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SINGULAR PERTURBATION FOR A BOUNDARY VALUE PROBLEM OF FOURTH ORDER NONLINEAR DIFFERENTIAL EQUATION

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Abstract

The singular perturbation for a boundary value problem of fourth order nonlinear differential equation is studied. Under suitable assumptions using differential inequalities the author finds a solution of the original problem and obtains the uniformly valid asymptotic expansions.

We consider the boundary value problem of fourth order nonlinear differential equation

$$\varepsilon^2 y^{(4)} = f(x, y, y'', \varepsilon), \quad 0<x<1, \tag{1}$$

$$y(0, \varepsilon) = A_1(\varepsilon), \quad y(1, \varepsilon) = B_1(\varepsilon), \tag{2}$$

$$y''(0, \varepsilon) = A_2(\varepsilon), \; y''(1, \varepsilon) = B_2(\varepsilon), \tag{3}$$

where ε is a small positive parameter. The case for second order differential equation has been studied (e.g. [1—6], [12—14]). For fourth order differential equation, only first approximation of the solution has been obtained. In this paper, a boundary value problem (1)—(3) of fourth order nonlinear differential equation is discussed by using differential inequalities. By introducing two functions that possess boundary layer behavior we obtain uniformly valid asymptotic expansions to any degree of precision for the sought function and its second order derivative on the entire interval $0 \leqslant x \leqslant 1$.

§ 1. Constructing Formal Solution

We now construct the outer solution of the original problem (1)—(3). The reduced problem is

$$f(x, y, y'', 0) = 0, \quad 0<x<1, \tag{4}$$

$$y(0) = A_1(0), \quad y(1) = B_1(0). \tag{5}$$

We assume that its solution is $y_0(x)$ and the known functions $f(x, y, y'', \varepsilon)$,

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$A_j(\varepsilon)$, $B_j(\varepsilon)(j=1, 2)$ all have continuous partial derivatives with respect to the variables involved up to $(n+1)$th-order and there is a positive constant m such that $f_{y''}\geqslant m$. Thus

$$f(x, y, y'', \varepsilon)\equiv F(\varepsilon)=\sum_{i=0}^{n} F_i\varepsilon^i+r, \tag{6}$$

$$A_j(\varepsilon)=\sum_{i=0}^{n} A_{ji}\varepsilon^i+r_{j1} \quad (j=1, 2),$$

$$B_j(\varepsilon)=\sum_{i=0}^{n} B_{ji}\varepsilon^i+r_{j2} \quad (j=1, 2),$$

where

$$F_0\equiv F(0)=f(x, y_0, y_0'', 0), \tag{7}$$

$$F_i\equiv\frac{1}{i!}F^{(i)}(0)=f_{y''}(x, y_0, y_0'', 0)y_i''+f_y(x, y_0, y_0'', 0)y_i+c_i(x) \quad (i=1, 2, \cdots, n), \tag{8}$$

$$A_{ji}=\frac{1}{i!}A_j^{(i)}(0) \quad (i=0, 1, \cdots, n;\ j=1, 2),$$

$$B_{ji}=\frac{1}{i!}B_j^{(i)}(0) \quad (t=0, 1, \cdots, n;\ j=1, 2),$$

$$r=O(\varepsilon^{n+1}) \quad (0<\varepsilon\ll 1),$$

$$r_{ji}=O(\varepsilon^{n+1}) \quad (i, j=1, 2;\ 0<\varepsilon\ll 1),$$

while $c_i(x)$ can be obtained by x and y_j, $j=0, 1, \cdots, i-1$ successively.

We now assume that the outer solution of the problem (1)—(3) has the following formal expansion:

$$Y(x, \varepsilon)=y_0(x)+y_1(x)\varepsilon+y_2(x)\varepsilon^2+\cdots, \tag{9}$$

where $y_i(x)$, $i=1, 2, \cdots$ are undetermined functions.

Substituting (9) into (1), we have

$$\varepsilon^2\frac{d^4Y}{dx^4}=f(x, Y, Y'', \varepsilon). \tag{10}$$

From (4), (6)—(8), collecting the terms of like powers of ε and equating the coefficients to zero, we have

$$f_{y''}(x, y_0, y_0'', 0)y_i''+f_y(x, y_0, y_0'', 0)y_i+c_i(x)=y_{i-2}^{(4)} \quad (0<x<1;\ i=1, 2, \cdots, n), \tag{11}$$

where $y_{-1}\equiv 0$. In order to obtain $y_i(x)$ from (11), we need the suitable boundary conditions $y_i(0)$ and $y_i(1)$. They will be given below. Substituting $y_i(x)$ into (9), we obtain the n-th approximation of the outer solution of the problem (1)—(3). Clearly, its second order derivative with respect to x may not satisfy boundary conditions (3) approximately. Then we shall construct the functions possessing boundary layer behavior near $x=0$ and $x=1$ respectively.

First, we construct the function $\xi(\tau, \varepsilon)$ possessing boundary layer behavior near $x=0$. Let

$$\overline{Y}=Y(x, \varepsilon)+\xi(\tau, \varepsilon), \quad \left(\tau=\frac{x}{\varepsilon}\right),$$

where τ is a stretched variable. We assume that $\xi(\tau, \varepsilon)$ has the following formal expansion:

$$\xi(\tau, \varepsilon)=\varepsilon^2(\xi_0(\tau)+\xi_1(\tau)\varepsilon+\xi_2(\tau)\varepsilon^2+\cdots). \tag{12}$$

Substituting $\overline{Y}$ into (1) we have

$$\varepsilon^2\frac{d^4\overline{Y}}{dx^4}=f(x, \overline{Y}, \overline{Y}'', \varepsilon). \tag{13}$$

From (10), (9) and (12) we get

$$\begin{aligned}\frac{1}{\varepsilon^2}\frac{d^4\xi}{d\tau^4}&=f\Big(\varepsilon\tau, Y+\xi, Y''+\frac{1}{\varepsilon^2}\frac{d^2\xi}{d\tau^2}, \varepsilon\Big)-f(\varepsilon\tau, Y, Y'', \varepsilon)\\ &\equiv\overline{F}(\varepsilon)=\sum_{i=0}^{n}\overline{F}_i\varepsilon^i+\bar{r},\end{aligned} \tag{14}$$

where

$$\begin{aligned}\overline{F}_0\equiv\overline{F}(0)&=f\Big(0, y_0, y_0''+\frac{d^2\xi_0}{d\tau^2}, 0\Big)-f(0, y_0, y_0'', 0)\\ &=f_{y''}\Big(0, y_0, y_0''+\theta_1\frac{d^2\xi_0}{d\tau^2}, 0\Big)\frac{d^2\xi_0}{d\tau^2}, \quad 0<\theta_1<1,\end{aligned}$$

$$\overline{F}_i\equiv\frac{1}{i!}\overline{F}^{(i)}(0)=f_{y''}\Big(0, y_0, y_0''+\frac{d^2\xi_0}{d\tau^2}, 0\Big)\frac{d^2\xi_i}{d\tau^2}+\bar{c}_i(\tau), \quad (i=1, 2,\cdots, n),$$

while $\bar{c}_i(\tau)$ are defined successively by polynomial of τ and y_j $(j=0, 1, \cdots, i)$ and ξ_j $(j=0, 1, \cdots, i-1)$. Equating the coefficients of like powers of ε will yield

$$\frac{d^4\xi_0}{d\tau^4}=f_{y''}\Big(0, y_0, y_0''+\theta_1\frac{d^2\xi_0}{d\tau^2}, 0\Big)\frac{d^2\xi_0}{d\tau^2}, \tag{15}$$

$$\frac{d^4\xi_i}{d\tau^4}=f_{y''}\Big(0, y_0, y_0''+\frac{d^2\xi_0}{d\tau^2}, 0\Big)\frac{d^2\xi_i}{d\tau^2}+\bar{c}_i(\tau), \quad (i=1, 2\cdots, n). \tag{16}$$

We shall give the suitable initial conditions of ξ_i below so as to solve for $\xi_i(\tau)$ $(i=0, 1, \cdots, n)$ from (15) and (16) successively.

Second, we construct the function $\eta(\sigma, \varepsilon)$ possessing boundary layer behavior near $x=1$. Let

$$\widetilde{Y}=\overline{Y}+\eta=Y(x, \varepsilon)+\xi(\tau, \xi)+\eta(\sigma, \varepsilon) \quad \Big(\sigma=\frac{1-x}{\varepsilon}\Big), \tag{17}$$

where σ is a stretched variable too. We assume that $\eta(\sigma, \varepsilon)$ has the following formal expansion:

$$\eta(\sigma, \varepsilon)=\varepsilon^2(\eta_0(\sigma)+\eta_1(\sigma)\varepsilon+\eta_2(\sigma)\varepsilon^2+\cdots). \tag{18}$$

Substituting $\widetilde{Y}$ into equation (1) we have

$$\varepsilon^2\frac{d^4\widetilde{Y}}{dx^4}=f(x, \widetilde{Y}, \widetilde{Y}'', \varepsilon).$$

From (13), (9), (12) and (18), we get

$$\begin{aligned}\frac{1}{\varepsilon^2}\frac{d^4\eta}{d\sigma^4}&=f\Big(1-\varepsilon\sigma, Y+\xi+\eta, Y''+\frac{1}{\varepsilon^2}\Big(\frac{d^2\xi}{d\tau^2}+\frac{d^2\eta}{d\sigma^2}\Big), \varepsilon\Big)\\ &\quad-f\Big(1-\varepsilon\sigma, Y+\xi, Y''+\frac{1}{\varepsilon^2}\frac{d^2\xi}{d\tau^2}, \varepsilon\Big)\\ &\equiv\widetilde{F}(\varepsilon)=\sum_{i=0}^{n}\widetilde{F}_i\varepsilon^i+\tilde{r},\end{aligned} \tag{19}$$

where

$$\tilde{F}_0\equiv\tilde{F}(0)=f\left(1,\ y_0,\ y_0''+\frac{d^2\xi_0}{d\tau^2}+\frac{d^2\eta_0}{d\sigma^2},\ 0\right)-f\left(1,\ y_0,\ y_0''+\frac{d^2\xi_0}{d\tau^2},\ 0\right)$$

$$=f_{y''}\left(1,\ y_0,\ y_0''+\frac{d^2\xi_0}{d\tau^2}+\theta_2\frac{d^2\eta_0}{d\sigma^2},\ 0\right)\frac{d^2\eta}{d\sigma^2},\quad 0<\theta_2<1,$$

$$\tilde{F}_i\equiv\frac{1}{i!}\tilde{F}^{(i)}(0)=f_{y''}\left(1,\ y_0,\ y_0''+\frac{d^2\xi_0}{d\tau^2}+\frac{d^2\eta_0}{d\sigma^2},\ 0\right)\frac{d^2\eta_i}{d\sigma^2}+\tilde{c}_i(\sigma),\quad (i=1,\ 2,\ \cdots,\ n),$$

while $\tilde{c}_i(\sigma)$ are defined successively by polynomial of σ, y_j, $\xi_j(j=0,\ 1,\ \cdots,\ i)$ and $\eta_j(j=0,\ 1,\ \cdots,\ i-1)$. Here we have assumed that ξ_j are functions possessing boundary layer behavior. Equating the coefficiients of like powers of ε, we have

$$\frac{d^4\eta_0}{d\sigma^4}=f_{y''}\left(1,\ y_0,\ y_0''+\frac{d^2\xi_0}{d\tau^2}+\theta_2\frac{d^2\eta_0}{d\sigma^2},\ 0\right)\frac{d^2\eta_0}{d\sigma^2},\tag{20}$$

$$\frac{d^4\eta_i}{d\sigma^4}=f_{y''}\left(1,\ y_0,\ y_0''+\frac{d^2\xi_0}{d\tau^2}+\frac{d^2\eta_0}{d\sigma^2},\ 0\right)\frac{d^2\eta_i}{d\sigma^2}+\tilde{c}_i(\sigma),\quad (i=1,\ 2,\ \cdots,\ n).\tag{21}$$

We shall also give the suitable initial conditions of η_i below so as to solve for $\eta_i(\sigma)$ $(i=0,\ 1,\ \cdots,\ n)$ from (20) and (21) successively.

Now we give the boundary conditions of y_i and the initial conditions of ξ_i and η_i as below.

We first define the initial conditions of $\xi_0(\tau)$ and $\eta_0(\sigma)$:

$$\frac{d^2\xi_0(0)}{d\tau^2}=A_2(0)-y_0''(0),\tag{15'}$$

$$\frac{d^2\eta_0(0)}{d\sigma^2}=B_2(0)-y_0''(1).\tag{20'}$$

In (15′) and (20′), we have deleted the small quantities of higher orders. We note that ξ_0 and η_0 satisfy the equations (15) and (20) respectively. It is not difficult to see that there exists a pair of the functions $\xi_0(\tau)$, $\eta_0(\sigma)\in C^{(4)}$ possessing boundary layer behavior[7]:

$$\frac{d^j\xi_0(\tau)}{d\tau^j}=O(e^{-\sqrt{m}(1-k_0)\tau})\quad (\tau\gg1,\ j=0,\ 1,\ 2)$$

$$\frac{d^j\eta_0(\sigma)}{d\sigma^j}=O(e^{-\sqrt{m}(1-k_0)\sigma})\quad (\sigma\gg1,\ j=0,\ 1,\ 2),$$

where k_0 is an arbitrary small positive constant.

Next, we define the conditions of $y_i(x)$, $\xi_i(\tau)$ and $\eta_i(\sigma)$ $(i=1,\ 2,\ \cdots,\ n)$:

$$y_i(0)=A_{1i}-\xi_{i-2}(0),\quad y_i(1)=B_{1i}-\eta_{i-2}(0),\tag{11'}$$

$$\frac{d^2\xi_i(0)}{d\tau^2}=A_{2i}-y_i''(0),\tag{16'}$$

$$\frac{d^2\eta_i(0)}{d\sigma^2}=B_{2i}-y_i''(1),\tag{21'}$$

in which the letters with negative subscript are taken to be zero. In (11′), (16′) and (21′), we have deleted the small quantities of higher orders. From the equations (11), (16). and (21) the $y_i(x)$, $\xi_i(\tau)$ and $\eta_i(\sigma)$ can be obtbined successively, where

ξ_i and η_i satisfy also the conditions

$$\frac{d^j\xi_i(\tau)}{d\tau^j}=O(e^{-\sqrt{m}(1-k_i)\tau}),$$

$$\frac{d^j\eta_i(\sigma)}{d\sigma^j}=O(e^{-\sqrt{m}(1-k_i)\sigma}),$$

$$(\tau,\ \sigma\gg 1,\quad i=1,\ 2,\ \cdots,\ n,\quad j=0,\ 1,\ 2,),$$

where k_i $(i=1, 2, \cdots, n)$ are arbitrary small positive constants.

Substituting the above defined y_i, ξ_i and $\eta_i (i=0, 1, \cdots, n)$ into (9), (12), (18) and (17) successively and replacing the stretched variables by x, we obtain the sum $\widetilde{Y}_n$ of the first n terms of formal asymptotic expansion which is the solution $y(x, \varepsilon)$ of the problem (1)—(3):

$$\widetilde{Y}_n=\sum_{i=0}^{n}\Big[y_i(x)+\varepsilon^2\Big(\xi_i\Big(\frac{x}{\varepsilon}\Big)+\eta_i\Big(\frac{1-x}{\varepsilon}\Big)\Big)\Big]\varepsilon^i. \tag{22}$$

§ 2. Estimate of the Remainders

Now we prove that under the suitable conditions original problem (1)—(3) has a solution $y(x, \varepsilon)\in C^{(4)}$ which can be represented by the following uniformly valid expansions:

$$y(x,\ \varepsilon)=\sum_{i=0}^{n}\Big[y_i(x)+\varepsilon^2\Big(\xi_i\Big(\frac{x}{\varepsilon}\Big)+\eta_i\Big(\frac{1-x}{\varepsilon}\Big)\Big)\Big]\varepsilon^i+R_1,\quad 0\leqslant x\leqslant 1, \tag{23}$$

and

$$y''(x,\ \varepsilon)=\sum_{i=0}^{n}\Big[y_i''(x)+\frac{d^2}{d\tau^2}\,\xi_i\Big(\frac{x}{\varepsilon}\Big)+\frac{d^2}{d\sigma^2}\,\eta_i\Big(\frac{1-x}{\varepsilon}\Big)\Big]\varepsilon^i+R_2,\quad 0\leqslant x\leqslant 1, \tag{24}$$

where R_1 and R_2 are remainders satisfying

$$R_i=O(\varepsilon^{n+1})\quad (0\leqslant x\leqslant 1,\ 0<\varepsilon\ll 1;\ i=1,\ 2).$$

We first state a lemma as follows.

Lemma. *We consider the boundary value problem for system of nonlinear differential equations*

$$\begin{cases} y_1''=f_1(x,\ y_1,\ y_2), \\ y_2''=f_2(x,\ y_1,\ y_2), \end{cases}\quad (0<x<1)$$

$$y_1(0)=A_1,\quad y_1(1)=B_1,$$

$$y_2(0)=A_2,\quad y_2(1)=B_2.$$

If there are functions $\alpha_i(x)$, $\beta_i(x)\in C^{(2)}[0, 1]$ $(i=1, 2)$, which satisfy the following conditions

$$\begin{cases} \alpha_i(0)\leqslant A_i\leqslant\beta_i(0), \\ \alpha_i(1)\leqslant B_i\leqslant\beta_i(1), \end{cases}\quad i=1,\ 2,$$

$$\begin{cases} \alpha_1''(x)\geqslant f_1(x,\ \alpha_1(x),\ y_2), \\ \beta_1''(x)\leqslant f_1(x,\ \beta_1(x),\ y_2), \end{cases}\quad 0<x<1,\quad \alpha_2(x)\leqslant y_2\leqslant\beta_2(x),$$

$$\begin{cases} \alpha_2''(x) \geqslant f_2(x, y_1, \alpha_2(x)), \\ \beta_2''(x) \leqslant f_2(x, y_1, \beta_2(x)), \end{cases} \quad 0<x<1, \quad \alpha_1(x) \leqslant y_1 \leqslant \beta_1(x),$$

and $f_i(x, y_1, y_2) \in C^{(1)}(D)$, *where*

$$D: \{0 \leqslant x \leqslant 1, \alpha_i(x) \leqslant y_i \leqslant \beta_i(x), i=1, 2\},$$

then the original problem can besolxed by a pair of functions $y_1(x)$, $y_2(x) \in C^{(2)}[0,1]$ *which satisfy the following conditions:*

$$\alpha_i(x) \leqslant y_i(x) \leqslant \beta_i(x), \quad 0 \leqslant x \leqslant 1, \ i=1, 2.$$

For the proof of this lemma please refer to [8—11].

Using the above lemma, we can prove the following theorem.

Theorem. *Assume:*

[I] *reduced problem* (4)—(5) *has a solution* $y_0(x) \in C^{2n+4}[0, 1]$;

[II] $f(x, y, y'', \varepsilon) \in C^{(n+1)}(D_\varepsilon)$, *where*

D_ε: $\{0 \leqslant x \leqslant 1, |y - \tilde{Y}_n| \leqslant d_1, |y'' - \tilde{Y}_n''| \leqslant d_2, 0 \leqslant \varepsilon \leqslant \varepsilon_1\}$ $(\varepsilon_1 > 0, d_i > 0, i=1, 2)$;

[III] $f_{y''} \geqslant m$, $(x, y_0, y'', \varepsilon) \in D_\varepsilon$, *where* m *is a certain positive constant;*

[IV] $A_i(\varepsilon)$, $B_i(\varepsilon) \in C^{(n+1)}[0, \varepsilon_1]$.

Then the boundary value problem (1)—(3) *has a solution* $y(x, \varepsilon) \in C^{(4)}$ $(0 \leqslant x \leqslant 1, 0 < \varepsilon \leqslant \varepsilon_0)$ *where* ε_0 *is a certain positive constant which can be represented by the uniformly valid asymptotic expansion* (23), *and whose second order derivative by* (24), *in* $0 \leqslant x \leqslant 1$.

Proof Let $y'' = z$. Thus the original problem (1)—(3) becomes the following boundary value problem:

$$y'' = z, \tag{25}$$

$$\varepsilon^2 z'' = f(x, y, z, \varepsilon), \tag{26}$$

$$y(0, \varepsilon) = A_1(\varepsilon), \quad y(1, \varepsilon) = B_1(\varepsilon), \tag{27}$$

$$z(0, \varepsilon) = A_2(\varepsilon), \quad z(1, \varepsilon) = B_2(\varepsilon). \tag{28}$$

Now we construct the $\alpha_i(x, \varepsilon)$ and $\beta_i(x, \varepsilon)$ as follows:

$$\alpha_1(x, \varepsilon) = \tilde{Y}_n(x, \varepsilon) - \varepsilon^{n+1}\gamma\left(\cos\sqrt{\frac{l}{m}}\,x + \cos\sqrt{\frac{l}{m}}(1-x) - 1\right), \tag{29}$$

$$\beta_1(x, \varepsilon) = \tilde{Y}_n(x, \varepsilon) + \varepsilon^{n+1}\gamma\left(\cos\sqrt{\frac{l}{m}}\,x + \cos\sqrt{\frac{l}{m}}(1-x) - 1\right), \tag{30}$$

$$\alpha_2(x, \varepsilon) = \tilde{Y}_n''(x, \varepsilon) - \frac{\varepsilon^{n+1}\gamma l}{m}\left(\cos\sqrt{\frac{l}{m}}\,x + \cos\sqrt{\frac{l}{m}}(1-x)\right), \tag{31}$$

$$\beta_2(x, \varepsilon) = \tilde{Y}_n''(x, \varepsilon) + \frac{\varepsilon^{n+1}\gamma l}{m}\left(\cos\sqrt{\frac{l}{m}}\,x + \cos\sqrt{\frac{l}{m}}(1-x)\right), \tag{32}$$

where $|f_y| \leqslant l$, $(x, y, y'', \varepsilon) \in D_\varepsilon$ and there is no harm in choosing the suitable l such that

$$2K\pi \leqslant \sqrt{\frac{l}{m}} \leqslant \left(2K + \frac{1}{3}\right)\pi \quad (K \text{ is a certain integer}),$$

and γ is a large enough positive constant, which will be chosen later.

From (29)—(30), (11′), (15′), (16′), (20′) and (21′) it is easy to see that

$$\alpha_i(x,\ \varepsilon)\in C^{(2)}(0\leqslant x\leqslant 1)\quad (i=1,\ 2),\tag{33}$$

$$\beta_i(x,\ \varepsilon)\in C^{(2)}(0\leqslant x\leqslant 1)\quad (i=1,\ 2),\tag{34}$$

and from the construction of α_i, β_i and $\widetilde{Y}_n(x, \varepsilon)$ we can get inequalities:

$$\alpha_i(0,\ \varepsilon)\leqslant A_i(\varepsilon)\leqslant \beta_i(0,\ \varepsilon)\quad (i=1,\ 2),\tag{35}$$

$$\alpha_i(1,\ \varepsilon)\leqslant B_i(\varepsilon)\leqslant \beta_i(1,\ \varepsilon)\quad (i=1,\ 2),\tag{36}$$

$$\alpha_1''\geqslant z,\quad (\alpha_2(x)\leqslant z\leqslant \beta_2(x)),\tag{37}$$

$$\beta_1''\leqslant z,\quad (\alpha_2(x)\leqslant z\leqslant \beta_2(x)),\tag{38}$$

for small enough positive constant ε' and large enough $\gamma_0>0$ as $0<\varepsilon\leqslant\varepsilon'$, $\gamma\geqslant\gamma_0$.

Since $y_i(x)$, $\xi_i(\tau)$ and $\eta_i(\sigma)$ satisfy (4), (11), (15), (16), (20) and (21), taking into account that $y_{n-1}^{(4)}(x)$ and $y_n^{(4)}(x)$ are bounded, we can easily obtain

$$\begin{aligned} f(x,\ \widetilde{Y}_n,\ \widetilde{Y}_n'',\ \varepsilon)=&\Big[f(x,\ y_0,\ y_0'',\ 0)+\sum_{i=1}^{n}(f_{y''}(x,\ y_0,\ y_0'',\ 0)y_i''\\ &+f_y(x,\ y_0,\ y_0'',\ 0)y_i+c_i(x)-y_{i-2}^{(4)}+y_{i-2}^{(4)})\varepsilon^i\Big]\\ &+\Big[\Big(f_{y''}\Big(0,\ y_0,\ y_0''+\theta_1\frac{d^2\xi_0}{d\tau^2},\ 0\Big)\frac{d^2\xi_0}{d\tau^2}-\frac{d^4\xi_0}{d\tau^4}+\frac{d^4\xi_0}{d\tau^4}\Big)\\ &+\sum_{i=1}^{n}\Big(f_{y''}\Big(0,\ y_0,\ y_0''+\frac{d^2\xi_0}{d\tau^2},\ 0\Big)\frac{d^2\xi_i}{d\tau^2}+\bar{c}_i(\tau)-\frac{d^4\xi_i}{d\tau^4}+\frac{d^4\xi_i}{d\tau^4}\Big)\varepsilon^i\Big]\\ &+\Big[\Big(f_{y''}\Big(1,\ y_0,\ y_0''+\frac{d^2\xi_0}{d\tau^2}+\theta_2\frac{d^2\eta_0}{d\sigma^2},\ 0\Big)\frac{d^2\eta_0}{d\sigma^2}-\frac{d^4\eta_0}{d\sigma^4}+\frac{d^4\eta_0}{d\sigma^4}\Big)\\ &+\sum_{i=1}^{n}\Big(f_{y''}\Big(1,\ y_0,\ y_0''+\frac{d^2\xi_0}{d\tau^2}+\frac{d^2\eta_0}{d\xi^2}+\frac{d^2\eta_0}{d\sigma^2},\ 0\Big)\frac{d^2\eta_i}{d\sigma^2}\\ &+\tilde{c}_i(\sigma)-\frac{d^4\eta_i}{d\sigma^4}+\frac{d^4\eta_i}{d\sigma^4}\Big)\varepsilon^i\Big]+O(\varepsilon^{n+1})\\ =&\,\varepsilon^2\widetilde{Y}_n^{(4)}(x,\ \varepsilon)+O(\varepsilon^{n+1}),\quad 0<\varepsilon\ll 1. \end{aligned}$$

Tnerefore, there exist $\delta>0$, $\varepsilon''>0$ such that the inequality

$$|f(x,\ \widetilde{Y}_n,\ \widetilde{Y}_n'',\ \varepsilon)-\varepsilon^2\widetilde{Y}_n^{(4)}(x,\ \varepsilon)|\leqslant\delta\varepsilon^{n+1}\tag{39}$$

holds for $0<\varepsilon\leqslant\varepsilon''$.

From mean value theorem we have

$$\begin{aligned} f(x,\ y,\ \alpha_2,\ \varepsilon)=&f(x,\ \widetilde{Y}_n,\ \widetilde{Y}_n'',\ \varepsilon)+f_y(x,\ \widetilde{Y}_n+\theta_3(y-\widetilde{Y}_n),\ \alpha_2,\ \varepsilon)(y-\widetilde{Y}_n)\\ &+f_{y''}(x,\ y,\ \widetilde{Y}_n''+\theta_4(\alpha_2-\widetilde{Y}_n''),\ \varepsilon)(\alpha_2-\widetilde{Y}_n''),\quad 0<\theta_3,\ \theta_4<1, \end{aligned}\tag{40}$$

$$\begin{aligned} f(x,\ y,\ \beta_2,\ \varepsilon)=&f(x,\ \widetilde{Y}_n,\ \widetilde{Y}_n'',\ \varepsilon)+f_y(x,\ \widetilde{Y}_n+\bar{\theta}_3(x-\widetilde{Y}_n),\ \beta_2,\ \varepsilon)(y-\widetilde{Y}_n)\\ &+f_{y''}(x,\ y,\ \widetilde{Y}_n''+\bar{\theta}_4(\beta_2-\widetilde{Y}_n''),\ \varepsilon)(\beta_2-\widetilde{Y}_n''),\quad 0<\bar{\theta}_3,\ \bar{\theta}_4<1. \end{aligned}\tag{41}$$

When $\alpha_1(x)\leqslant y\leqslant\beta_1(x)$, we can get

$$|f_y(y-\widetilde{Y}_n)|\leqslant\varepsilon^{n+1}\gamma 1\Big(\cos\sqrt{\frac{l}{m}}\,x+\cos\sqrt{\frac{l}{m}}(1-x)-1\Big)$$

from (29), (30) and $|f_y|\leqslant 1$. And we have

$$f(x,\ y,\ \alpha_2,\ \varepsilon)\leqslant\varepsilon^2\widetilde{Y}_n^{(4)}+\delta\varepsilon^{n+1}-\gamma l\varepsilon^{n+1},$$

$$f(x,\ y,\ \beta_2,\ \varepsilon)\geqslant\varepsilon^2\widetilde{Y}_n^{(4)}-\delta\varepsilon^{n+1}+\gamma l\varepsilon^{n+1},$$

from (39)—(41). Thus we obtain

$$\varepsilon^2\alpha_2''-f(x,\ y,\ \alpha_2,\ \varepsilon)\geqslant(\gamma l-\delta)\varepsilon^{n+1},$$

$$\varepsilon^2\beta_2''-f(x,\ y,\ \beta_2,\ \varepsilon)\leqslant(\delta-\gamma l)\varepsilon^{n+1}.$$

Taking $\varepsilon_0=\min(\varepsilon_1,\ \varepsilon',\ \varepsilon'')$, $\gamma=\max(\gamma_0,\ \delta/l)$ we have

$$\varepsilon^2\alpha_2''\geqslant f(x,\ y,\ \alpha_2,\ \varepsilon), \tag{42}$$

$$\varepsilon^2\beta_2''\leqslant f(x,\ y,\ \beta_2,\ \varepsilon), \tag{43}$$

$$(\alpha_1(x)\leqslant y\leqslant\beta_1(x),\ 0<x<1,\ 0<\varepsilon\leqslant\varepsilon_0).$$

From the relations (33)—(38), (42) and (43), using the lemma, we get a pair of functions $y(x,\ \varepsilon)$, $z(x,\ \varepsilon)\in C^{(2)}$ $(0\leqslant x\leqslant 1)$ for the boundary value problem (25)—(28), which satisfy inequalities

$$\alpha_1(x,\ \varepsilon)\leqslant y(x,\ \varepsilon)\leqslant\beta_1(x,\ \varepsilon),\quad 0\leqslant x\leqslant 1,\ 0<\varepsilon\leqslant\varepsilon_0,$$

$$\alpha_2(x,\ \varepsilon)\leqslant z(x,\ \varepsilon)\leqslant\beta_2(x,\ \varepsilon),\quad 0\leqslant x\leqslant 1,\ 0<\varepsilon\leqslant\varepsilon_0.$$

From (29)—(30) we obtain

$$y(x,\ \varepsilon)=\sum_{i=0}^{n}\left[y_i(x)+\varepsilon^2\left(\xi_i\left(\frac{x}{\varepsilon}\right)+\eta_i\left(\frac{1-x}{\varepsilon}\right)\right)\right]\varepsilon^i+O(\varepsilon^{n+1}),$$

$$z(x,\ \varepsilon)=\sum_{i=0}^{n}\left[y_i''(x)+\frac{d^2\xi_i\left(\dfrac{x}{\varepsilon}\right)}{d\tau^2}+\frac{d^2\eta_i\left(\dfrac{1-x}{\varepsilon}\right)}{d\sigma^2}\right]\varepsilon^i+O(\varepsilon^{n+1}),$$

$$(0\leqslant x\leqslant 1,\ 0<\varepsilon\ll 1).$$

Since $z(x,\ \varepsilon)=y''(x,\ \varepsilon)$, we then have $y\in C^{(4)}(0\leqslant x\leqslant 1,\ 0<\varepsilon\leqslant\varepsilon_0)$, and the relations (23) and (24) hold uniformly in $0\leqslant x\leqslant 1$.The proof of Theorem is now completed.

References

[1] Yarmish, J., Aspects of the Numerical and Theoretical Treatment of Singular Perturbations, Doctoral Dissertation, New York Univ., New York, 1972.

[2] Fife, P. C., Semilinear elliptic boundary value problems with small parameter, *Arch Ration, Mech. Anal.*, **52** (1973), 205—232.

[3] Dorr, F. W., Parter, S. V. and Shampine, L. F., Applications of the maximum principle to singular perturbation problems, *SIAM Rev.*, **15** (1973), 43—88.

[4] Howes, F. A., Differential inequalities and applications to nonlinear singular perturbation problems, *J. Differential Equations*, **20** (1976), 133—149.

[5] Howes. F, A., The asymptotic solution of a class of singularly perturbed nonlinear boundary value problems via differential inequalities, *SIAM J. Math. Anal.*, **9** (1978), 215—249.

[6] O'Malley, R. E. Jr., On a boundary value problem for a nonlinear differential equation with a small parameter, *SIAM J. Appl. Math.*, **17** (1969), 569—581.

[7] O'Malley, R. E. Jr., Introduction to Singular Perturbations, Academic Press, New York, 1974.

[8] Nagumo, M., Über die Differentialgleichung $y''=f(x,\ y,\ y')$, *Proc. Phys. Math. Soc. Japan*, **19** (1937), 861—866.

[9] Jackson, L. K., Subfunctions and second-order ordinary differential inequalities, *Advances in Math.*, **2** (1968), 307—363.

[10] Bernfeld, S. R. and Lashmikantham, V., An Introduction to Nonlinear Boundary Value Problems, Academic Press, New York, 1974.

[11] Schmitt, K., Periodic solution of systems of second order differential equations, *J. Differential Equations*, **11** (1972), 180—192.

[12] Mo Jiaqi (莫嘉琪), The estimation of solution of singularly perturbed two-point problem for a class of

nonlinear differential equations (in Chinese), *J. of Anhui Normal University (Natural Science)*, 1982, I, 1—6.

[13] Mo Jiaqi(莫嘉琪)and Li Xingzhi(李兴治), The estimation of solution of singularly perturbed two-point problem for a class of nonlinear differential equation (II) (in Chinese), *J. of Anhui Normal University (Natural Science)*, 1983 6—13.

[14] Mo Jiaqi (莫嘉琪), The estimation of solution of singularly perturbed boundary value problem for the nonlinear equation $\varepsilon y''=f(x, y, y', \varepsilon)$ (in Chinese), *Chin. Ann. of Math.*, **5A**: 1 (1984), 73—78.