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REFLECTION ON BOUNDARY OF SINGULARITIES

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Abstract

This paper discusses the reflection on degenerated surface of singularities for Tricomi operator. Two kind of boundary conditions: Dirichlet condition and general first order boundary conditions have been investigated.

§ 1. Introduction

[1, 2] studied the reflection of singularities on noncharacteristic boundary for operator of principal type and later, [3, 4, 5] considered the glancing case. The present paper is concerned with the same problem for Tricomi operators, which is of real non-principal type. Let us consider a differential operator of second order

$$Pu=D_t^2u-t\sum_{j=1}^{n-1}D_{x_j}^2u+\sum_{j=1}^{n-1}a_jD_{x_j}u+au \tag{1.1}$$

defined in $R_+^n=R^{n-1}\times R_+^1$. Through any point $(x_0, 0, \xi_0, 0)\in\partial T^*(\bar{R}_+^n)$ there is a Hamilton flow, denoted by $\gamma(x_0, \xi_0)$ and tangential to the boundary $t=0$. The problem we are interested in is how the singularity developes along $\gamma(x_0, \xi_0)$. $\mathscr{D}'(\bar{R}_+^n)$ is the dual space of $\{u\in C_c^\infty(R^n)\mid \operatorname{supp} u\subset\bar{R}_+^n\}$. In this paper, we shall often use the notations and notions given in [5]. Denote by $\tilde{T}(\bar{R}_+^n)$ the vector bundle on $\bar{R}_+^n$ with $\partial/\partial x_j$ $(j=1, \cdots, n-1)$ and $x_n\,\partial/\partial x_n$ as a base. Its coordinates may be expressed as $(x_1, \cdots, x_n, a_1, \cdots, a_n)=(x, a)$. The smooth map π: $\tilde{T}(\bar{R}_+^n)\ni(x, a)\mapsto(x_1, \cdots, x_n, a_1, \cdots, x_na_n)\in T(\bar{R}_+^n)$ induces a natural map π^* from $T^*(\bar{R}_+^n)$ into the dual vector bundle of $\tilde{T}(\bar{R}_+^n)$, $\tilde{T}^*(\bar{R}_+^n)$. The range of π^*, $R(\pi^*)$ is homeomorphic to $T^*(R_+^n)\cup T^*(\partial R_+^n)$. Therefore, later we will always identify $R(\pi^*)$ with $T^*(R_+^n)\cup T^*(\partial R_+^n)$.

If $u\in\mathscr{D}'(\bar{R}_+^n)$ and $Pu\in C^\infty(\bar{R}_+^n)$, it follows that $WF_b(u)\subset R(\pi^*)$. By $z\bar{\in}WF_b(u)$ we mean that $z\bar{\in}WF(u)$ if $z\in T^*(R_+^n)$ and u is smooth up to the boundary $t=0$ at point z if $z=(x_0, \xi_0)\in T^*(\partial R_+^n)$, i.e., there is a pseudodifferential operators $A(x, D)$ elliptic at (x_0, ξ_0) such that Au is smooth near $(x_0, 0)$. Our main result is the following

Theorem. *Assume that $u\in\mathscr{D}'(\bar{R}_+^n)$ and $Pu\in C^\infty$ near $(x_0, 0)$. If either of the*

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following two conditions:

(1) $(x_0, \xi_0)\overline{\in} WF(u(x, 0))$,

(2) $B\in \mathrm{OPS}^1(R^{n-1})$ *is elliptic at* (x_0, ξ_0) *or* $B\in\mathrm{OPS}^0(\Gamma)$ *for a conical neighbourhood* Γ *of* (x_0, ξ_0) *such that* $(x_0, \xi_0)\overline{\in} WF(D_t u+Bu|_{t=0})$, *is fulfilled, then there exists an interval containing* (x_0, ξ_0) *of orbit* $\gamma(x, \xi_0)\subset\tilde{T}^*(\overline{R}^n_+)$, *denoted by* $\gamma(x_0, \xi_0)$ *still, such that the alternation of* $\gamma(x_0, \xi_0)\cap WF_b(u)=\phi$ *and* $\gamma(x_0, \xi_0)\subset WF_b(u)$ *is valid.*

After the present paper was accepted for publication, the author recently see Hormander's new book (The Analysis of Linear Partial Differential Operators IV), in which the part (1) of the previous theorem was obtained for more general differential operators of 2nd order.

§ 2. A Special Differential Operator

This section studies the special case of (1.1) of the form

$$\mathring{P}u=D_t^2u-t\sum_{j=1}^{n-1}D_{x_j}^2u,\quad t>0, \tag{2.1}$$

which is a degenerated hyperbolic operator. Application of the energy estimates introduced in [7, 8] and the technique of mollifier gives at once

Lemma 2.1 *Assume that* $u\in\mathscr{D}'(\overline{R}^n_+)$ *satisfies* $\mathring{P}u=0$ *in a neighbourhood in* $\overline{R}^n_+$ *of* $(x_0, 0)$, $O(x_0)\times(0, \delta)$, *where* δ *is a positive constant and that* $u(x, 0)=0$, $\partial u/\partial t$ $(x, 0)=0$ *on* $O(x_0)$. *Then* $u=0$ *in another smaller neighbourhood* $O'(x_0)\times(0, \delta')$.

We also have a regularity theorem in microlocal version of Cauchy problem for (2.1).

Lemma 2. 2. *If* $\mathring{P}u\in C^\infty(x_0, 0)$, *where* $u\in\mathscr{D}'(\overline{R}^n_+)$ *and* $(x_0, \xi_0)\overline{\in} WF(D_t^i u|_{t=0})$ $(i=0, 1)$, *it follows that for some* $\delta>0$,

$$WF_b(u)\cap\gamma(x_0, \xi_0)\cap\{0\leqslant t\leqslant\delta\}=0. \tag{2.2}$$

Proof By the uniqueness theorem of Cauchy problem for (2.1) it suffices to prove the case that

$$f \text{ and } u \text{ are compactly supported, respectively, in } \overline{R}^n_+ \text{ and } R^{n-1} \tag{2.3}$$

and

$$f \text{ is smooth and } A(D_x)D_t^i u(x, 0)\in H_\infty(R^{n-1})\quad (i=1.2) \tag{2.4}$$

for some pseudodifferential operator $A\in\mathrm{OPS}^0(R^{n-1})$, elliptic at (x_0, ξ_0). Let $\hat{u}(\xi, t)$ denote the Fourier transformation of u with respect to variables $(x_1,\cdots, x_{n-1})$. Then we have

$$\hat{u}_{tt}(\xi, t)+t|\xi|^2\hat{u}(\xi, t)=-\hat{\mathring{P}}u(\xi, t)=f(\xi, t), \tag{2.1'}$$

$$\hat{u}(\xi, 0)=\hat{u}_0(\xi),\quad \hat{u}_t(\xi, 0)=\hat{u}_1(\xi). \tag{2.5}$$

Split the solution into two parts: $u=V+W$, where

$$V(\xi,\ t)=t^{1/2}\Big[C_0|\xi|^{1/3}J_{-1/3}\Big(\frac{2}{3}|\xi|t^{3/2}\Big)\hat{u}_0(\xi)+C_1|\xi|^{-1/3}J_{1/3}\Big(\frac{2}{3}|\xi|t^{3/2}\Big)\hat{u}_1(\xi)\Big], \tag{2.6}$$

$$W(\xi,\ t)=t^{1/2}\int_0^t C_2 s^{1/2}\Big[J_{1/3}\Big(\frac{2}{3}|\xi|t^{3/2}\Big)J_{-1/3}\Big(\frac{2}{3}|\xi|s^{3/2}\Big)$$
$$-J_{1/3}\Big(\frac{2}{3}|\xi|s^{3/2}\Big)J_{-1/3}\Big(\frac{2}{3}|\xi|t^{3/2}\Big)\Big]f(\xi,\ s)ds. \tag{2.7}$$

Here C_0, C_1 and C_2 are all constants independent of t, ξ, s. In the sequal, without otherwise statement, all C_i have just the same meaning. Evidently, the first is the solution to $Pu=0$ with $u(x,\ 0)=u_0(x)$ and $u_t(x,\ 0)=u_1(x)$, and the second is that to $Pu=f$ with $u(x,\ 0)=u_t(x,\ 0)=0$. In view of the property of Bessel's functions: $|J_\lambda(z)|\leqslant C_4|z|^\lambda$ if Re $\lambda>-1/2$, and by means of (2.3), (2.4) it is not difficult to prove that $A(D_x)V$ and $W\in H_{(0,\infty)}(R^{n-1}\times(0,\ 1))$. The theorem on partial hypoellipticity provides that $A(D_x)V$ and $W\in H_\infty(R^{n-1}\times(0,\ 1))$, which implies (2.2) immediately. This proves the present lemma.

As is well known, the usual way to analysis the propagation of singularities of Cauchy data is to construct a parametrix for Cauchy problem by Fourier integral operator. But at the glancing case the phase function of this Fourier integral operator is not smooth up to the bourdary $t=0$. In order to overcome such a difficulty, we shall study some properties of the asymptotic expansion of Bessel's functions. Referring to [9, p. 526], we have

$$J_\lambda(z)=z^{-1/2}(e^{-iz}e(\lambda,\ z)+e^{iz}q(\lambda,\ z)). \tag{2.9}$$

Here when $z\neq0$, $e(\lambda,\ z)$, $q(\lambda,\ z)$ have the asymptotic expansions, respectively,

$$e(\lambda,\ z)\sim e_0(\lambda)+\frac{1}{z}e_1(\lambda)+\cdots, \tag{2.8}$$

$$q(\lambda,\ z)\sim q_0(\lambda)+\frac{1}{z}q_1(\lambda)+,\cdots \tag{2.9}$$

with

$$e_0(\lambda)\neq0,\quad q_0(\lambda)\neq0 \text{ for all } \lambda\in\mathbb{C}^1. \tag{2.10}$$

Let us denote by $E_\pm(t)$ and $Q_\pm(t)$ the pseudodifferential operators with t as smooth parameter when $t>0$ and with $e(\pm1/3,\ (2|\xi|t^{3/2})/3)$ and $q(\pm1/3,\ (2|\xi|t^{3/2})/3)$ as their total symbols, respectively. Hence, we can rewrite (2.6) in the form

$$V(x,\ t)=t^{-1/4}\int e^{i\varphi_-}(|\xi|^{-1/6}\widehat{E_-(t)u_0}(\xi)+|\xi|^{-5/6}\widehat{E_+(t)u_1}(\xi))d\xi$$
$$+t^{-1/4}\int e^{i\varphi_+}(|\xi|^{-1/6}\widehat{Q_-(t)u_0}(\xi)+|\xi|^{-5/6}\widehat{Q_+(t)u_1}(\xi))d\xi$$
$$=t^{-1/4}[I_-(\varLambda^{-1/6}E_-u_0+\varLambda^{-5/6}E_+u_1)+I_+(\varLambda^{-1/6}Q_-u_0+\varLambda^{-5/6}Q_+u_1)], \tag{2.11}$$

where $\varphi_\pm=x\cdot\xi\pm(2|\xi|t^{3/2})/3$, $\varLambda^\alpha$ is the pseudodifferential operator with the total symbol $|\xi|^\alpha$ and $E_\pm$, $Q_\pm$ are elliptic pseudodifferential operators because of (2.10). Obviously, $I_\pm$ are the Fourier integral operators with phase function $(x-y)\cdot\xi\pm(2|\xi|t^{3/2})/3$ (sometimes, we denote them by $\varphi_\pm$ still). The homogeneous canonical relation associated with $I_\pm$

$$\Lambda_{\varphi\pm}(t)=\{(x, \eta, y, \xi)\,|\,x=y\mp(2\xi t^{3/2})/3|\xi|,\ \eta=\xi\}. \tag{2.12}$$

If we regard $I_\pm$ as the Fourier integral operators from $\mathscr{E}'(R^{n-1})$ into $\mathscr{D}'(R^{n-1}\times(0, 1))$, this relation may be written in

$$\tilde{\Lambda}_{\varphi\pm}=\{(x, t, \eta, \tau, y, \xi)\,|\,x=y\mp(2\xi t^{3/2})/3|\xi|,\ \eta=\xi,\ \tau=\pm|\xi|t^{1/2}\}. \tag{2.13}$$

Integration of the Hamilton equation for (2.1)

$$\begin{aligned}&\frac{dx}{ds}=-2t\xi, \quad \frac{d\xi}{ds}=0,\\ &\frac{dt}{ds}=2\tau, \quad \frac{d\tau}{ds}=|\xi|^2\end{aligned} \tag{2.14}$$

with the initial data $x=y$, $t=0$, $\xi=\xi_0$, $\tau=0$ gives its Hamilton flow

$$\gamma(y, \xi_0)=\{y-(2\xi_0|\xi_0|^2s^3)/3,\ |\xi_0|^2s^2,\ \xi_0,\ |\xi_0|^2s\},\ s\in R^1. \tag{2.15}$$

With $\gamma_\pm(y, \xi_0)=\{\gamma(y, \xi_0)\}\cap\{\tau\gtrless 0\}$ and by (2.13), (2.15) we can conclude that for $I_\pm(g)$, where $g\in\mathscr{E}'(R^{n-1})$, the singularity of g at (y, ξ_0) developes along $\gamma_\pm(y, \xi_0)$ only.

Lemma 2.3. *Assume that $\mathring{P}u=0$ where $u\in\mathscr{D}'(\bar{R}^n_+)$. Then*

(1) $(x_0,\xi_0)\bar{\in}WF(\Lambda^{-1/6}Q_-(t^*)u_0+\Lambda^{-5/6}Q_+(t^*)u_1)$ $(WF(\Lambda^{-1/6}E_-(t^*)u_0+\Lambda^{-5/6}E_+(t^*)u_1))$ *for some positive constant t^* if $\gamma_+(x_0, \xi_0)(\gamma_-(x_0, \xi_0))$ does not meet $WF(u)$,*

(2) (x_0, ξ_0) *does not meet $WF_b(u)$ if $\gamma_+(x_0, \xi_0)(\gamma_-(x_0, \xi_0))\cap WF(u)=\varnothing$ and $(x_0, \xi_0)\bar{\in}WF(u(x, 0))$ or $(x_0, \xi_0)\bar{\in}WF(RD_tu+u|_{t=0})$ for a pseudodifferential operator $R\in OPS^{-1}(R^{n-1})$.*

Proof Let us first prove the assertion (1). From the hypothesis: $WF(u)\cap\gamma_+(x_0, \xi_0)=\varnothing$ it fillows that there exists a constant $t^*>0$ (indeed, any $t^*>0$) such that

$$\begin{aligned}&(x_0-(2\xi_0(t^*)^{3/2})/3|\xi_0|,\ t^*,\ \xi_0,\ |\xi_0|\sqrt{t^*})\\ &\quad=(x^*, t^*, \xi^*, \tau^*)\in\gamma_+(x_0, \xi_0),\ (x^*, t^*, \xi^*, \tau^*)\bar{\in}WF(u)\end{aligned} \tag{2.16}$$

and

$$(x^*,t^*, \xi^*, \tau^*, y, \xi)\bar{\in}\tilde{\Lambda}_{\varphi-} \quad \text{for all } (y, \xi)\in T^*(R^{n-1})\backslash\{0\}. \tag{2.17}$$

From (2.11) we can get

$$I_+(\Lambda^{-1/6}Q_-(t)u_0+\Lambda^{-5/6}Q_+(t)u_1)=t^{1/4}u-I_-(\Lambda^{-1/6}E_-(t)u_0+\Lambda^{-5/6}E_+(t)u_1),$$

which implies

$$(x^*, t^*, \xi^*, \tau^*)\bar{\in}WF(I_+(\Lambda^{-1/6}Q_-(t)u_0+\Lambda^{-5/6}Q_+(t)u_1))$$

in view of (2.16), (2.17).

Let map i be the inclusion of $t=t^*$ into R^n. Using (2.16) and the well known fact that $WF(u|_{t=t_*})\subset i^*\,WF(u)$ one can derive

$$(x^*, \xi^*)\bar{\in}WF(I_+(\Lambda^{-1/6}Q_-(t)u_0+\Lambda^{-5/6}Q_+(t)u_1|_{t=t_*})).$$

From the ellipticity of I_+ and the homogeneous canonical relation about the propagation singularities of I_+: $WF(I_+(g))=\tilde{\Lambda}_{\varphi_+}\circ WF(g)$, it is easily seen that

$$(x_0, \xi_0)\bar{\in}WF(\Lambda^{-1/6}Q_-(t^*)u_0+\Lambda^{-5/6}Q_+(t^*)u_1), \tag{2.18}$$

which proves the assertion (1).

Now we turn to the verification of (2). Assume that $\gamma_+(x_0, \xi_0)$ does not meet $WF(u)$. Then the assertion (1) provides

$$(x_0, \xi_0)\overline{\in}WF(\Lambda^{-1/6}Q_-(t^*)u_0+\Lambda^{-5/6}Q_+(t^*)u_1).$$

By means of the ellipticity of $Q_+(t^*)$ and the hypothesis: $(x_0, \xi_0)\overline{\in}WF(u(x, 0))$, we have $(x_0, \xi_0)\overline{\in}WF(u_1)$. Now the assertion that $\gamma(x_0, \xi_0)$ does not meet $WF_b(u)$ is the immediate consequence of Lemma 2.2. Assume that $(x_0, \xi_0)\overline{\in}WF(Ru_1+u_0)$. Substituting $-Ru_1$ for u_0 in (2.18) we can get

$$(x_0, \xi_0)\overline{\in}WF([\Lambda^{-1/6}Q_-(t^*)R-\Lambda^{-5/6}Q_+(t^*)]u_1).$$

The principal symbol of $(\Lambda^{-1/6}Q_-(t^*)R-\Lambda^{-5/6}Q_+(t^*))$ equals $|\xi|^{-5/6}q_+(1/3)$ multiplied by a non-null constant. So this is also an elliptic pseudodifferential operator and $(x_0, \xi_0)\overline{\in}WF(u_1)$, which implies that $WF(u_0)=WF(u_0+Ru_1-Ru_1)\overline{\in}(x_0, \xi_0)$. The remainder of the proof is the same as the preceding one.

Remark. If $(x_0, \xi_0)\overline{\in}WF(D_tu+Ru)|_{t=0})$ with $R\in OPS^0(R^{n-1})$ instead of the hypothesis in (2) that $(x_0, \xi_0)\overline{\in}WF(u_0)$, then $\gamma(x_0, \xi_0)$ does not meet $WF_b(u)$ either.

The crucial step of proving the main theorem is

Lemma 2.4. *For any given $\xi_0\in R^{n-1}\backslash\{0\}$ and any given neighbourhood in $\bar{R}^n_+$ of $(x_0, 0)$, $O(x_0)\times[0, \delta)$, there exist pseudodifferential operators smoothly depending on t, $A(t)\in \mathrm{OPS}^{-1}(R^{n-1})$, and $B(t)$, $C(t)\in\mathrm{OPS}^0(R^{n-1})$, where $B(0)$ and $C(0)$ are elliptic at (x_0, ξ_0) such that*

$$(AD_t+B)P=\mathring{P}(AD_t+C)+R^2_{-\infty}D_t^2+R^1_{-\infty}D_t+R_{-\infty},\quad 0\leqslant t<\delta_1, \tag{2.19}$$

and

$$\mathrm{Supp}\,\sigma(A)(\sigma(B),\sigma(C),\sigma(R^i_{-\infty})\ (i=0, 1, 2))\subset O'(x_0)\times[0, \delta_1] \tag{2.20}$$

for a smaller neighbourhood $O'(x_0)\times[0, \delta_1)\subset O(x_0)\times[0, \delta)$. Here P is the operator of the form (1.1), $R^i_{-\infty}\in C^\infty(\bar{R}^1_+, \mathrm{OPS}^{-\infty}(R^{n-1}))$ and $\sigma(A)(\sigma(B), \sigma(C)$, etc.) is the total symbol of $A(B, C$, etc.).

The proof of Lemma 2.4 is rather long and is postponed until in section 3.

The proof of Theorem From the hypotheses of the present lemma, it follows that there exists a neighbourhood $\mathcal{N}=0(x_0)\times[0, \delta)$ such that $Pu\in C^\infty(\mathcal{N})$. The theorem on partial hypoellipticity gives $u\in H^{loc}_{(k,s-k)}(\mathcal{N})$ for some real s and all real k. With $AD_tu+Cu=W$, applying Lemma 2.4 to $\mathcal{N}$ we have

$$\mathring{P}W=(AD_t+B)Pu-\sum_{i=0}^{2}R^i_{-\infty}D_t^iu\in C^\infty(R^{n-1}\times[0, \delta_1)). \tag{2.21}$$

We need only to prove the case of $\mathring{P}W=0$ since the explicit formula (2.7) enables us to make (2.21) homogeneous without affecting the conditions and the conclusions of the present theorem.

Let us now consider (1). If $(x_0, \xi_0)\overline{\in}WF(u(x, 0))$, then application of the original equation $Pu=f$ shows that on the boundary $t=0$,

$$(C+D_tA)D_tu=D_tW-A\Big(f-\sum_{j=1}^{n-1}a_jD_{x_j}-a\Big)u-(D_tC)u=D_tW+g_1. \tag{2.22}$$

Here $g_1\in\mathscr{D}'(\partial R^n_+)$ and $(x_0,\ \xi_0)\overline{\in}WF(g_1)$ and later all $g_i(i=2.3,\ \cdots)$ have the same meaning. By (2.22) and the ellipticity of $C(0)$ we have

$$D_tu=(C+D_tA)^{-1}D_tW+g_2. \tag{2.23}$$

The equality that $A(C+D_tA)^{-1}D_tW-W=g_3$ follows from inserting (2.23) into the expression of W. Therefore Lemma 2.3(2) provides $\gamma(x_0,\ \xi_0)\cap WF_b(w)=\varnothing$. This means that D_tW is smooth up to boundary at $(x_0,\ \xi_0)$ and $(x_0,\ \xi_0)\overline{\in}WF_b(D_tW|_{t=0})$. In view of (2.23) we have $(x_0,\ \xi_0)\overline{\in}WF(D_tu|_{t=0})$. Now the assertion: $(x_0,\ \xi_0)\overline{\in}WF_b(u)$ is the consequence of Lemma 2.2.

The proof of the case (2) Let $b_1(x,\ \xi)$ be the principal symbol of pseudodifferential operator B. After substituting $g-Bu$ for D_tu in (2.22) we get

$$-\Big[(C+D_tA)B+A\sum_{j=1}^{n-1}a_jD_{x_j}+Aa-D_tC\Big]u=D_tW+g_4 \tag{2.24}$$

and

$$(-AB+C)u=W+g_5. \tag{2.25}$$

The principal symbol of the operator in the left hand side of (2.24) is

$$-b_1c_0, \tag{2.26}$$

where c_0 is that of operator $C(0)$. The ellipticity at $(x_0,\ \xi_0)$ of B guarantees $-b_1c_0\neq 0$. So there exists $R_{-1}\in\mathrm{OPS}^{-1}$ such that

$$u=R_{-1}D_tW+g_6.$$

The remainder of the proof is similar to that for the case (1).

If $B\in\mathrm{OPS}^0(\Gamma)$ for a conical neighbourhood Γ of $(x_0,\ \xi_0)$, then the equality

$$u=(C-AB)^{-1}W+g_7\quad\text{near } (x_0,\ \xi_0),$$

follows immediately from (2.25). After inserting it into (2.24), no doubt, we can see that there exists a pseudodifferential operator $R_0\in\mathrm{OPS}^0$, such that

$$D_tW+R_0W=g_8,\quad\text{near } (x_0,\ \xi_0).$$

The proof of Theorem will be completed if we note the remark of Lemma 2.3 and repeat the argument in proving the case (1).

§ 3. The Proof of Lemma 2.4

Let operators A, B and C have, respectively, the asymptotic expansions $\sigma(A)\sim a_{-1}+a_{-2}+\cdots$, $\sigma(B)\sim b_0+b_{-1}+b_{-2}+\cdots$ and $\sigma(C)\sim c_0+c_{-1}+c_{-2}+\cdots$. We write (1.1) in the slightly general form

$$P=D_t^2+R, \tag{1.1'}$$

where $R\in\mathrm{OPS}^2$ has the asymptotic expansion $\sigma(R)\sim r_2+r_1+\cdots$. With $\mathring{R}$ being in OPS^2 with r_2 as its total symbol, we evaluate

$$
\begin{aligned}
&(AD_t+B)(D_t^2+R)-(D_t^2+\mathring{R})(AD_t+C)\\
&\quad=(B-C-(2D_tA))D_t^2+(AR-\mathring{R}A-(D_t^2A+2D_tC))D_t\\
&\qquad+(BR-\mathring{R}C+A(D_tR)-D_t^2C).
\end{aligned}
\tag{3.1}
$$

Hence the transport equations are of the form

$$b_0=c_0, \tag{3.2}$$

$$2\partial a_{-l}/\partial t+\sqrt{-1}(c_{-l}-b_{-l})=0, \tag{3.3}$$

$$\partial c_{-l}/\partial t+\frac{1}{2}\{r_2,a_{-l-1}\}-\frac{\sqrt{-1}}{2}a_{-1-l}r_1=F_l^1, \tag{3.4}$$

$$-\sqrt{-1}(c_{-1-l}-b_{-1-l})r_2-\{r_2,\ c_{-l}\}+\sqrt{-1}b_{-l}r_1+a_{-1-l}\partial r_2/\partial t=F_l^2. \tag{3.5}$$

Here F_l^i $(i=1,\ 2)$ depend only on c_{-l+k}, b_{-l+k}, a_{-l+k-1} $(l\geqslant k\geqslant 1)$ and the operator P, moreover, $F_0^1=F_0^2=0$. Setting $r_2=-t|\xi|^2$ and $r_1=a_j\xi_j$ in (3.4), (3.8) and combining (3.3)—(3.5), one can get the systems of differential equations:

$$
\left\{
\begin{aligned}
&\partial c_{-l}/\partial t-t\sum_{j=1}^{n-1}\frac{\xi_j}{|\xi|}\partial(a_{-1-l}|\xi|)/\partial x_j-\frac{\sqrt{-1}}{2}(a_{-1-l}|\xi|)\frac{r_1}{|\xi|}=F_l^1,\\
&t\partial(a_{-1-l}|\xi|)/\partial t-t\sum_{j=1}^{n-1}\frac{\xi_j}{|\xi|}\ \partial c_{-l}/\partial x_j-\frac{(\sqrt{-1}c_{-l}r_1-a_{-1-l}|\xi|^2)}{2|\xi|}=F_l^2/|\xi|,
\end{aligned}
\right.
\tag{3.6$_l$}
$$

which are degenerated symmetric systems.

Let us consider the Cauchy problem of $(3.6)_l$ with initial data

$$c_{-l}(x,\ t,\ \xi)|_{t=0}=c_{-l}(x,\ \xi)\in C^\infty(R^{n-1}\times R^{n-1}\backslash\{0\}), \tag{3.7$_l$}$$

homogeneous in ξ of degree $-l$. The proof of Lemma 2.4 is based on the existence of C^∞ solutions of $(3.6)_l$, $(3.7)_l$.

Lemma 3.1. *For every l there exists a unique solution to $(3.6)_l$, $(3.7)_l$, $c_{-l},a_{-1-l}|\xi|\in C^\infty(R^{n-1}\times[0,\ 1]\times R^{n-1}\backslash\{0\})$ which is homogeneous in ξ of degree $-l$.*

Proof With $W_l^\pm=c_{-l}\pm\sqrt{t}\,a_{-1-l}|\xi|$ we may transform $(3.6)_l$, $(3.7)_l$ into

$$\pm\sqrt{t}\,\partial_tW_l^\pm-t\sum_{j=1}^{n-1}\frac{\xi_j}{|\xi|}\partial x_jW_l^\pm-\frac{\sqrt{-1}}{2}\frac{r_1}{|\xi|}W_l^\pm=F_l^2\pm\sqrt{t}\,F_l^1, \tag{3.6$'_l$}$$

$$W_l^\pm|_{t=0}=c_{-l}(x,\ \xi). \tag{3.7$'_l$}$$

The uniqueness of $(3.6)_l$, $(3.7)_l$ is equivalent to that of $(3.6')_l$ $(3.7')_l$. Furthermore, the solution to $(3.6)_l$, $(3.7)_l$ as well as $(3.6')_l$, $(3.7')_l$ propagates with finite speed and for any ξ every characteristic curve of $(3.6')_l$ is nothing else but that of (1.1).

We now proceed to study the existence of $(3.6')_l$, $(3.7')_l$. The coefficients in $(3.6')_l$ are all smooth in $(x,\ \xi)\in R^{n-1}\times R^{n-1}\backslash\{0\}$ and continuous in $t\in[0,\ 1]$. Let us consider the case of $l=0$. Integration of $(3.6')_l$, $(3.7')_l$ gives the solution $W_0^\pm$ smooth as a function of x, ξ and $t>0$ and continuous in $t\geqslant 0$. Evidently, $c_0=(W_0^++W_0^-)/2$, $a_{-1}=(W_0^+-W_0^-)/2\sqrt{t}\,|\xi|$ satisfy $(3.6)_0$ in $t>0$, and $(3.7)_l$. The rest task is to show that c_0, a_{-1} are smooth up to the boundary $t=0$. From the first equotion of $(3.6)_l$ it follows that $c_0\in C^1([0,\ 1],\ C^\infty(R^{n-1}\times R^{n-1}\backslash\{0\})$. By means of the second of $(3.6)_l$ we can obtain

$$a_{-1}=\frac{1}{|\xi|}\int_0^1 \lambda^{-1/2}\left(\frac{\sqrt{-1}r_1}{2|\xi|}c_0+t\sum_{j=1}^{n-1}\frac{\xi_j}{|\xi|}\partial_{x_j}c_0\right)\Big|_{t\to\lambda t}d\lambda, \tag{3.8}$$

which implies that $a_{-1}\in C^1([0, 1], C^\infty(R^{n-1}\times R^{n-1}\backslash\{0\}))$ too.

Obviously, repeated application of $(3.6)_0$, $(3.7)_0$ leads to the conclusion that the solutions c_0, a_{-1} we obtained are smooth up to the boundary $t=0$.

When $l>0$, we can prove the assertion of the present lemma by induction with respect to l. No difficulty in principle occurs. So the details of the proof need not be repeated. This completes the proof.

The end of the proof of Lemma 2.4 For given neighbourhood of $(x_0, 0)$, $\mathcal{N}=O(x_0)\times[0, \delta)$, choose a cutoff function $\varphi(x)\in C_0^\infty(0(x_0))$ with $\varphi(x)\equiv 1$ near x_0. Set

$$c_0=\varphi(x) \quad \text{and} \quad c_{-l}=0\ (l=1, 2, \cdots) \tag{3.7''}$$

as the initial data of $(3.6)_l$. Hence Lemma 3.1 shows that for every l the system of transport equations has C^∞ solution, which is homogeneous in ξ of degree $-l$. The property of propagation with finite speed ensures that one can find another neighbourhood of $(x_0, 0)$, $O'(x_0)\times[0, \delta_1)\subset O(x_0)\times[0, \delta)$ such that

$$\operatorname{Supp} c_{-l}(x, t, \xi)(a_{-1-l}(x, t, \xi))\subset O'(x_0)\times[0, \delta_1) \quad \text{if } 0\leqslant t\leqslant\delta_1.$$

Utilizing (3.2), (3.3) we can obtain b_{-l} $(l=0, \cdots)$. Now the total symbols of operators A, B and C expected are determined by the asymptotic sum of $\{a_{-j}\}$, $\{b_{-j}\}$ and $\{c_{-j}\}$. Furthermore, (3.2), (3.7″) guarantee the ellipticity of B and C. The proof is complete.

Remark 1. Let $\psi(\xi)$ be a homogeneous function of degree zero with support contained in a conical neighbourhood Γ. Take $c_0=\varphi(x)\psi(\xi)$ and $c_{-l}=0$ as initial data in $(3.7'')_l$. Then we have

$$\operatorname{Supp}\sigma(A)\ (\sigma(B), \sigma(C), \sigma(R^t_{-\infty}))\subset 0(x_0)\times[0, \delta_1)\times\Gamma.$$

Remark 2. The method used here is also applicable for more general degenerated hyperbolic operators, for example

$$P=D_t^2-t\sum_{j=1}^{n-1}a_{ij}(x, t)D_{x_i}D_{x_j}+R_1,$$

where (a_{ij}) is positive definite and $R_1\in \mathrm{OPS}^1$ (R^{n-1}) smoothly depends on t.

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