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THE GENERALIZED p-NORMAL OPERATORS AND p-HYPONORMAL OPERATORS ON BANACH SPACE

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Abstract

In this paper, the authors discuss the generalized p-normal and p-hyponormal operators on Banach space. Some results in this paper are the generalization of Sen's results on generalized p-selfadjoint operator and some open questions of Sen's are answered.

For the generalized p-normal operators, the following formulae are obtained:

$$r(T)=\|T\|, \quad \|(T-\lambda I)^{-1}\|=\frac{1}{\operatorname{dist}(\lambda, \sigma(T))}.$$

§ 1. Introduction

We recall from Nath[4] that a complex Banach space X is called a complex generalized semi-inner product space if correspooding to an arbitrary pair of elements $x, y\in X$, there is a complex number $[x, y]_p$ which satisfies the following properties for any $x, y, z\in X$ and $\lambda\in C$ (C denotes the complex field):

(1) $[x+y, z]_p=[x, z]_p+[y, z]_p$. $[\lambda x, y]_p=\lambda[x, y]_p$;

(2) $[x, x]_p>0$, for $x\neq 0$;

(3) $|[x, y]_p|\leqslant[x, x]_p^{1/p}[y, y]_p^{\frac{p-1}{p}}$, $1<p<\infty$.

A generalized semi-inner product (briefly g. s. i. p) $[x, y]_p$ which generates the norm $\|,\|$ means that for any $x\in X$, $\|x\|=[x, x]_p^{\frac{1}{p}}$.

Sen [7, Corollary 1 and 10, Note 1.1] has proved the following result.

Proposition 1.1. *If X is a complex Banach space with norm $\|\cdot\|$, then for each $p\in(1, +\infty)$, there exists a g. s. i, p $[x, y]_p$ which generates norm $\|\cdot\|$, and in this case we have*

$$[x, \lambda y]_p=|\lambda|^{p-2}\bar{\lambda}[x, y]_p, \quad \text{for any } x, y\in X, \lambda\in C. \tag{1.1}$$

Moreover if $p\neq p'$, $p, p'\in(1, +\infty)$ and $[\ ,\ ]_p$, $[\ ,\ ]_{p'}$ are respectively the corresponding g. s, i, p which generates the norm $\|\cdot\|$, then for all $x, y\in X$, $y\neq 0$

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$$[x, y]_p = \|y\|^{p-p'}[x, y]_{p'}. \tag{1.2}$$

Milicié [6, Theorem 4] has proved that if X is a smooth strictly convex and reflexive Banach space, then there is a unique $g.\ s.\ i.\ p[\ ,\ ]_p$ which generates the norm and for each $f \in X^*$ there is a unique $y \in X$ such that $f(x) = [x, y]_p$ for all $x \in X$, and in this case we have

$$\|f\| = \|y\|^{p-1}.$$

From Milicié's result and Proposition 1.1, we have

Proposition 1.2. *For each $f \in X^*$ and $p' \in (1, +\infty)$, there is a unique $y' \in X$ such that $f(x) = [x, y']_{p'}$ for all $x \in X$, where the g. s. i. $p[\ ,\ ]_{p'}$ generates the norm.*

Throughout this paper, we shall always assume that X is a Banach space which is smooth, strictly convex and reflexive. We also assume that the $g,\ s,\ i,\ p[\ ,\ ]_p$ which appears in this paper always generates the norm.

Sen[7], by the above result of Milicié, has defined the generalized adjoint operator T_p^* of an operator T on X.

Suppose $p \in (1, +\infty)$. For $T \in B(X)$ and $y \in X$, by $g(x) = [Tx, y]_p$, we obtain a linear continuous functional $g \in X^*$. From Proposition 1.2 there is a unique $y^* \in X$ such that

$$[Tx, y]_p = [x, y^*]_p. \tag{1.3}$$

Therefore, $T_p^* y = y^*$ well defines a mapping which maps X into X, T_p^* is called the generalized p-adjoint operator of T.

Generally, T_p^* is not a linear operator and is depending on p. For any mapping A which maps X into X, let

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

Evidently, we have

Proposition 1.3. *Let $T \in B(X)$. $p \in (1, +\infty)$. Then*

(1) $\|T_p^*\| = \|T\|^{\frac{1}{p-1}}$,

(2) $\|T_p^* T\| \leqslant \|T_p^*\| \|T\|$,

(3) $\|T T_p^*\| \leqslant \|T\| \|T_p^*\|$.

T is celled a generalized p-self-adjoint operator if $T_p^* = T$.

Sen[10] has discussed some spectral properties of generalized 2-selfadjoint operators and has given some open problems. In this paper, we shall introduce the concept of generalized p-normal and p-hyponormai operators which extends the concept of generalized p-selfadjoint operators and answer the Sen's questions.

Proposition 1.4. *Let $A, B \in B(X)$ such that A_p^*, B_p^* are both linear bouoded operators and there are constants M_A, M_B such that $(A_p^*)_p^* = M_A A$, $(B_p^*)_p^* = M_B B$. Then*

(1) $[Ax, y]_p = [x, A_p^* y]_p$, $[A_p^* x, y]_p = |M_A|^{p-2} \overline{M}_A [x, Ay]_p$;

(2) $((A_p^*)_p^*)_p^* = M^{\frac{2-p}{p-1}} \overline{M}_A A_p^*$;

(3) $\|A_p^*\| = \|M_A\|^{\frac{p-1}{p}} \|A\|$;

(4) $(AB)_p^* = B_p^* A_p^*$;

(5) *A is a regular operator if and only if A_p^* is a regular operator.*

Proof We only prove (3). For convenience, write

$$A^* = A_p^*,\ A^{**} = (A_p^*)_p^*,\ A^{***} = ((A_p^*)_p^*)_p^*,\ [x, y] = [x, y]_p.$$

Since

$$\|Ax\|^p = [Ax, Ax] = |M_A|^{1-p}[A^*Ax, x]| \le |M_A|^{1-p}[A^*Ax, A^*Ax]^{\frac{1}{p}}[x, x]^{\frac{p-1}{p}},$$

we also have

$$\|Ax\|^p = [x, A^*Ax] \le [x, x]^{\frac{1}{p}}[A^*Ax, A^*Ax]^{\frac{p-1}{p}}.$$

Thus

$$\|Ax\|^{2p} \le |M_A|^{1-p}\|A^*Ax\|^p\|x\|^p,$$

so that $\|Ax\|^2 \le |M_A|^{\frac{1-p}{p}}\|A^*Ax\|\|x\|$. Hence $\|A\| \le |M_A|^{\frac{1-p}{p}}\|A^*\|$.

Similarly, since

$$\|A^*x\|^p = [A^*x, A^*x] = [AA^*x, x] \le [AA^*x, AA^*x]^{\frac{1}{p}}[x, x]^{\frac{p-1}{p}}$$

and

$$\begin{aligned}\|A^*x\|^p = [x, A^{**}A^*x] &= [x, M_A AA^*x] = |M_A|^{p-1}|[x, AA^*x]| \\ &\le |M_A|^{p-1}[x, x]^{\frac{1}{p}}[AA^*x, AA^*x]^{\frac{p-1}{p}},\end{aligned}$$

we have $\|A^*x\|^2 \le |M_A|^{\frac{p-1}{p}}\|x\|\|AA^*x\|$. From this, we have

$$\|A^*\| \le |M_A|^{\frac{p-1}{p}}\|A\|.$$

Hence $\|A^*\| = |M_A|^{\frac{p-1}{p}}\|A\|$, this completes the proof of (3).

Dnfinition 1.5. *Let $x, y \in X\setminus\{0\}$. The vector x is said to be generalized p-orthogonal to y if $[y, x]_p = 0$. If N is a subset of X, let*

$$N^\perp = \{x \mid [y, x]_p = 0 \text{ for any } y \in N\}.$$

$N^\perp$ is called the generalized p-orthogonal complement of N.

Proposition 1.6. *Let $A \in B(X)$, $A \ne 0$. If A_p^* is a linear bounded operator and $(A_p^*)_p^* = MA$, where M is a constant, then*

(1) $N(A) = \overline{R(A_p^*)}^\perp$,

(2) $N(A_p^*) = \overline{R(A)}^\perp$,

(3) $R(A) \subset \overline{N(A_p^*)}^\perp$,

(4) $R(A_p^*) \subset \overline{N(A)}^\perp$,

where $N(A)$ and $R(A)$ denote respectively the null space and the range of A, $\overline{\Omega}$ denotes the closure of Ω.

The proof of Proposition 1.6 is completely analogous to the proof in the Hilbert space, so we omit it.

§ 2. Generalized p-Normal and p-Hyponormal Operators

Definition 2.1. *Let $T\in B(X)$ and let T_p^* be a linear bounded operator. If there is a constant M (relative to p) such that $(T_p^*)_p^*=MT$ and $T_p^*T=TT_p^*$, then T is said to be a generalized p-normal operator.*

Evidently, generalized p-selfadjoint operators must be generalized p-normal operators.

Example 2.2. Let $X=l^p$, $1<p<+\infty$, $p\neq 2$. Then X is a smooth, strictly convex and reflexive Banach space. For arbitrary $x, y\in X$, $x=\{x_i\}_{i=1}^\infty$, $y=\{y_i\}_{i=1}^\infty$, let

$$[x, y]_p=\sum_{i=1}^\infty x_i|y_i|^{p-2}\bar{y}_i. \tag{2.1}$$

Then $[x, y]_p$ is a generalized semi-inner product which generates the norm. For any $f\in X^*$, $f=\{f_i\}_{i=1}^\infty\in l^q$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, let

$$y_j=\begin{cases}0 & \text{if } f_j=0,\\ |f_j|^{\frac{1}{p-1}}e^{-i\alpha_j} & \text{if } f_j=|f_j|e^{i\alpha_j}\neq 0.\end{cases}$$

Evidently, $y\in X$ and

$$f(x)=[x. y]_p \quad \text{for any } x\in X.$$

If $x=\{x_i\}_{i=1}^\infty$ and $y=\{y_i\}_{i=1}^\infty$, let

$$T_1x=\{x_i'\}_{i=1}^\infty, \quad T_2x=\{x_i''\}_{i=1}^\infty,$$

where

$$x_i'=\begin{cases}x_1 & i=1,\\ 0 & i\neq 1,\end{cases} \quad x_i''=\begin{cases}x_2 & i=2,\\ 0 & i\neq 2.\end{cases}$$

Since $[T_1x. y]_p=x_1|y_1|^{p-2}\bar{y}_1$ and $[x, T_1y]_p=x_1|y_1|^{p-2}\bar{y}_1$, T_1 is a generalized p-selfadjoint operator. Similarly, T_2 is also a generalized p-selfadjoint operator. Let $T=T_1+iT_2$ $A=T_1-iT_2$. Since

$$[Tx, y]_p=x_1\bar{y}_1|y_1|^{p-2}+ix_2\bar{y}_2|y_2|^{p-2},$$
$$[x, Ty]_p=x_1\bar{y}_1|y_1|^{p-2}-ix_2\bar{y}_2|y_2|^{p-2},$$

we see that T is not a generalized p-selfadjoint operator. But since

$$[x, Ay]_p=x_1\bar{y}_1|y_1|^{p-2}+ix_2\bar{y}_2|y_2|^{p-2}=[Tx, y]_p=[x, T_p^*y]_p,$$

we have $T_p^*=A$. Since $[Ax, y]_p=x_1|y_1|^{p-2}\bar{y}_1-ix_2|y_2|^{p-2}\bar{y}_2=[x, Ty]_p$, we have $(T_p^*)_p^*=T(M=1)$. On the other hand, since

$$T_p^*Tx=T_p^*\{x_1, ix_2, 0, \cdots\}=\{x_1, x_2, 0, \cdots\},$$
$$TT_p^*x=T\{x_1. -ix_2, 0, \cdots\}=\{x_1, x_2, 0, \cdots\},$$

we have $T_p^*T=TT_p^*$, that is to say that T is a generalized p-normal operator.

Definition 2.3. *Let $T\in B(X)$, $p\in(1, +\infty)$. Write*

$$W_p(T)=\{[Tx, x]_p \mid \|x\|=1\},$$

which is called the generalized p-numeral range of T. If $W_p(T)\subset R^1$, where R^1 denotes the real field, then T is said to be a generalized p-Hermidt operator; if for any $x\in X$, $[Tx, x]_p\geqslant 0$, then T is said to be a positive operator (briefly writing $T\geqslant 0$).

Definition 2.4. *Let $T\in B(X)$. If T_p^* is a linear bounded operator and there is a constant M (relative to p) such that $(T_p^*)_p^*=MT$ and $T_p^*T-TT_p^*\geqslant 0$, then T is called a generalized p-hyponormal operator.*

Evidently, a generalized p-normal operator must be a generalized p-hyponormal operator.

Example 2.5. Let $X=l^p$. Define g, s, i, $p[\ ,\ ]_p$ as in Example 2.2. Let Π be the unilateral shift operator, i.e, for any $x=\{x_i\}_{i=1}^\infty\in X, \Pi x=\{0, x_1, x_2, \cdots\}$. Moreover, let

$$\Pi' x=\{x_2, x_3, \cdots\}.$$

Then Π' is a linear bounded operator on X and since

$$[\Pi x, y]_p=\sum_{i=1}^\infty x_i|y_{i+1}|^{p-2}\bar{y}_{i+1}=[x, \Pi' y]_p,$$

we have $\Pi_p^*=\Pi'$. On the other hand

$$[x, (\Pi_p^*)_p^* y]_p=[\Pi_p^* x, y]_p=\sum_{i=1}^\infty x_{i+1}|y_i|^{p-2}\bar{y}_i=[x, \Pi y]_p.$$

Thus $(\Pi_p^*)_p^*=\Pi(M=1)$. But since $\Pi_p^*\Pi=I$, $\Pi_p^*\Pi-\Pi\Pi_p^*=T_1$ and $[T_1 x, x]_p=|x_1|^p\geqslant 0$, i.e, T_1 is positive, we know

$$\Pi_p^*\Pi-\Pi\Pi_p^*\geqslant 0.$$

So Π is a generalized p-hyponormal operator, but is not a generalized p-normal operator.

From Definitions 2.1 and 2.3, we obtain easily the following result.

Proposition 2.6. *If T is a generalized p-normal (or p-hyponormal) operator, then for any $\lambda\in C$, λT is also a generalized p-normal (p-hyponormal) operator.*

In what follows, we discuss the constant M in Definitions 2.1 and 2.3.

Proposition 2.7. *If T is a generalized p-hyponormal operator and $T\neq 0$,* then $M>0$.

Proof For convenience, write $T^*=T_p^*$, $T^{**}=(T_p^*)_p^*$, $D=T_p^*T-TT_p^*$. Since

$$\begin{aligned}\|T^2x\|^p&=[T^2x, T^2x]_p=[T^2x, M^{-2}T^{**2}x]_p=[T^{*2}T^2x, M^{-2}x]_p\\&=[T^*(TT^*+D)Tx, M^{-2}x]_p=[(T^*T)^2x, M^{-2}x]_p+[T^*DTx, M^{-2}x]_p\\&=[T^*Tx, M^{-1}T^*Tx]_p+[DTx, M^{-1}Tx]_p\\&=|M^{-1}|^{p-2}\overline{M^{-1}}\{[T^*Tx, T^*Tx]_p+[DTx, Tx]_p\}\end{aligned}$$

but $D\geqslant 0$, we have $\overline{M^{-1}}>0$. Thus $M>0$.

From Propositions 1.3 and 1.2, we have

Theorem 2.8. *Let $T\in B(X)$, $p\in(1, +\infty)$. If T_p^* is a linear bounded operator and $(T_p^*)_p^*=MT$, then*

$$M=\|T\|^{\frac{p(2-p)}{(p-1)^2}}.$$

Corollary 2.9. *Let T be a generalized p-selfadjoint operator and $p\neq 2$, then $\|T\|=1$.*

Corollary 2.10. *Let T be a generalized 2-normal operator, then $M=1$, so that $(T_2^*)_2^*=T$.*

Corollary 2.11. *Let T be a generalized p-normal operator. If $\|T\|=1$, then $(T_p^*)_p^*=T$, i.e, $M=1$.*

Sen [10, Note 2.5] asked: if T is a generalized p-selfadjoint operator for any $p\in(1,\infty)$, is T an isometric operator? The above Corollary 2.9 provides a partial answer to the problem.

§ 3. The Properties of the Generalized p-Normal Operators

For convenience, in what follows, we always write briefly

$$T^*=T_p^*,\ T^{**}=(T_p^*)_p^*,\ [x,y]=[x,y]_p.$$

Theorem 3.1. *If T is a generalized p-normal operator, then*

(1) *for any $x\in X$, $\|Tx\|=M^{\frac{1-p}{p}}\|T^*x\|$, and from this, we have $N(T)=N(T^*)$.*

(2) *for any $\lambda\in C$, $N(T-\lambda I)=N((T-\lambda I)^*)$.*

Proof (1) Since

$$\begin{aligned}\|Tx\|^p&=[Tx,Tx]=[Tx,M^{-1}T^{**}x]=M^{1-p}[T^*Tx,x]=M^{1-p}[TT^*x,x]\\&=M^{1-p}[T^*x,T^*x]=M^{1-p}\|T^*x\|^p,\end{aligned}$$

we have $\|Tx\|=M^{\frac{1-p}{p}}\|T^*x\|$.

(2) From Proposition 1.2 and the fact that T is a generalized p-normal operator, we have

$$\begin{aligned}(T-\lambda I)^*(T-\lambda I)&=(T^*(T-\lambda I))^*M^{-1}-\lambda(T-\lambda I)^*\\&=T(T-\lambda I)^*-\lambda(T-\lambda I)^*=(T-\lambda I)(T-\lambda I)^*.\end{aligned}$$

If $x\in N(T-\lambda I)$, then

$$\begin{aligned}\|(T-\lambda I)^*x\|^p&=[(T-\lambda I)^*x,(T-\lambda I)^*x]=[(T-\lambda I)(T-\lambda I)^*x,x]\\&=[(T-\lambda I)^*(T-\lambda I)x,x]=0,\end{aligned}$$

so that $x\in N((T-\lambda I)^*)$. Hence $N(T-\lambda I)\subset N((T-\lambda I)^*)$.

Similarly, since

$$\|(T-\lambda I)x\|^p=[x,(T-\lambda I)(T-\lambda I)^*x],$$

it follows that $N((T-\lambda I)^*)\subset N((T-\lambda I))$. Thus (2) holds.

Corollary 3.2. *If T is a generalized p-normal operator, then eigenvectors corresponding to distinct eigenvalues of T are orthogonal mutually.*

Theorem 3.4. *Let T be a generalized p-normal operator for all $p\in(1,+\infty)$ and $T_p^*=T_{p'}^*$ for all $p,p'\in(1,+\infty)$. If $(T_p^*)_p^*=T$ and $N(T)=\{0\}$, then T is an* *isometric operator on* X.

Proof Since

$$[Tx, y]_p=[x, T^*y]_p=\|T^*y\|^{p-p'}[x, T^*y]_{p'}$$

and

$$[Tx, y]_p=\|y\|^{p-p'}[Tx, y]_{p'}=\|y\|^{p-p'}[x. T^*y]_{p'},$$

if $y\in\overline{R(T)}^\perp$, then $\|T^*y\|=\|y\|$. By Theorem 3.1, it follows that $\|Ty\|=\|y\|$. By Proposition 1.6, if $y\in\overline{R(T)}^\perp$, then $y\in N(T^*)$. Thus, from the assumption and Theorem 3.1, it follows that $y\in N(T)$. But $N(T)=\{0\}$, so that $y=0$, the proof is complete.

Corollary 3.5. *If T is a generalized p-selfadjoint operctor for all $p\in(1, +\infty)$ and $N(T)=\{0\}$, then T is an isometric operator on X.*

The above corollary partially answers the Sen's question [10, Note 2.5].

Theorm 3.6. *Let T be a generalized p-normal operator. Then $N(T)\cup R(T)$ is dense in X.*

Proof If $N(T)\cup R(T)$ is not dense in X, then by Sen's result [7, Corollary 3], there is a vector z_0, $z_0\neq 0$, such that $z_0\in(N(T)\cup R(T))^\perp$. From this it follows that for any $x\in X$,

$$[x, T^*z_0]=[Tx, z_0]=0.$$

Thus $T^*z_0=0$. By assumption and Theorem 3.1, $Tz_0=0$. Thus $z_0\in N(T)$. But since $z_0\in(N(T)\cup R(T))^\perp$, we have $\|z_0\|^p=[z_0, z_0]=0$, this contradicts $z_0\neq 0$.

Lemma 3.8. *Let $T\in B(X)$. Then $\sigma_\Pi(T)\subset\overline{W_p(T)}$. Particularly, we have $\partial\sigma(T)\subset\overline{W_p(T)}$, where $\partial\sigma(T)$ denotes the boundary of $\sigma(T)$.*

The proof is completely analogous to the proof in the Banach space, so we omit it.

From Lemma 3.8 and Sen [10, Corollary 2.2], we have

Theorem 3.9. *If T is a generalized p-normal operator, then*

$$\sigma_\Pi(T)=\sigma(T)\subset\overline{W_p(T)}.$$

From the definition of single-valued extension property of operator and (1.1), (1.2), we obtain easily the following theorem.

Theorem 3.10. *If T is a generalized p-normal operator, then T has the single-valued extension property.*

§ 4. Further Properties of Generalized p-Normal Operators

In this paragraph, we will prove that for any generalized p-hyponormal operator T, the following formula holds:

$$r(T)=|W_p(T)|=\|T\|$$

and for any gneralized p-normal operator T, we have

$$\|(T-\lambda I)^{-1}\|=1/\text{dist}(\lambda, \sigma(T)) \quad \text{for all } \lambda\in\rho(T).$$

The first result is the affirmative answer to Sen's problem [10, Note 2.19].

On the Hilbert space, Li shaokuan[8] has defined the concept of class (N)-operators. Evidently, his definition may be extended to the case of Banach space.

Let $T\in B(X)$, T is called a class (N)-operator on X if for any $x\in X$,

$$\|Tx\|^2\leqslant\|T^2x\|\,\|x\|.$$

Proposition 4.1. *If T is a class (N)-operator on X, then for any $p\in(1, +\infty)$*

$$r(T)=\|T\|=|W_p(T)|.$$

Proof The proof of $r(T)=\|T\|$ is completely analogous to the proof in [3], we omit it. From Lumer [5, Theorem 4] and Sen [10, Note 2.9], it follows that

$$r(T)\leqslant|W_p(T)|\leqslant\|T\|,$$

which completes the proof.

Theorem 4.2. *Let T be a generalized p-normal operator, Then*

$$r(T)=|W_p(T)|=\|T\|.$$

Proof By Proposition 4.1, we only need to prove that T is a class (N)-operator. Without loss of generality, we may assume $T\neq 0$. If $D=T^*T-TT^*\geqslant 0$ and $T^{**}=MT$, by Proposition 2.6 it follows that $M>0$ and

$$\begin{aligned}\|T^2x\|^p&=[T^2x, T^2x]=[T^2x, M^{-2}(T^{**})^2x]=[T^{*2}T^2x, M^{-2}x]\\&=[T^*(TT^*, +D)Tx, M^{-2}x]=[(T^*T)^2x, M^{-2}x]+[T^*DTx, M^{-2}x],\\&\geqslant[(T^*T)^2x, M^{-2}x]=[(T^*T)x, T^*T(M^{-1}x)]=M^{1-p}\|T^*Tx\|^p.\end{aligned}$$

Moreover, since

$$\|Tx\|^p=[Tx, Tx]\leqslant[x, x]^{\frac{1}{p}}[T^*Tx, T^*Tx]^{\frac{p-1}{p}}$$

and

$$\|Tx\|^p=[T^*Tx, M^{-1}x]=M^{1-p}[T^*Tx, x]\leqslant M^{1-p}\|T^*Tx\|\,\|x\|^{p-1},$$

we have

$$\|Tx\|^{2p}\leqslant M^{1-p}\|T^*Tx\|^p\|x\|^p,$$

Thus $\|Tx\|^{2p}\leqslant\|T^2x\|^p\|x\|^p$. It follows that

$$\|Tx\|^2\leqslant\|T^2x\|\,\|x\|.$$

Theorem 4.3. *Let $T\in B(X)$, $p\in(1, +\infty)$, If $TT_p^*=T_p^*T$, then*

$$r(T)=\|T\|.$$

Proof For convenience, write $T^*=T_p^*$, $[\ ,\ ]=[\ ,\ ]_p$.

Since

$$\|Tx\|^p=[Tx, Tx]=[x, T^*Tx]\leqslant\|x\|\,\|T^*Tx\|^{p-1},$$

we have $\|T\|^p\leqslant\|T^*T\|^{p-1}$. Moreover, since

$$\|T^*T\|^{p-1}\leqslant\|T^*\|^{p-1}\|T\|^{p-1},$$

by Proposition 1.3 it follows that $\|T^*T\|^{p-1}\leqslant\|T\|^p$. Thus

$$\|T\|^p=\|T^*T\|^{p-1}. \tag{4.1}$$

On the other hand, since

$$\|T^*Tx\|^p=[T^*Tx,\ T^*Tx]=[TT^*Tx,\ Tx]=[T^*T^2x,\ Tx]\leqslant\|T^*T^2x\|\,\|Tx\|^{p-1},$$

we have

$$\|T^*T\|^p\leqslant\|T\|^{p-1}\|T^*\|\,\|T^2\|=\|T\|^{p-1}\|T\|^{\frac{1}{p-1}}\|T^2\|=\|T\|^{\frac{p^2-2p+2}{p-1}}\|T^2\|. \qquad (4.2)$$

By (4.1) and (4.2), it follows that

$$\|T\|^{\frac{p^2-(p^2-2p+2)}{p-1}}\leqslant\|T^2\|,$$

i.e $\|T\|^2\leqslant\|T^2\|$. Hence $\|T\|^2=\|T^2\|$.

In what follows, we prove that equality $\|T^n\|=\|T\|^n$ holds for any positive integral number n. If for any $n\leqslant k$, equality $\|T^n\|=\|T\|^n$ holds, since

$$\|T^kx\|^p=[T^kx,\ T^kx]=[T^{k-1}x,\ T^*T^kx]\leqslant\|T^{k-1}x\|\,\|T^*T^kx\|^{p-1}$$

we have $\|T^k\|^p=\|T^{k-1}\|\,\|T^*T^k\|^{p-1}$. By assumption, it follows that

$$\|T\|^{kp-k+1}\leqslant\|T^*T^k\|^{p-1}.$$

But since

$$\|T^*T^k\|^{p-1}\leqslant\|T^*\|^{p-1}\|T^k\|^{p-1}=\|T\|\,\|T\|^{k(p-1)},$$

we have

$$\|T^*T^k\|^{p-1}=\|T\|^{kp-k+1}. \qquad (4.3)$$

On the other hand, since

$$\|T^*T^kx\|^p=[T^*T^kx,\ T^*T^kx]=[T^*T^{k+1}x,\ T^kx]\leqslant\|T^*\|\,\|T^{k+1}x\|\,\|T^kx\|^{p-1}, \qquad (4.4)$$

we have

$$\|T^*T^k\|^p\leqslant\|T^*\|\,\|T^{k+1}\|\,\|T^k\|^{p-1}. \qquad (4.4)$$

From (4.3) and (4.4), it follows that

$$\|T\|^{\frac{kp-k+1}{p-1}}\leqslant\|T^*\|\,\|T^{k+1}\|\,\|T^k\|^{p-1}\leqslant\|T\|^{\frac{1}{p-1}}\|T^{k+1}\|\,\|T\|^{k(p-1)}.$$

Then

$$\|T\|^{k+1}=\|T\|^{\frac{p(kp-k+1)1}{p-1}-(kp-k)}\leqslant\|T^{k+1}\|.$$

Thus $\|T^{k+1}\|=\|T\|^{k+1}$. From this, it follows that $r(T)=\|T\|$.

Corollary 4.4. *Let T be a generalized p-normal operator and $T\neq 0$. Then for any $\lambda\in C$,* we have

$$r(T-\lambda I)=\|T-\lambda I\|.$$

Proof Let $T^{**}=MT$. From Sen [7, Theorem 3], it follows that

$$(T-\lambda I)^*T^{**}=(T^*(T-\lambda I))^*=((T-\lambda I)T^*)^*=T^{**}(T-\lambda I)^*.$$

Thus

$$(T-\lambda I)^*(T-\lambda I)=M^{-1}(T-\lambda I)^*T^{**}-\lambda(T-\lambda I)^*=M^{-1}T^{**}(T-\lambda I)^*-\lambda(T-\lambda I)^*$$
$$=(T-\lambda I)(T-\lambda I)^*.$$

By Theorem 4.3, the required result holds.

Corollary 4.5. *If T is a generalized p-normal operator, then for any $\lambda\in\rho(T)$ we have*

$$r((T-\lambda I)^{-1})=\|(T-\lambda I)^{-1}\|.$$

Hence by spectral mapping theorem, we have

$$\|(T-\lambda I)^{-1}\|=1/\text{dist }(\lambda, \sigma(T)).$$

Proof Under the conditions of the corollary, we can easily prove that equality

$$(T-\lambda I)^{-1}((T-\lambda I)^{-1})_p^*=((T-\lambda I)^{-1})_p^*(T-\lambda I)^{-1}$$

holds. Hence by Theorem 4.4, it follows that

$$r((T-\lambda I)^{-1})=\|(T-\lambda I)^{-1}\|.$$

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