, \cdots, 4)$(see Fig. 2 and 3)defined by

$$\begin{cases} p_{jk}(A_{jk})=1, & p_{jk}(A_m)=0, \quad (m=1,2,3), \\ p_{jk}(A_{mn})=0. & (m,n)\neq(j,k). \end{cases} \tag{3.10}$$

It is easy to prove

$$|p_{jk}|_{1,K_i}\leqslant C, \quad 1\leqslant j<k\leqslant 3. \tag{3.11}$$

Let f be a function on K_i. $\hat{f}$ is a function on $\hat{K}_i=F_i^{-1}(K_i)$ defined by

$$\hat{f}=f\cdot F_i, \tag{3.12}$$

where $\cdot$ denote composition of function. By (3.9) we have

$$e_h=\sum_{2\leqslant j<k\leqslant 3} e_h(A_{jk})\ \hat{p}_{jk}. \tag{3.13}$$

Obviously, $\hat{p}_{jk}(\in H^1(\hat{K}_i)$, where $\hat{K}_i=F_i^{-1}(K_i))$ is a piecewise linear function. By (3.13) we obtain:

1. K_i is an interior element. We have $\hat{K}_i=\hat{K}$ and

$$e_h(A_{jk})=\left[\int_{\widetilde{\hat{A}_j\hat{A}_k}}\hat{p}_{jk}\,ds\right]^{-1}\int_{\widetilde{\hat{A}_j\hat{A}_k}}e_h\,ds, \quad 1\leqslant j<k\leqslant 3. \tag{3.14}$$

By (3.5) we have

$$\int_{\widetilde{\hat{A}_j\hat{A}_k}}e_h ds=\int_{\widetilde{\hat{A}_j\hat{A}_k}}e ds, \quad 1\leqslant j<k\leqslant 3. \tag{3.15}$$

By (3.10) and (3.12)

$$\left[\int_{\widetilde{\hat{A}_j\hat{A}_k}}\hat{p}_{jk}\,ds\right]^{-1}\leqslant C. \tag{3.16}$$

Then by (3.13), (3.14), (3.15) and (3.16) we have

$$\begin{aligned}\|e_h(A_{jk})\|&\leqslant C\|e\|_{0,\widetilde{\hat{A}_j\hat{A}_k}} \text{(by (1.2.3) in [3])}\\ &\leqslant C(\|e\|^2_{0,\hat{K}}+|e|^2_{1,\hat{K}})^{\frac{1}{2}} \quad \text{(by (3.1.20) in [3])}\\ &\leqslant C|\det(B_i)|^{-\frac{1}{2}}(\|e\|^2_{0,\hat{K}_i}+\|B_i\|^2|e|^2_{1,\hat{K}})^{\frac{1}{2}},\end{aligned} \tag{3.17}$$

where $\|e_h(A_{jk})\|$ denote the Euclidean norm of $e(A_{jk})$in R^2. Using Theorem 3.13 in [3], from (3.11) and (3.17) we have

$$|e_h|_{1,K_i}\leqslant C(h_i^{-2}\|e\|^2_{0,\hat{K}_i}+|e|^2_{1,\hat{K}_i})^{\frac{1}{2}}. \tag{3.18}$$

2. K_i is a boundary element. We have $\hat{K}_i\neq\hat{K}$ and $e_h(A_{23})=0$. Thus we only evalute $e_h(A_{12})$ and $e_h(A_{13})$. Using the above method (3.13)—(3.17), we obtain ($(j,k)=(1,2),(1,3)$)

$$\begin{aligned}\|e_h(A_{jk})\|&\leqslant C\|e\|_{0,\partial(A_1\hat{O}\hat{A}\hat{B})}\text{(Fig 4, by (1, 2, 4) in [3])}\\ &\leqslant C(\|e\|^2_{0,A_1\hat{O}\hat{A}\hat{B}}+|e|^2_{1,A_1\hat{O}\hat{A}\hat{B}})^{\frac{1}{2}}\\ &\leqslant C(h_i^{-2}\|e\|^2_{0,K_i}+|e|^2_{1,K_i})^{\frac{1}{2}}.\end{aligned} \tag{3.19}$$

By (3.19) and (3.11) we obtain

$$|e_h|_{1,K_i}\leqslant C(h_i^{-2}\|e\|_{0,K_i}+|e|^2_{1,K_i})^{\frac{1}{2}}. \tag{3.20}$$

Thus, by (3.18)、(3.20)、(3.2) and (3.4) we have

$$|e_h|_1\leqslant C(h^{-2}\|e\|^2_0+\|q_h\|^2_{0/1})^{\frac{1}{2}}. \tag{3.21}$$

Now we evalute $\|e\|_0$. Since Γ is sufficiently smooth, is is well known that there exists a unique $z\in(H_0^1\cap H^2)^2$ such that

$$-\Delta z=g,\quad \text{in } \Omega, \tag{3.22}$$

where $g\in L^2)^2$. Moreovere, there exists a positive constant C satisfying

$$\|z\|_2\leqslant C\|g\|_0. \tag{3.23}$$

By § 5 in [4], there exists a unique $z_h\in X_h^0$ such that

$$a(z_h, v_h)=(g, v_h),\quad \forall v_h\in X_h^0. \tag{3.24}$$

Moreover, we have

$$\|z-z_h\|_1\leqslant Ch\|z\|_2\leqslant Ch\|g\|_0. \tag{3.25}$$

From (3.22) we obtain

$$a(z, v_h)=(g, v_h)+\int_\Gamma \frac{\partial z}{\partial n}\cdot v_h ds,\quad \forall v_h\in X_h^0. \tag{3.26}$$

We have

$$\|e\|_0=\sup_{g\in(L^2)^2}\frac{\left|\int_\Omega e\cdot z\,dx\right|}{|\,\|g\|_0}. \tag{3.27}$$

By (3.3) we have

$$\int_\Omega e\cdot g dx=a(e, z)=a(e, z-z_h).$$

From above equality, we have

$$\left|\int_\Omega e\cdot g\,dx\right|\leqslant Ch|e|_1\|g\|_0. \tag{3.28}$$

From (3.27) and (3.28) we derive

$$\|e\|_0\leqslant Ch|e|_1\leqslant Ch\|q_h\|_{0\backslash 1}. \tag{3.29}$$

From (3.8)、(3.21)、(3.27)、(3.28) and (3.29)we obtain (3.7).

Let S_h be an arbitrary element of Q_{h0}. Let S_{hi} denote the value of S_h on K_i. Giving $q_h\in Q_{h0}$, from (3.2)—(3.6) we obtain v_h. By (3.5) and (3.6) we have

$$\begin{aligned}(\mathrm{div} v_h-q_h, S_h)| &=\left|\sum_{i=1}^{m_0+m_1} S_{hi}\int_{K_i}(\mathrm{div}\, v_h-\mathrm{div} v)dx\right|\\ &=\left|\sum_{i=1}^{m_0}\sum_{j=3}^{4} S_{hi}\int_{e_{ij}}(v_h-v)ds\right|\\ &\leqslant C_1h^2\sum_{i=1}^{m_0}\sum_{j=3}^{4}|S_{hi}|\,|v_h|_{1,K_{ij}}\\ &\leqslant C_1h\Big(\sum_{i=1}^{m_0}|S_{hi}|^2\mathrm{meas}(K_i)\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^{m_0}|v_{h1,K_i}^2\Big)^{\frac{1}{2}}\\ &\leqslant C_1h\|q_h\|_{0\backslash 1}|v_h|_1.\end{aligned} \tag{3.30}$$

Taking $S_h=q_h$ in (3.20), from Lemma 3.2 we obtain

$$\begin{aligned}(\mathrm{div}\, v_h, q_h) &=\|q_h\|_{0\backslash 1}^2+(\mathrm{div} v_h-q_h, k_h)\geqslant\|q_h\|_{0\backslash 1}^2-C_1h|v_h|_1\|q_h|_{0\backslash 1}\\ &\geqslant\frac{1}{C}\|q_h\|_{0\backslash 1}|v_h|_1-C_1h|v_h|_1\|q_h\|_{0\backslash 1}.\end{aligned}$$

When h is sufficiently small, there exists a positive constant $\alpha>0$ satisfying

$$(\operatorname{dix} \boldsymbol{v}_h, q_h) \geqslant \alpha \|q_h\|_{0\backslash 1} |\boldsymbol{v}_h|_1.$$

From here we obtain (3.1).

§ 4. Error Estimate of Stokes Problem

Theorem 4.1. *Let* $\boldsymbol{u} \in (H_0^1)^2$ *and* div $\boldsymbol{u}=0$. *Then there exists a positive constant* C *satisfying*

$$\inf_{\boldsymbol{v}_h \in V_h^0} |\boldsymbol{u}-\boldsymbol{v}_h|_1 \leqslant C \inf_{\boldsymbol{v}_h \in X_h^0} |\boldsymbol{u}-\boldsymbol{v}_h|_1. \tag{4.1}$$

Proof Let $\boldsymbol{w}_h$ be an arbitrary element of $\boldsymbol{X}_h^0$. By Lemma 2.1 and Lax-Milgram Lemma, there exists a unique $\boldsymbol{u}_h \in \boldsymbol{V}_h^0$ satisfying

$$(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) = (\nabla \boldsymbol{w}_h, \nabla \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^0. \tag{4.2}$$

Arguing as in the proofs of Theorem 2.1—Theorem 2.3, we see that there exists a unique $p_h \in Q_{ho}$ satisfying

$$\begin{cases} (\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - (\operatorname{div} \boldsymbol{v}_h, p_h) = (\nabla \boldsymbol{w}_h, \nabla \boldsymbol{v}_h), & \forall \boldsymbol{v}_h \in \boldsymbol{X}_h^0, \\ (\operatorname{div} \boldsymbol{u}_h, \mu_h) = 0, & \forall \mu_h \in Q_{ho}. \end{cases} \tag{4.3}$$

From (4.3) and Brezzi inequality (3.1) we know that $(\boldsymbol{w}_h - \boldsymbol{u}_h)$ and p_h are simultaneously different from zero or equal to 0.

From (4.3) we obtain

$$\frac{\left|\int_\Omega \nabla(\boldsymbol{u}_h - \boldsymbol{w}_f) \cdot \nabla \boldsymbol{u}_h \, dx\right|}{|\boldsymbol{v}_h|_1} = \frac{\left|\int_\Omega p_h \operatorname{div} \boldsymbol{v}_h \, dx\right|}{|\boldsymbol{v}_h|_1}, \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h^0. \tag{4.4}$$

Using (3.1), from (4.4) we derive

$$|\boldsymbol{u}_f - \boldsymbol{w}_h|_1 \geqslant C \|p_h\|_{0\backslash 1}. \tag{4.5}$$

Taking $\boldsymbol{v}_h = \boldsymbol{u}_h - \boldsymbol{w}_h$ in (4.3), from (4.5) we obtain

$$\int_\Omega p_h \operatorname{div}(\boldsymbol{u}_h - \boldsymbol{w}_h) dx = |\boldsymbol{u}_h - \boldsymbol{w}_h|_1^2 \geqslant C |\boldsymbol{u}_h - \boldsymbol{w}_h|_1 \|p_h\|_{0\backslash 1}. \tag{4.6}$$

Since div $\boldsymbol{u}=0$ and $\int_\Omega p_h \operatorname{div} \boldsymbol{u}_h \, dx = 0$ we have

$$\int_\Omega p_h \operatorname{div}(\boldsymbol{u}_h - \boldsymbol{w}_h) dx = \int_\Omega p_h \operatorname{div}(\boldsymbol{u} - \boldsymbol{w}_h) dx. \tag{4.7}$$

Erom (4.6) and (4.7) we derive

$$|\boldsymbol{u}_h - \boldsymbol{w}_h|_1 \|p_h\|_{0\backslash 1} \leqslant C |\boldsymbol{u} - \boldsymbol{w}_h|_1 \|p_h\|_{0\backslash 1}, \tag{4.8}$$

$$|\boldsymbol{u}_h - \boldsymbol{w}_h|_1 \leqslant C |\boldsymbol{u} - \boldsymbol{w}_h|_1. \tag{4.9}$$

Using triangle inequality, we obtain

$$\inf_{\boldsymbol{u}_h \in V_h^0} |\boldsymbol{u} - \boldsymbol{v}_h|_1 \leqslant |\boldsymbol{u} - \boldsymbol{u}_h|_1 \leqslant |\boldsymbol{u} - \boldsymbol{w}_h|_1 + |\boldsymbol{u}_h - \boldsymbol{w}_h|_1 \leqslant (1+C) |\boldsymbol{u} - \boldsymbol{w}_h|_1. \tag{4.10}$$

Since $\boldsymbol{w}_h$ is an arbitrary element of $\boldsymbol{X}_h^0$, from (4.10) we obtain (4.1).

Theorem 4.2. *Let* $p \in H^1 \backslash R$. *There exists* $q_h \in Q_{ho}$ *satisfying*

$$\|p - q_h\|_{0\backslash 1} \leqslant Ch \|p\|_{1/1}. \tag{4.11}$$

Proof of Theorem 4.2 may be found in [3].

Theorem 4.3. *Let $\boldsymbol{u}\in H^2\cap H_0^1$. There exists $\boldsymbol{u}_h\in \boldsymbol{X}_h^0$ satisfying*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_1\leqslant Ch\|\boldsymbol{u}\|_2. \tag{4.12}$$

Proof of Theorem 4.3 may be found in [4].

Theorem 4.4. *Let Γ be sufficiently smooth and $\boldsymbol{f}\in(L^2)^2$. Let $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}_h, p_h)$ be solutions of (1.1)—(1.3) and variational P_{h2} respectively. Then we have*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_1+\|p-p_h\|_{0\backslash 1}\leqslant Ch(\|\boldsymbol{u}\|_2+\|p\|_{1\backslash 1})\leqslant Ch\|\boldsymbol{f}\|_0. \tag{4.13}$$

Proof By hypotheses and Proposition 2.2 in [5] (see Chapter 1) we obtain $\boldsymbol{u}\in(H^2\cap H_0^1)^2$ and $p\in H^1\backslash R$,

$$\|\boldsymbol{u}\|_2+\|p\|_{1\backslash 1}\leqslant Ch\|\boldsymbol{f}\|_0. \tag{4.14}$$

Let $\boldsymbol{w}_h$ be an arbitrary element of $\boldsymbol{V}_h^0$. Thus $\boldsymbol{v}_h=\boldsymbol{u}_h-\boldsymbol{w}_h\in \boldsymbol{V}_h^0$. Then we have

$$\nu a(\boldsymbol{v}_h, \boldsymbol{v}_h)=(\boldsymbol{f}, \boldsymbol{v}_h)-\nu a(\boldsymbol{w}_h, \boldsymbol{v}_h). \tag{4.15}$$

Using Green formula, from (1.1) we obtain

$$\nu a(\boldsymbol{u}, z_h)-b(z_h, p)=(\boldsymbol{f}, \boldsymbol{z}_h)+\nu\int_\Gamma \frac{\partial \boldsymbol{u}}{\partial n}\cdot \boldsymbol{z}_h ds-\int_\Gamma p(\boldsymbol{z}_h\cdot n)ds, \quad \forall \boldsymbol{z}_h\in \boldsymbol{X}_h^0. \tag{4.16}$$

From (4.15) and (4.16) we obtain

$$\begin{aligned}\nu a(\boldsymbol{v}_h, \boldsymbol{v}_h)&=\nu a(\boldsymbol{u}-\boldsymbol{w}_h, \boldsymbol{v}_h)-b(\boldsymbol{v}_h, p-\mu_h)\\&\quad-\nu\int_\Gamma \frac{\partial \boldsymbol{u}}{\partial n}\cdot \boldsymbol{v}_h ds+\int_\Gamma p(\boldsymbol{v}_h\cdot n)ds, \quad \forall \mu_h\in Q_{ho}.\end{aligned} \tag{4.17}$$

Using Lemma 3.1 Cauchy inequality. (4.14), from (4.17) we obtain

$$\nu|\boldsymbol{v}_h|_1^2\leqslant\nu|\boldsymbol{u}-\boldsymbol{w}_h|_1|\boldsymbol{v}_h|_1+\|p-\mu_h\|_{0\backslash 1}|\boldsymbol{v}_h|_1+Ch^{\frac{3}{2}}\|\boldsymbol{f}\|_0|\boldsymbol{v}_h|_1$$

$$|\boldsymbol{v}_h|_1\leqslant C_1(|\boldsymbol{u}-\boldsymbol{w}_h|_1+\|p-\mu_h\|_{0\backslash 1})+C_2h^{\frac{3}{2}}\|\boldsymbol{f}\|_0.$$

By above inequality and triangle inequality, we have

$$|\boldsymbol{u}-\boldsymbol{u}_h|_1\leqslant(1+C_1)\Big(\inf_{\boldsymbol{w}_h\in V^0_h}|\boldsymbol{u}-\boldsymbol{w}_h|_1+\inf_{\mu_h\in Q_{ho}}\|p-\mu_h\|_{0\backslash 1}\Big)+C_2h^{\frac{3}{2}}\|\boldsymbol{f}\|_0. \tag{4.18}$$

By (2.1) and (4.16) we obtain

$$\begin{aligned}-(\operatorname{div}\boldsymbol{z}_h, p_h-\mu_h)&=\nu a(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{z}_h)-(\operatorname{div}\boldsymbol{z}_h, p-\mu_h)\\&\quad-\nu\int_\Gamma \frac{\partial \boldsymbol{u}}{\partial n}\cdot \boldsymbol{z}_h ds+\int_\Gamma p(\boldsymbol{z}_h\cdot n)ds. \quad \forall \boldsymbol{z}_h\in \boldsymbol{X}_h^0,\ \mu_h\in Q_{ho}.\end{aligned}$$

Using (3.1), from above equality we derive

$$\|p_h-\mu_h\|_{0\backslash 1}\leqslant C(|\boldsymbol{u}-\boldsymbol{u}_h|_1+\|p-\mu_h\|_{0\backslash 1}+h^{\frac{3}{2}}\|\boldsymbol{f}\|_0), \quad \forall \mu_h\in Q_{ho}.$$

Using triangle inequality we have

$$\|p-p_h\|_{0\backslash 1}\leqslant C\Big(|\boldsymbol{u}-\boldsymbol{u}_h|_1+\inf_{\mu_h\in Q_{ho}}\|p-\mu_h\|_{0\backslash 1}+h^{\frac{3}{2}}\|\boldsymbol{f}\|_0\Big). \tag{4.19}$$

By (4.18)、(4.19)、Theorem 4.2 and Theorem 4.3 we obtain (4.14),

Theorem 4.5. *Under the hypotheses of Theorem 4.3, we have*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0\leqslant Ch^2(\|\boldsymbol{u}\|_2+\|p\|_{1\backslash 1})\leqslant Ch^2\|\boldsymbol{f}\|_0. \tag{4.20}$$

Proof We only give a sketch of proving Theorem 4.5.

Let $\boldsymbol{\eta}\in(L^2)^2$. There exists a unique $(\boldsymbol{\phi}_\eta, \xi_\eta)\in(H^2\cap H_0^1)^2\times(H^1\backslash R)$ satisfying (see [5])

$$\begin{cases}-\nu\Delta\boldsymbol{\phi}_\eta+\mathbf{grad}\,\xi_\eta=\boldsymbol{\eta}; & \text{in } \Omega,\\ \operatorname{div}\boldsymbol{\phi}_\eta=0, & \text{in } \Omega,\\ \boldsymbol{\phi}_\eta=0, & \text{on } \Gamma,\end{cases}\tag{4.21}$$

$$\|\boldsymbol{\phi}_\eta\|_2+\|\xi_\eta\|_{0\backslash 1}\leqslant C\|\boldsymbol{\eta}\|_0.\tag{4.22}$$

We have

$$\begin{aligned}(\boldsymbol{u}-\boldsymbol{u}_h,\ \boldsymbol{\eta})&=(\boldsymbol{u}-\boldsymbol{u}_h,\ -\nu\Delta\boldsymbol{\phi}_\eta+\mathbf{grad}\,\xi_\eta)\\ &=\nu a(\boldsymbol{u}-\boldsymbol{u}_h,\ \boldsymbol{\phi}_\eta-\boldsymbol{\phi}_h)+\nu a(\boldsymbol{u}-\boldsymbol{u}_h,\ \boldsymbol{\phi}_h)\\ &\quad-(\operatorname{div}(\boldsymbol{u}-\boldsymbol{u}_h),\ \xi_\eta-\xi_h)-\nu\int_\Gamma\frac{\partial\boldsymbol{\phi}_\eta}{\partial n}\cdot\boldsymbol{u}_h\,ds\\ &\quad-\int_\Gamma\xi_\eta(\boldsymbol{u}_h\cdot n)ds,\quad\forall\boldsymbol{\phi}_h\in\boldsymbol{V}_h^0,\quad\xi_h\in Q_{h0},\end{aligned}\tag{4.23}$$

$$\begin{aligned}\nu a(\boldsymbol{u}-\boldsymbol{u}_h,\ \boldsymbol{\phi}_\eta)&=(\operatorname{div}(\boldsymbol{\phi}_h-\boldsymbol{\phi}_\eta),\ p-p_h)\\ &\quad+\nu\int_\Gamma\frac{\partial\boldsymbol{u}}{\partial n}\cdot\boldsymbol{\phi}_h ds-\int_\Gamma p(\boldsymbol{\phi}_h\cdot n)ds,\quad\forall\boldsymbol{\phi}_h\in\boldsymbol{V}_h^0.\end{aligned}\tag{4.24}$$

Let $G=\bigcup_{i=1}^{m_0}K_i$. By (4.23) and (4.44) we obtain

$$\begin{aligned}|(\boldsymbol{u}-\boldsymbol{u}_h,\ \boldsymbol{\eta})|&\leqslant C_1(|\boldsymbol{u}-\boldsymbol{u}_h|_1+\|p-p_h\|_{0\backslash 1})(|\boldsymbol{\phi}_\eta-\boldsymbol{\phi}_h|_1+\|\xi_\eta-\xi_h\|_{0\backslash 1})\\ &\quad+C_2h^{\frac{3}{3}}\|\boldsymbol{f}\|_0(|\boldsymbol{\phi}_h-\boldsymbol{\phi}_\eta|_{1,G}+|\boldsymbol{\phi}_\eta|_{1,G})\\ &\quad+C_3h^{\frac{3}{2}}\|\boldsymbol{\eta}\|_0(|\boldsymbol{u}-\boldsymbol{u}_h|_{1,G}+|\boldsymbol{u}|_{1,G}).\end{aligned}\tag{4.25}$$

Let $\boldsymbol{\phi}_h\in\boldsymbol{V}_h^0$ and $\xi_h\in Q_{h0}$ be such that

$$|\boldsymbol{\phi}_\eta-\boldsymbol{\phi}_h|_1+\|\xi_\eta-\xi_h\|_{0\backslash 1}\leqslant Ch(\|\boldsymbol{\phi}_\eta\|_2+\|\xi_\eta\|_{1\backslash 1}).\tag{4.26}$$

Using inequality (see [6])

$$|\tilde{u}|_{1,G}\leqslant Ch^{\frac{1}{2}}\|\tilde{u}\|_2,\quad\forall\tilde{u}\in H_0^1(\Omega)\cap H^2(\Omega),\tag{4.27}$$

from (4.25) and (4.26) we obtain

$$\begin{aligned}|(\boldsymbol{u}-\boldsymbol{u}_h,\ \boldsymbol{\eta})|&\leqslant C_1h(|\boldsymbol{u}-\boldsymbol{u}_h|_1+\|p-p_h\|_{0\backslash 1})\|\boldsymbol{\eta}\|_0+C_2h^2\|\boldsymbol{f}\|_0\|\boldsymbol{\eta}\|_0\\ &\leqslant Ch^2\|\boldsymbol{f}\|_0\|\boldsymbol{\eta}\|_0.\end{aligned}\tag{4.28}$$

We have

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0=\sup_{\eta\in(L^2)^2-\{0\}}\frac{|(\boldsymbol{u}-\boldsymbol{u}_h,\ \boldsymbol{\eta})|}{\|\boldsymbol{\eta}\|_0}.\tag{4.29}$$

Using (4.28), from (4.29) we obtain (4.20).

§ 5. Error Estimate of Navier-Stokes

Variational problem NS: Find $(\boldsymbol{u}, p)\in(H_0^1)^2\times L_0^2$ satisfying

$$\begin{cases}\nu a(\boldsymbol{u},\boldsymbol{v})-b(\boldsymbol{v},\ p)+c(\boldsymbol{u};\ \boldsymbol{u},\ \boldsymbol{v})=(\boldsymbol{f},\ \boldsymbol{v}),\quad\forall\boldsymbol{v}\in(H_0^1)^2,\\ b(\boldsymbol{u},\ \mu)=0,\quad\forall\mu\in L_0^2,\end{cases}\tag{5.1}$$

where

$$c(\boldsymbol{u};\ \boldsymbol{v},\ \boldsymbol{w})=\int_{\Omega}(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}dx, \tag{5.2}$$

Define

$$\gamma(\boldsymbol{f})=\sup_{\boldsymbol{w}\ni(H_0^1)^2-\{0\}}\frac{(\boldsymbol{f},\ \boldsymbol{w})}{|\boldsymbol{w}|_1}, \tag{5.3}$$

$$\rho(\beta)=\sup_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in(H_0^1)^2-\{0\}}\frac{((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\ \boldsymbol{w})}{|\boldsymbol{u}|_1|\boldsymbol{v}|_1|\boldsymbol{w}|_1}, \tag{5.4}$$

$$\gamma_h(\boldsymbol{f})=\sup_{\boldsymbol{w}_h\in X_h^0-\{0\}}\frac{(\boldsymbol{f},\ \boldsymbol{w}_h)}{|\boldsymbol{w}_h|_1}, \tag{5.5}$$

$$\rho_{1h}(\beta)=\sup_{\boldsymbol{u}_h,\boldsymbol{v}_h,\boldsymbol{w}_h\in X_h^0-\{0\}}\frac{((\boldsymbol{u}_h\cdot\nabla)\boldsymbol{v}_h,\ \boldsymbol{w}_h)}{|\boldsymbol{u}_h|_1|\boldsymbol{v}_h|_1|\boldsymbol{w}_h|_1}, \tag{5.6}$$

$$\rho_{2h}(\beta)=\sup_{\boldsymbol{u}_h,\boldsymbol{w}_h\in X_h^0-\{0\},\ \boldsymbol{v}\in(H_0^1)^2-\{0\}}\frac{((\boldsymbol{u}_h\cdot\nabla)\ \boldsymbol{v},\ \boldsymbol{w}_h)}{|\boldsymbol{u}_h|_1|\boldsymbol{v}|_1|\boldsymbol{w}_h|_1}. \tag{5.7}$$

Theorem 5.1. *We have*

$$\lim_{h\to 0}\gamma_h(\boldsymbol{f})=\gamma(\boldsymbol{f}), \tag{5.8}$$

$$\lim_{h\to 0}\rho_{1h}(\beta)=\lim_{h\to 0}\rho_{2h}(\beta)=\rho(\beta). \tag{5.9}$$

This Theorem 5.1 has been proved in [7].

Variational problem NS_{h1}: Find $(\boldsymbol{u}_h,\ p_h)\in \boldsymbol{X}_h^0\times Q_{ho}$ satisfying

$$\begin{cases}\nu a(\boldsymbol{u}_h,\ \boldsymbol{v}_h)-b(\boldsymbol{v}_h,\ p_h)+c(\boldsymbol{u}_h;\ \boldsymbol{u}_h,\ \boldsymbol{v}_h)=(\boldsymbol{f},\ \boldsymbol{v}_h), & \forall \boldsymbol{v}_h\in \boldsymbol{X}_h^0,\\ b(\boldsymbol{u}_h,\ \mu_h)=0, \quad \forall \mu_h\in Q_{ho}.\end{cases} \tag{5.10}$$

Variational problem NS_{h2}: Find $\boldsymbol{u}_h\in \boldsymbol{V}_h^0$ satisfying

$$\nu a(\boldsymbol{u}_h,\ \boldsymbol{v}_h)+c(\boldsymbol{u}_h;\ \boldsymbol{u}_h,\ \boldsymbol{v}_h)=(\boldsymbol{f},\ \boldsymbol{v}_h),\ \forall \boldsymbol{v}_h\in \boldsymbol{V}_h^0. \tag{5.11}$$

Define

$$B_0=\{\boldsymbol{v}_h\,|\,\boldsymbol{v}_h\in \boldsymbol{V}_h^0,\ |\boldsymbol{v}_h|_1\leqslant\frac{2}{\nu}\ \gamma_h(\boldsymbol{f})\},$$

$$B_1=\{(\boldsymbol{v}_h,\ \mu_h)\,|\,(\boldsymbol{v}_h,\ \mu_h)\in \boldsymbol{V}_h^0\times Q_{h0},\ |\boldsymbol{v}_h|_1\leqslant\frac{2}{\nu}\ \gamma_h(\boldsymbol{f})\}.$$

Theorem 5.2. *Variational problem NS_{h2} has a unique solution in B_0, if the following condition holds:*

$$\gamma_h(\boldsymbol{f})\rho_{1h}(\beta)<\frac{\nu^2}{4}. \tag{5.12}$$

Proof We propose the following problem: Given $\boldsymbol{u}_h\in \boldsymbol{V}_h^0$, find $\boldsymbol{w}_h\in \boldsymbol{V}_h^0$ satisfying

$$\nu a(\boldsymbol{w}_h,\ \boldsymbol{v}_h)=(\boldsymbol{f},\ \boldsymbol{v}_h)-c(\boldsymbol{u}_h;\ \boldsymbol{u}_h,\ \boldsymbol{v}_h),\quad \forall \boldsymbol{v}_h\in \boldsymbol{V}_h^0. \tag{5.13}$$

Arguing as in the case of Theorem 2.1, we see that there exists a unique $\boldsymbol{w}_h$ satisfying (5.13). Let us call F the map: $\boldsymbol{u}_h\to F\boldsymbol{u}_h=\boldsymbol{w}_h$ solution of (5.13).

In order to prove Theorem 5.2, it suffices to prove that F is a map from B_0 into itself and a contraction on B_0. Arguing as in the case of Theorem 2.5 in [8] for drawing the above conclusion, we may come to the above conclusion.

Theorem 5.3. *Variational problem SN_{h1} has a unique solution in B_1, if* (5.12) *holds.*

Both variational problems NS_{h1} and NS_{h2} can be characterized by the following

Theorem 5.4.

Theorem 5.4. *Let $\boldsymbol{u}_h$ be a solution of variational problem NS_{h2}, then there exists a unique $p_h \in Q_{ho}$ such that $(\boldsymbol{u}_h, p_h)$ is a solution of variational problem NS_{h1}. Conversely, let $(\boldsymbol{u}_h, p_h)$ be a solution of variational problem NS_{h1}, then $\boldsymbol{u}_h$ is a solution of variational problem NS_{h2}.*

Proof of these theorems is easy. We skip the proof.

Theorem 5.5. *Let Γ be sufficiently smooth and $\boldsymbol{f} \in (L^2)^2$. We assume that the following inequalities hold:*

$$\gamma_h(\boldsymbol{f})\rho_{1h}(\beta) < \frac{\nu^2}{4}, \quad \rho_{2h}(\beta)\gamma(\boldsymbol{f}) < \frac{\nu^2}{4}. \tag{5.14}$$

Let $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}_h, p_h)$ be solutions of (1.4)—(1.6) and variational problem NS_{h1}. Then we have: 1. $(\boldsymbol{u}, p) \in (H_0^1 \cap H^2)^2 \times (H^1 \backslash R)$. 2. There exists a constant C such that

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_1 + \|p-p_h\|_{0\backslash 1} \leqslant Ch\|\boldsymbol{f}\|_0. \tag{5.15}$$

Proof The first part has been proved in [5].

We only give the sketch of the proof of the second part.

(a). By Theorem 2.2 there exists a unique $(\boldsymbol{w}_h, r_h) \in \boldsymbol{X}_h^0 \times Q_{ho}$ (in fact, $(\boldsymbol{w}_h, r_h) \in \boldsymbol{V}_h^0 \times Q_{ho}$) satisfying

$$\begin{cases} \nu a(\boldsymbol{w}_h, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, r_h) = \nu a(\boldsymbol{u}, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p), & \forall \boldsymbol{v}_h \in \boldsymbol{X}_h^0, \\ b(\boldsymbol{w}_h, \mu_h) = 0 \,(\equiv b(\boldsymbol{u}, \mu_h)), & \forall \mu_h \in Q_{ho}. \end{cases} \tag{5.16}$$

Moreover, we have

$$\|\boldsymbol{u}-\boldsymbol{w}_h\|_1 + \|p-r_h\|_{0\backslash 1} \leqslant Ch\|\boldsymbol{f}\|_{0\backslash 1}. \tag{5.17}$$

(b). We have

$$\begin{cases} \nu a(\boldsymbol{u}_h-\boldsymbol{w}_h, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h-r_h) = c(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}_h) - c(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) \\ -\nu\displaystyle\int_\Gamma \frac{\partial \boldsymbol{u}}{\partial n}\cdot\boldsymbol{v}_h ds + \int_\Gamma p(\boldsymbol{v}_h\cdot n) ds, \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h^0, \\ b(\boldsymbol{u}_h-\boldsymbol{w}_h, \mu_h) = 0, \quad \forall \mu_h \in Q_{ho}. \end{cases} \tag{5.18}$$

Taking $\boldsymbol{v}_h = \boldsymbol{u}_h - \boldsymbol{w}_h$, we have

$$\begin{aligned} &|c(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}_h) - c(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h)| \\ &\quad = |c(\boldsymbol{u}-\boldsymbol{w}_h; \boldsymbol{u}, \boldsymbol{v}_h) - c(\boldsymbol{v}_h; \boldsymbol{u}, \boldsymbol{v}_h) + c(\boldsymbol{u}_h; \boldsymbol{u}-\boldsymbol{w}_h, \boldsymbol{v}_h) \\ &\qquad - c(\boldsymbol{u}_h; \boldsymbol{v}_h, \boldsymbol{v}_h)| \\ &\quad \leqslant c_1\|\boldsymbol{u}-\boldsymbol{w}_h\|_1 |\boldsymbol{u}|_1 |\boldsymbol{v}_h|_1 + c_2|\boldsymbol{u}_h|_1 |\boldsymbol{u}-\boldsymbol{w}_h|_1 |\boldsymbol{v}_h|_1 \\ &\qquad + \rho_{1h}(\beta)|\boldsymbol{u}_h|_1 |\boldsymbol{v}_h|_1^2 + \rho_{2h}(\beta)|\boldsymbol{v}_h|_1^2 |\boldsymbol{u}|_1. \end{aligned} \tag{5.19}$$

Using $|\boldsymbol{u}|_1 \leqslant \frac{1}{\nu}\gamma(\boldsymbol{f})$ (see [5]), $|\boldsymbol{u}_h|_1 \leqslant \frac{2}{\nu}\gamma_h(\boldsymbol{f})$, (5.14), we obtain

$$\rho_{1h}(\beta)|\boldsymbol{u}_h|_1 + \rho_{2h}(\beta)|\boldsymbol{u}|_1 < \frac{3}{4}\nu. \tag{5.20}$$

Taking $\boldsymbol{v}_h = \boldsymbol{u}_h - \boldsymbol{w}_h$ and Using (5.19), (5.20), Lemma 3.1, we obtain

$$\begin{cases} |\boldsymbol{u}_h-\boldsymbol{w}_h|_1 \leqslant Ch\|\boldsymbol{f}\|_0, \\ \|p_h-r_h\|_{0\backslash 1} \leqslant Ch\|\boldsymbol{f}\|_0, \end{cases} \tag{5.21}$$

where C may depends on $\gamma(\boldsymbol{f})$ and $\gamma_h(\boldsymbol{f})$. From (5.2) we obtain (5.21).

Theorem 5.6. *Under the hypotheses of Theorem 5.5, we have*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0 \leqslant Ch^2\|\boldsymbol{f}\|_0. \tag{5.22}$$

Proof Arguing as in the case of Theorem 3.3 in [8], we may obtain (5.22).

References

[1] Bercovier, M. and Pironneau, O., Error estimates for finite element method solution of the Stokes problem in the primitive variables, *Numer. Math.*, **33**(1979), 211—224.

[2] Girault, V. and Raviart, P. A., Finite element approximation of the Navier-Stokes equations, Berlin Heideberg New York: Springer, 1979.

[3] Ciarlet, P. G., The finite element method for elliptic problems, Amsterdam, North-Holland, 1978

[4] Li Likang, A finite element method for the stationary Stokes equations, *Numer. Math., A Journal of Chinese Universities*, **6**(1984), 104—115.

[5] Teman, R., Navier-Stokes equations, Amsterdam, North-Holland, 1979.

[6] Blair, J. J., High order approximations to the boundary conditions for the F. E. M., *Math. Comp.*, **30** (1976), 250—262

[7] Li Likang, A finite element method for stationary Navier-Stokes equation, *Numer. Math., A Journal of Chinese Universities*, **7**(1985), 350—359.

[8] Li Likang, An analysis of finite element method for the stationary Navier-Stokes equation, *Mathematica Numerica Sinica*, **8**(1985), 27—40.