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THE ANALYTIC INVARINT SUBSPACE OF THE n–TUPLE OF COMMUTING OPERATORS

Zou Chengzu (邹承祖)* Li Liangqing (李良青)*

Abstract

In this paper, the authors extend the concept of analytic invariant subspace to the case of n–tuple of commuting operators. The analytic invariant subspace is better than spectral maximal space in some aspects. This provides a class of invariant subspace, which is helpful to the study of decomposable theory for n–tuple operators.

Let X be a complex Banach space and $a=(a_1, \cdots, a_n)$ be a commuting n–tuple of bounded linear operators on X. We denote by Lat a the collection of all subspaces which are invariant under a_i $(i=1, \cdots, n)$. Obviously, Lat $a=\bigcap_{i=1}^{n}$ Lat a_i.

If $G\subset\mathbb{C}^n$ is an open set, we denote by $\mathscr{A}(G, X)$ and $C^\infty(G, X)$ the spaces of X–valued analytic functions and C^∞–functions on G, respectively. It has been proved in [1] that $C^\infty(G, X)=\mathscr{B}(G, X)$, where $\mathscr{B}(G, X)$ is the space of all continuous X–valued functions on G being infinitely differentiable with respect to $\bar{z}_1, \cdots, \bar{z}_n$ in the distribution sense.

If $\sigma=(s_1, \cdots, s_n)$ is n–tuple of indeter minates and Y is one of the spaces X, $\mathscr{A}(G, X)$ and $C^\infty(G, X)$, we shall denote by $\Lambda^p[\sigma, Y]$ the set of all exterior forms of degree p in σ, having coefficients in Y. We define

$$\Lambda^p[\sigma, Y]=0, \quad \text{when} \quad p<0.$$

For $z\in\mathbb{C}^n$, let $\alpha_i(z)=z_i-a_i$, $(i=1, \cdots, n)$ and

$$\alpha(z)=\alpha_1(z)s_1+\cdots+\alpha_n(z)s_n,$$

$$\bar{\partial}=\frac{\partial}{\partial\bar{z}_1}d\bar{z}_1+\cdots+\frac{\partial}{\partial\bar{z}_n}d\bar{z}_n.$$

α^p: $\Lambda^p[\sigma, Y]\to\Lambda^{p+1}[\sigma, Y]$ is a homomorphism in the sense that $\alpha^p\psi=\alpha\wedge\psi$ for every $\psi\in\Lambda^p[\sigma,Y]$. Usually α^p is written as α.

Moreover, if $Y=\mathscr{A}(G, X)$ or $Y=C^\infty(G, X)$, then $(\alpha\psi)(z)=\alpha(z)\wedge\psi(z)$ $\forall z\in G$, $\psi\in\Lambda^p[\sigma, Y]$. We define $\alpha\oplus\bar{\partial}$: $\Lambda^p[\sigma\cup d\bar{z}, Y]\to\Lambda^{p+1}[\sigma\cup d\bar{z}, Y]$, where

$$d\bar{z}=(d\bar{z}_1, \cdots, d\bar{z}_n),$$

by $\alpha\oplus\bar{\partial}\psi=\alpha\psi+\bar{\partial}\psi=\alpha\wedge\psi+\bar{\partial}\wedge\psi$ for $\psi\in\Lambda^p[\sigma\cup d\bar{z}, Y]$. $\Lambda[\sigma, Y]$ is a Koszul complex

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* Department of Mathematics, Jilin University, Changchun, China.

and the cohomology of $\Lambda[\sigma, Y]$ is the graded module $H(Y, \alpha)=\{H^p(Y, \alpha)\}$, where $H^p(Y, \alpha)=\mathrm{Ker}\ \alpha^p/\mathrm{Im}\ \alpha^{p-1}$. By J. L. Taylor [2], the joint spectrum of a is defined as

$$\mathrm{Sp}(a, X)=\{z \mid z\in\mathbb{C}^n.\ \exists P \text{ such that } H^p(X, \alpha)\neq 0\}$$

and the resolvent set of a is $r(a, X)=C^n\backslash\mathrm{Sp}(a, X)$. For $x\in X$, S. Frunză[3] defined the local spectrum of a at x as $\mathrm{Sp}(a, x)=C^n\backslash r(a, x)$, where $r(a, x)=\{z \mid z\in\mathbb{C}^n$, there exists an open set $G\to z$ such that $\exists\psi\in\Lambda^{n-1}[\sigma\cup d\bar{z}, C^\infty(G, X)]$ having the property that $sx=xs, \wedge\cdots\wedge s_n=(\alpha\oplus\bar{\partial})\psi$ on $G\}$ is the local resolvent set of a at x.

S. Frunză[3] introduced the decomposable theory of n-tuple of operators too. At present stage, it seems to be imperfect because some of the properties in the decomposable theory of single operator can not be extended. The property of spectral maximal space for n-tuple of operators is not as good as that for single operator. In this paper, we extend the concept of analytic invariant subspace to the case of n-tuple $a=(a_1, \cdots, a_n)$. The analytic invariant sub-space is better than spectral maximal space in some aspects. This provides a class of invariant subspaces which is helpful to the study of decomposable theory for n-tuple operators that will be done in another paper.

Definition 1. *$Y\in\mathrm{Lat}\ a$ will be called analytic invariant subspace of a, if for any polydisc $D\subset\mathbb{C}^n$, integer p, $0\leqslant p\leqslant n-1$ and $\psi\in\Lambda^p[\sigma, \mathscr{A}(D, X)]$ with $\alpha\psi\in\Lambda^{p+1}[\sigma, \mathscr{A}(D, Y)]$, there are $\varphi\in\Lambda^p[\sigma, \mathscr{A}(D,Y)]$ and $\xi\in\Lambda^{p-1}[\sigma,\mathscr{A}(D, X)]$ such that*

$$\psi=\varphi+\alpha\xi.$$

Definition 2. *a is said to have the single valued extension property, or $a\in(A)$ for short, if $H^p(\mathscr{A}(D, X), \alpha)=0$ for all polydisc $D\subset\mathbb{C}^n$ and each p, $0\leqslant p\leqslant n-1$.*

It should be noted that if $n=1$ then property (A) of $a=(a_1)$ and of the single operator a_1 coincides.

Proposition 3. *If $a=(a_1)$, then the analytic invariant subspace of a is exactly the analytic invariant subspace of a_1.*

Proof Let G be an open set in $\mathbb{C}^n$ and $f\in\mathscr{A}(G, X)$ such that

$$(z-a_1)f(z)\in Y, \quad \forall z\in G,$$

where Y is an analytic invariant subspace of a. Then for every polydisc $D\subset G$,

$$f\in\mathscr{A}(D, X)=\Lambda^0[\sigma,\mathscr{A}(D, X)]$$

and $\alpha f\in\Lambda^1[\sigma, \mathscr{A}(D, Y)]$. By Definition 1, there exist

$$\varphi\in\Lambda^0[\sigma, \mathscr{A}(D, Y)]=\mathscr{A}(D, Y)$$

and $\xi\in\Lambda^{0-1}[\sigma, \mathscr{A}(D, X)]=0$ such that $\varphi=f-\alpha\xi=f$. Hence $f(z)\in Y$, $\forall z\in D$. Since D is arbitrary, $f(z)\in Y$, $\forall z\in G$.

Theorem 4. *Let $a\in(A)$ and Y be an analytic invariant subspace of a. Then $a_Y\in(A)$ and*

$$\mathrm{Sp}(a, y)=\mathrm{Sp}(a_Y, y), \quad \forall y\in Y,$$

where $a_Y=(a_1|Y, \cdots, a_n|Y)$.

Proof Choose any $\psi\in\Lambda^p[\sigma, \mathscr{A}(D, Y)]$ with $\alpha\psi=0$, where $0\leqslant p\leqslant n-1$ and polydisc $D\subset\mathbb{C}^n$. Since $a\in(A)$, there exists $\varphi\in\Lambda^{p-1}[\sigma, \mathscr{A}(D, X)]$ with $\alpha\varphi=\psi$. Therefore $\alpha\varphi\in\Lambda^p[\sigma, \mathscr{A}(D, Y)]$. Because Y is an analytic invariant subspace of a, we can choose $\eta\in\Lambda^{p-1}[\sigma, \mathscr{A}(D, Y)]$ such that $\alpha\eta=\alpha\varphi=\psi$. It follows that

$$H^p(\mathscr{A}(D, Y), \alpha)=0. \quad 0\leqslant p\leqslant n-1.$$

That is $a_Y\in(A)$.

If $x\in X$ and $\rho(a, x)=\{z|z\in\mathbb{C}^n$, there exists $z\in D$ and $f_1, \cdots, f_n\in\mathscr{A}(D, X)$ with $(\zeta_1-a_1)f_1(\zeta)+\cdots+(\zeta_n-a_n)f_n(\zeta)=x, \forall\zeta\in D\}$, then $\sigma(a, x)=\mathbb{C}^n\backslash\rho(a, x)$ is said to be the analytic local spectrum of a at x. It follows by J. Eschmeier[4] that

$$\sigma(a, x)=\mathrm{Sp}(a,x), \quad \forall x\in X.$$

Therefore we only prove that $\sigma(a, y)=\sigma(a_Y, y)$, $\forall y\in Y$.

Choose any polydisc $D\subset\rho(a, y)$. Then there are $f, \cdots, f_n\in\mathscr{A}(D, X)$ with

$$(z_1-a_1)f_1(z)+\cdots+(z_n-a_n)f_n(z)\equiv y, \quad \forall z\in D.$$

Let $\psi=f_1\hat{s}_1-f_2\hat{s}_2+\cdots+(-1)^{n-1}f_n\hat{s}_n$, where

$$\hat{s}_i=s_1\wedge\cdots\wedge s_{i-2}\wedge s_{i+1}\wedge\cdots\wedge s_n=s_1\wedge\cdots\wedge\hat{s}_i\wedge\cdots\wedge s_n.$$

Then $\psi\in\Lambda^{n-1}[\sigma, \mathscr{A}(D, X)]$ and $\alpha\psi=Sy\in\Lambda^n[\sigma, \mathscr{A}(D, Y)]$. Since Y is an analytic invariant sub-space of a, it follows by Definition 1 that there is a form

$$\varphi\in\Lambda^{n-1}[\sigma, \mathscr{A}(D,Y)]$$

with $\alpha\psi=\alpha\varphi$. φ can be written as

$$\varphi=\varphi_1\hat{s}_1-\varphi_2\hat{s}_2+\cdots+(-1)^{n-1}\varphi_n\hat{s}_n, \ \varphi_i\in\mathscr{A}(D, Y), \quad i=1, \cdots, n.$$

Hence

$$y\equiv(z_1-a_1)\varphi_1(z)+\cdots+(z_n-a_n)\varphi_n(z), \quad \forall z\in D,$$

and $D\subset\rho(a_Y,y)$. It follows that $\rho(a, y)\subset\rho(a_Y, y)$, or $\sigma(a_Y, y)\subset\sigma(a, y)$.

On the other hand, it is clear that $\sigma(a, y)\subset\sigma(a_Y,y)$, $\forall y\in Y$. Therefore

$$\sigma(a, y)=\sigma(a_Y, y), \quad \forall y\in Y.$$

Thus the proof is completed.

It can be seen from the following example that Theorem 4 is not generally true for any invariant subspace of a. This is one of the differences between commuting n-tuple operators aud single operator.

Example. Let $\mathscr{H}$ be a Hilbert space and $\{\psi_{ij}\}_{i,j=-\infty}^{\infty}$ be one of its orthogonal bases. a_1, a_2 are two operators defined by the following

$$a_1\psi_{ij}=\psi_{i+1,j}, \quad a_2\psi_i=\psi_{i,j+1}, \quad \text{for all } i, j.$$

Clearly a_1, a_2 are all bilateral shifts of infinite multiplicity and $a_1a_2=a_2a_1$. Thus (a_1, a_2) is commuting 2-tuple of normal operators.

a_1, a_2 are all identity decomposable and thus (a_1, a_2) is identity decomposable 2-tuple (see [7]). Hence $(a_1, a_2)\in(A)$.

Let Y be a subspace of $\mathscr{H}$ and $\{\psi_{ij}\}_{i>0}\cup\{\psi_{ij}\}_{j>0}$ its basis. Define

$$\varphi(\lambda_1, \lambda_2)=\sum_{i,j<0}\psi_{ij}\lambda_1^{-i}\lambda_2^{-j}.$$

Then φ is analytic in the polydisc

$$D=\{(\lambda_1, \lambda_2)\,|\,\lambda_i\in\mathbb{C}, |\lambda_i|<\frac{1}{2}, i=1, 2\},\ \varphi\notin\mathscr{A}(D,Y).$$

But

$$\begin{aligned}(\lambda_1-a_1)\varphi(\lambda_1, \lambda_2)&=\sum_{i,j\le 0}(\psi_{ij}\lambda_1^{-i+1-j}\lambda_2-\psi_{i+1,j}\lambda_1^{-i}\lambda_2^{-j})\\&=\sum_{i,j\le 0}\psi_{ij}\lambda_1^{-i+1}\lambda_2^{-j}-\sum_{i\le 1,j\le 0}\psi_{ij}\lambda_1^{-i+1}\lambda_2^{-j}\\&=-\sum_{j\le 0}\psi_{1j}\lambda_2^{-j}\in Y.\end{aligned}$$

Similarly, we obtain

$$(\lambda_2-a_2)\varphi(\lambda_1, \lambda_2)=-\sum_{i\le 0}\psi_{i1}\lambda_1^{-i}\in Y.$$

Hence

$$\alpha\varphi=a_1\varphi S_1+a_2\varphi S_2\in\Lambda^1[\sigma, \mathscr{A}(D, Y)].$$

But if $\alpha\varphi=\alpha\eta$ for $\eta\in\mathscr{A}(D, X)$, then $\varphi=\eta$. Thus there is no $\eta\in\mathscr{A}(D, Y)$ with $\alpha\eta=\alpha\varphi$. On the other hand, clearly $\alpha(\alpha\varphi)=0$. It follows that $(a_1|_Y, a_2|_Y)\notin(A)$.

This example shows that the restriction of a commuting n-tuple a having property (A) on its ordinary invariant subspace may not have the property (A), even if a is identity decomposable. It also shows that the difference between the property(A) of commuting n-tuple $a=(a_1, \cdots, a_n)$ and of each a_i is very large. In the above example, (a_1, a_2) is identity decomposable and hence decomposable. Therefore for every analytic function f on $\mathrm{Sp}((a_1. a_2), \mathscr{H})$, $f(a_1, a_2)$ is decomposable (see [3]). Thus $f(a_1, a_2)|_Y\in(A)$, since $f(a_1, a_2)$ is a single operator on $\mathscr{H}$ and $Y\in\mathrm{Lat}\,(a_1. a_2)$. So for every polynomial $P=P(z_1, z_2,)$

$$P(a_1, a_2)|_Y=P(a_1|_Y, a_2|_Y)\in(A).$$

But $(a_1|_Y, a_2|_Y)=(a_1, a_2)|_Y\notin(A)$.

Corollary 5. *If $a\in(A)$ and Y is an analytic invariant subspace of a, then*

$$\mathrm{Sp}\,(a, Y)=\bigcup_{y\in Y}\mathrm{Sp}(a, y)\subseteq\mathrm{Sp}(a, X).$$

Proof Theorem 4 shows that $a_Y\in(A)$ and by the Remark 3.1 of S. Frunza[3] we know that if $a\in(A)$then $\mathrm{Sp}\,(a, X)=\bigcup_{x\in X}\mathrm{Sp}(a,x)$. Thus it follows from Theorem[4] that

$$\mathrm{Sp}(a, Y)=\mathrm{Sp}(a_Y, Y)=\bigcup_{y\in Y}\mathrm{Sp}(a_Y, y)=\bigcup_{y\in Y}\mathrm{Sp}(a,y)\subset\bigcup_{x\in X}\mathrm{Sp}(a, x)=\mathrm{Sp}(a, X).$$

For $Y\in\mathrm{Lat}\,a$, let $a^Y=(a_1^Y, \cdots, a_n^Y)$, where a_i^Y is the induced operator by a_i on X/Y, $i=1, 2, \cdots, n$. Since

$$0\to Y\to X\to X/Y\to 0$$

is exact,

$$0\to C^\infty(G, Y)\to C^\infty(G, X)\to C^\infty(G, X/Y)\to 0$$

is also exact. Therefore, for any $\psi\in\Lambda^p[\sigma\cup d\bar{z}, C^\infty(G, X/Y)]$ there is a form

$$\varphi\in\Lambda^p[\sigma\cup d\bar{z}, C^\infty(G, X)]$$

such that $\varphi/Y=\psi$, where G is any one of the open sets in $\mathbb{C}^n$. Similarly, it can be proved that for any $\psi\in\Lambda^p[\sigma,\mathscr{A}[G, X/Y)]$ there is a form $\varphi\in\Lambda^p[\sigma, \mathscr{A}(G, X)]$ such that $\varphi/Y=\psi$.

Theorem 6. *If $Y\in\mathrm{Lat}\, a$, then Y is an analytic invariant subspace of a iff $a^Y\in(A)$.*

Proof Let $a^Y\in(A)$. Then for every polydisc $D\subset\mathbb{C}^n$,

$$H^p(\mathscr{A}(D, X/Y), \alpha^Y)=0,\quad 0\leqslant p\leqslant n-1,$$

where

$$\alpha^Y(z)=(z_1-a_Y^1)s_1+\cdots+(z_n-a_n^Y)s_n,$$

$$\alpha^Y(\psi/Y)(z)=(\alpha\psi)/_Y(z)=\alpha^Y(z)\wedge(\psi/Y)(z).$$

Now assume $\psi\in\Lambda^p[\sigma, \mathscr{A}(D,X)]$, $0\leqslant p\leqslant n-1$ and $\alpha\psi\in\Lambda^{p+1}[\sigma, \mathscr{A}(D, Y)]$. Then $\alpha^Y(\psi/Y)=(\alpha\psi)/Y=0$ on D. Because $a^Y\in(A)$, there exists $\varphi\in\Lambda^{p-1}[\sigma, \mathscr{A}(D, X)]$ such that $\psi/Y=\alpha^Y(\varphi/Y)=(\alpha\varphi)/Y$ on D. Hence $\eta=\psi-\alpha\varphi\in\Lambda^p[\sigma, \mathscr{A}(D, Y)]$, that is $\psi=\eta+\alpha\varphi$. Therefore Y is an analytic invariant subspace of a.

On the other hand, if Y is an analytic invariant subspace of a,

$$\psi\in\Lambda^p[\sigma, \mathscr{A}(D, X)]$$

with $\alpha^Y(\psi/Y)=(\alpha\psi)/Y=0$ $(0\leqslant p\leqslant n-1)$, then $\alpha\psi\in\Lambda^{p+1}[\sigma, \mathscr{A}(D, Y)]$. Hence there are $\varphi\in\Lambda^p[\sigma, \mathscr{A}(D, Y)]$ and $\xi\in\Lambda^{p-1}(\sigma, \mathscr{A}(D, Y)]$ such that $\psi=\varphi+\alpha\xi$. Thus $\psi/Y=\alpha^Y(\xi/Y)$. That is $H^p(\mathscr{A}(D, X/Y), \alpha^Y)=0$, $0\leqslant p\leqslant n-1$.

Proposition 7. *If D is a polydisc in $\mathbb{C}^n$ and p is an integer with $0\leqslant p\leqslant n-1$, then $H^p(\mathscr{A}(D, X), \alpha)=0$ iff $H^p(C^\infty(D, X), \alpha\oplus\bar\partial)=0$.*

Proof Let $H^p(C^\infty(D, X), \alpha\oplus\bar\partial)=0$ and $\psi\in\Lambda^p[\sigma, \mathscr{A}(D, X)]$ with $\alpha\psi=0$ on D. Since $\bar\partial\psi=0$, we may regard ψ as a form in $\Lambda^p[\sigma\cup d\bar{z}, C^\infty(D, X)]$ and $(\alpha\oplus\bar\partial)\psi=0$. Hence there exists $\varphi\in\Lambda^{p-1}[\sigma\cup d\bar{z}, C^\infty(D, X)]$ with $\psi=(\alpha\oplus\bar\partial)\varphi$. φ can be uniquely written as $\varphi=\varphi_0+\varphi_1+\cdots+\varphi_{p-1}$, where φ_i has the degree i in $S_1, \cdots, S_n$. Since

$$\psi=(\alpha\oplus\bar\partial)\varphi,$$

we obtain $\psi=\alpha\varphi_{p-1}$, $\alpha\varphi_{p-2}+\bar\partial\varphi_{p-1}=0$, $\alpha\varphi_{p-3}+\bar\partial\varphi_{p-2}=0$, $\cdots$, $\alpha\varphi_1+\bar\partial\varphi_2=0$, $\alpha\varphi_0+\bar\partial\varphi_1=0$ $\bar\partial\varphi_0=0$. By the Lemma 2.1 of S. Frunză [3], we canfind a form

$$\xi_0\in\Lambda^{p-2}[d\bar{z}, C^\infty[D, X)]$$

such that $\varphi_0=\bar\partial\xi_0$. Replacing this to the equation $\alpha\varphi_0+\bar\partial\varphi_1=0$, we obtain

$$\bar\partial(\varphi_1-\alpha\xi_0)=0.$$

Successively we can choose $\xi_0, \xi_1, \cdots,\xi_{p-2}$ such that

$$\varphi_i=\alpha\xi_{i-1}+\bar\partial\xi_i,\ i=1, \cdots, p-2,$$

where ξ_i has degree $p-2-i$ in $d\bar{z}_1,\cdots, d\bar{z}_n$. Since $\varphi_{p-1}-\alpha\xi_{p-2}$ has degree 0 in $d\bar{z}_n$, $\bar\partial(\varphi_{p-1}-\alpha\xi_{p-2})=0$. Hence $\varphi_{p-1}-\alpha\xi_{p-2}\in\Lambda^{p-1}[\sigma, \mathscr{A}(D, X)]$ and

$$\psi=\alpha\varphi_{p-1}=\alpha(\varphi_{p-1}-\alpha\xi_{p-2}).$$

Thus we have $H^p(\mathscr{A}(D, X), \alpha)=0$. The proof is concluded.

It follows directly from the Proposition 7 that $a\in(A)$ iff for any polydisc $D\subset\mathbb{C}^n$,

$$H^p(C^\infty(D,X), \alpha\oplus\bar\partial)=0, \quad 0\leqslant p\leqslant n-1.$$

This, together with the proof of Theorem 2.1 of S. Frunză [3],shows that $a\in(A)$ iff for any open set $G\subset\mathbb{C}^n$,

$$H^p(C^\infty(G, X), \alpha\oplus\bar\partial)=0, \quad 0\leqslant p\leqslant n-1.$$

For the converse see Proposition 2.1 of S. Frunză [3].

Theorem 8. *If $Y\in Lat\ a$, then Y is an analytic invariant subspace of a iff, for any open set $G\subset\mathbb{C}^n$ and p with $0\leqslant p\leqslant n-1$, the following statement holds: If*

$$\psi\in\Lambda^p[\sigma\cup d\bar z, C^\infty(G, X)], \quad \alpha\psi\in\Lambda^{p+1}[\sigma\cup d\bar z, C^\infty(G, Y)],$$

there must exist forms $\varphi\in\Lambda^p[\sigma\cup d\bar z, C^\infty(G, Y)]$ and $\eta\in\Lambda^{p-1}[\sigma\cup d\bar z, C^\infty(G, X)]$ such that

$$\varphi=\varphi+(\alpha\oplus\bar\partial)\eta.$$

Proof In the same way of proving Theorem 6, we can prove that the sufficient condition is equavalent to the statement that for any open set $G\subset\mathbb{C}^n$,

$$H^p(C^\infty(G, X/Y), \alpha\oplus\bar\partial)=0, \quad 0\leqslant p\leqslant n-1,$$

and this, according to Proposition 7, is true iff $a^Y\in(A)$. Now the proof can be obtained directly from Theorem 6.

Corollary 9. *If $a\in(A)$ and Y is an analytic invariant subspace of a, then*

$$\mathrm{Sp}\ (a, X)=\mathrm{Sp}(a, Y)\cup\mathrm{Sp}(a^Y, X/Y).$$

Proof First of all, from the exactness of

$$0\to Y\to X\to X/Y\to 0$$

we have

$$\mathrm{Sp}\ (a, X)\subset\mathrm{Sp}(a,Y)\cup\mathrm{Sp}(a^Y, X/Y),$$

and

$$\mathrm{Sp}(a^Y, X/Y)\subset\mathrm{Sp}(a, X)\cup\mathrm{Sp}(a, Y).$$

Since $a\in(A)$ and Y is an analytic invariant subspace of a, $\mathrm{Sp}\ (a, Y)\subset\mathrm{Sp}\ (a, X)$ by Corollary 5. Therefore

$$\mathrm{Sp}\ (a^Y, X/Y)\subset\mathrm{Sp}(a, X)\cup\mathrm{Sp}(a, Y)\subset\mathrm{Sp}(a, X)$$

and the proof is completed.

If Y is an analytic invariant subspace of a, then $a^Y\in(A)$. Hence for any $x\in X$, $\mathrm{Sp}\ (a^Y, X/Y)$ makes sense, that is, the equation $SX/Y=(\alpha\oplus\bar\partial)\psi$ has a global solution ψ on $r(a^Y, X/Y)$, where X/Y is the equavalent class of $x\in X$ in X/Y.

Theorem 10. *If $a\in(A)$ and Y is an analytic invariant subspace of a, then*

$$\mathrm{Sp}\ (a, x)=[\mathrm{Sp}(a, x)\cap\mathrm{Sp}(a, Y)]\cup\mathrm{Sp}(a^Y, X/Y), \quad \forall x\in X.$$

Proof In the first place, for any polydisc $D\subset r(a, x)$, there exist a form $\psi\in\Lambda^{n-1}[\sigma, \mathscr{A}(D, X)]$ such that $sx=\alpha\psi$ on D. Hence $SX/Y=\alpha^Y(\psi/Y)$ on D.

Therefore $\mathrm{Sp}(a^Y, X/Y)\subset \mathrm{Sp}(a, x)$, $\forall x\in X$. This shows that

$$\mathrm{Sp}(a, x)\supset [\mathrm{Sp}(a, x)\cap \mathrm{Sp}(a, Y)]\cup \mathrm{Sp}(a^Y, X/Y).$$

Conversely, if $z_0\notin [\mathrm{Sp}(a, x)\cap \mathrm{Sp}(a, Y)]\cup \mathrm{Sp}(a^Y, X/Y)$, then

$$z_0\in r(a^Y, x/Y)\cap [r(a, x)\cup r(a, Y)].$$

If $z_0\notin r(a, x)$, $z_0\in r(a^Y, x/Y)\cap r(a, Y)$. There exists $\psi\in \Lambda^{n-1}[\sigma, \mathscr{A}(D, X)]$ such that $sx/Y=\alpha^Y(\psi/Y)$ on D, where D is an open polydisc in $\mathbb{C}^n$,

$$z_0\in D\subset r(a, Y)\cup q(a^Y, x/Y).$$

Hence $sx-\alpha\psi\in \Lambda^n[\sigma, \mathscr{A}(D, Y)]$. Since $D\subset r(a, Y)$, $H^n(\mathscr{A}(D, Y),\alpha)=0$. There is a form $\varphi\in \Lambda^{n-1}[\sigma, \mathscr{A}(D, Y)]$ such that $Sx-\alpha\psi=\alpha\varphi$. Therefore

$$sx=\alpha(\varphi+\psi)(z),\quad \forall z\in D.$$

It follows that $z_0\in D\subset r(a, x)$. This contradicts the assertion $z_0\notin r(a, x)$. And the contradiction shows that if $z_0\notin [\mathrm{Sp}(a, x)\cap \mathrm{Sp}(a,Y)]\cup \mathrm{Sp}(a^Y, x/Y)$ then $z_0\notin \mathrm{Sp}(a, x)$. Consequently

$$\mathrm{Sp}(a, x)\subset [\mathrm{Sp}(a, x)\cap \mathrm{Sp}(a, Y)]\cup \mathrm{Sp}(a^Y, x/Y).$$

Proposition 11. *Suppose $a\in(A)$, Y is an analytic invariant subspace of a, and suppose that $f=(f_1, \cdots, f_m)$, where f_i is analytic on an open set U, $\mathrm{Sp}(a, x)\subset U\subset \mathbb{C}^n$. Then Y is an analytic invariant subspace of $f(a)$.*

Proof Since Y is an analytic invariant subspace of a and $a\in(A)$, $a^Y\in(A)$. We can apply Theorem 3.2 of J. Eschmeier [5] and conclude that $f(a^Y)\in(A)$; since every f_i is analytic on U and $U\supset \mathrm{Sp}(a, x)\supset \mathrm{Sp}(a^Y, X/Y)$. By Theorem 6, to conclude the proof, we need only to prove $f_i(a^Y)=f_i(a)^Y$, $0\leqslant i\leqslant m$.

Let $x\in X$, $x/Y\in X/Y$, $1\leqslant i\leqslant m$. Suppose $\psi\in \Lambda^{n-1}[\sigma\cup d\bar{z}, C^\infty(U, X)]$ such that $Sx-(\alpha\oplus\bar\partial)\psi$ has compact support in U. If $c_n=\dfrac{(-1)^n}{(2\pi i)^n}$, then for every $x\in X$,

$$\begin{aligned} f_i(a)^Y(x/Y) &=(f_i(a)x)/Y= c_n\int_U \pi(sx-(\alpha\oplus\bar\partial)\psi)f_i(z)\wedge dz/Y\\ &=c_n\int_U \pi(sx-(\alpha\oplus\bar\partial)\psi)f_i(z)/Y\wedge dz\\ &=c_n\int_U \pi[sX/Y-(\alpha^Y\oplus\bar\partial)(\psi/Y)]f_i(z)dz\\ &=f_i(a^Y(x/Y). \end{aligned}$$

In the above, we have used the fact that if $\varphi\in C^\infty(U, X)$ and $\varphi(z)\to\zeta$ $(z\to z_0)$ then $\varphi(z)/Y\to\zeta/Y$. But this is true because

$$\|\varphi(z)/Y-\zeta/Y\|=\|(\varphi(z)-\zeta)/Y\|\leqslant\|\varphi(z)-\zeta\|\to 0.$$

Thus the proof is compteted.

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