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ON METHODS OF SOLTUION FOR SOME KINDS OF SINGULAR INTEGRAL EQUTIONS WITH CONVOLUTION*

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Abstract

Methods of solution for some kinds of equations containing Cauchy principal value integral together with convolution are discussed. The general solutions and the conditions of solvability are obtained.

There were rather complete investigations on the method of solution for equations of Cauchy type as well as integral equations of convolution type[1,2]. The invertibility of Wiener-Hopf operators with discontinuous coefficients was considered in [3]. For operators containing both Cauchy principal value integral and convolution, the conditions of their Noethericity were discussed in [4, 5] in more general cases. For applications, the problem to find their solutions is very important. In this paper, we give effective methods of solution for certain basic kinds of such equations, including, besides the Cauchy principal value integral, equations with one or two convolution kernels, equations of Wiener-Hopf type and dual equations, in normal cases.

Some special kinds of Riemann boundary value problems with discontinuous coefficients appear in the course of solution, which are solved in the same time. It is necessary for us to introduce certain new classes of functions in advance and to point out some of their properties.

The Fourier transforms used in this paper are understood to be performed in $L_2(-\infty, +\infty)$ and the functions involved certainly belong to this space.

§ 1. Some Classes of Functions and Their Properties

In [2], the concepts of classes $\{0\}$ and $\{\{0\}\}$ were introduced as follows. A function $F(s)$ belongs to $\{\{0\}\}$, if the following two conditions are fulfilled:

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1) $F(s)\in\hat{H}$, that is, it satisfies the Hölder condition on the whole real axis, including ∞, i.e., $\pm\infty$ (notation used in [6]);

2) $F(s)\in L_2(-\infty, +\infty)$.

$f(t)\in\{0\}$ if its Fourier transform

$$F(s)=\mathbf{V}f=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(t)e^{ist}dt,\quad -\infty<s<+\infty, \tag{1.1}$$

belongs to $\{\{0\}\}$. On maintaining condition 1), we strengthen condition 2) slightly to

2′) $F(s)=O(1/|s|^{\mu})$, $\mu>\frac{1}{2}$, where $|s|$ is sufficiently large.

Then we call $F(s)\in((0))$ or $((0))^{\mu}$ and $f(t)\in(0)$ or $(0)^{\mu}$. From 2)′ it is assured that 2) is valid. If we strengthen 2)′ slightly again to

2)″ $F(s)\in H^{\mu}(N_\infty)$, $\mu>\frac{1}{2}$, i.e., it belongs to H in the neighborhood N_∞ of ∞, and $F(\infty)=0$,

then we call $F(s)\in$ «0» or «0»$^{\mu}$ and $f(t)\in\langle 0\rangle$ or $\langle 0\rangle^{\mu}$. From 2)″ it is assured that 2)′ is valid. Hence

$$«0»\subset((0))\subset\{\{0\}\},\quad \langle 0\rangle\subset(0)\subset\{0\}.$$

For two functions $k(t)$ and $f(t)$, if we use the notation of convolution

$$k*f=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k(t-\tau)f(\tau)d\tau, \tag{1.2}$$

then it is well known that

$$\mathbf{V}(k*f)=KF,$$

where K, F are the Fourier transforms of k, f respectively (we always use the capital letter represents the Fourier transform of the corresponding small letter). We know that $k, f\in\{0\}$ implies $k*f\in\{0\}$[2]. Obviously, when at least one of k and $f\in(0)$, then $k*f\in(0)$; when both k and $f\in\langle 0\rangle$, then $k*f\in\langle 0\rangle$.

We also introduce the operator $\mathbf{T}$ of Cauchy principal value integral

$$\mathbf{T}f=\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{f(\tau)}{\tau-t}d\tau,\quad -\infty<t<+\infty. \tag{1.3}$$

From [1, 7], $\mathbf{T}$ maps $\{\{0\}\}$ and «0» into themselves respectively and $\mathbf{T}^2=\mathbf{I}$ (identity).

We also introduce operators $\mathbf{N}$ and $\mathbf{S}$:

$$\mathbf{N}f(t)=f(-t),\quad \mathbf{S}f(t)=f(t)\operatorname{sgn}t. \tag{1.4}$$

Obviously $\mathbf{N}^2=\mathbf{S}^2=\mathbf{I}$ and $\mathbf{SN}=-\mathbf{NS}$.

For the inverse Fourier transform operator $\mathbf{V}^{-1}$:

$$\mathbf{V}^{-1}F=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}F(s)e^{-its}ds,\quad -\infty<t<+\infty, \tag{1.5}$$

it is evident that

$$\mathbf{V}^{-1}=\mathbf{NV}=\mathbf{VN},\quad \mathbf{V}^2=\mathbf{N}. \tag{1.6}$$

It was proved in [2], that when applying to functions in $\{0\}$,

$$\mathbf{VS}=\mathbf{TV}. \tag{1.7}$$

The following lemma plays an important role:

Lemma 1. *When applying to functions in* $\{0\}$,

$$\mathbf{VT}=-\mathbf{SV}, \tag{1.8}$$

i.e.,

$$\mathbf{V}\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{f(\tau)}{\tau-t}\,d\tau=-F(s)\,\mathrm{sgn}s. \tag{1.8$'$}$$

Proof From (1.7), $\mathbf{T}=\mathbf{VSV}^{-1}$ and so, by (1.6),

$$\mathbf{VT}=\mathbf{V}^2\mathbf{SV}^{-1}=\mathbf{NSNV}=-\mathbf{N}^2\mathbf{SV}=-\mathbf{SV}.$$

Note that, from $f\in\{0\}$, (0) or $\langle 0\rangle$, generally we could not assure that $\mathbf{T}f$ belongs to the same class. However, we have

Lemma 2. *If* $f\in\{0\}$, (0) or $\langle 0\rangle$ and $F(0)=0$, *then* $\mathbf{T}f$ *belongs to the same class.*

Proof By supposition, $\mathbf{V}f\in\{\{0\}\}$, $((0))$ or $\langle\!\langle 0\rangle\!\rangle$. From Lemma 1

$$\mathbf{VT}f=-F(s)\mathrm{sgn}s.$$

Noting that $F(\infty)=F(0)=0$, we know $\mathbf{VT}f\in\{\{0\}\}$, $((0))$ or $\langle\!\langle 0\rangle\!\rangle$. Therefore $\mathbf{T}f\in\{0\}$, (0) or $\langle 0\rangle$.

Besides, we note that, for the class (0) or $\langle 0\rangle$, the index μ is invariant, provided $\frac{1}{2}<\mu<1$.

Moreover, if $f(t)\in L_1(-\infty,+\infty)$, then $F(0)=0$ is actually

$$\int_{-\infty}^{+\infty}f(t)\,dt=0.$$

§ 2. Singular Integral Equations with One Convolution Kernel

Let us solve the following equation

$$a\varphi(t)+\frac{b}{\pi i}\int_{-\infty}^{+\infty}\frac{\varphi(\tau)}{\tau-t}\,d\tau+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k(t-\tau)\varphi(\tau)\,d\tau=g(t),$$
$$b\neq 0,\quad -\infty<t<+\infty, \tag{2.1}$$

where a and b are constants, k, $g\in\{0\}$ and the unknown function φ is required to be in $\{0\}$ too.

Taking Fourier transforms of both sides of (2.1), by Lemma 1, we get

$$[a-b\,\mathrm{sgn}\,s+K(s)]\Phi(s)=G(s),$$

which follows $G(0)=\Phi(0)=0$ since $G(s)$ is continuous at $s=0$.

Restricting ourselves to the normal type, i.e.,

$$K(s)\neq\begin{cases}-(a-b), & 0\leqslant s\leqslant+\infty,\\ -(a+b), & -\infty\leqslant s\leqslant 0,\end{cases} \tag{2.2}$$

we obtain

$$\Phi(s)=\frac{G(s)}{a-b\operatorname{sgn}s+K(s)}. \tag{2.3}$$

Since $G(0)=0$ and $G(s)\in\{\{0\}\}$, we conclude $\Phi(s)\in\{\{0\}\}$ and hence $\varphi=\mathbf{V}^{-1}\Phi$ is truly the unique solution of (2.1) in $\{0\}$.

We also see that $\varphi\in(0)^\mu$ if $g\in(0)^\mu$ and $\varphi\in\langle 0\rangle^\mu$ if k, $g\in\langle 0\rangle^\mu$, provided $\mu<1$. Thus, we obtain

Theorem 1. *If k, $g\in\{0\}$, in case of normal type, i.e. (2.2) to be valid, then (2.1) is solvable if and only if $G(0)=0$ and has the unique solution $\varphi=\mathbf{V}^{-1}\Phi$ in $\{0\}$, where Φ is given by (2.3). Moreover, $g\in(0)$ implies $\varphi\in(0)$ and k, $g\in\langle 0\rangle$ implies $\varphi\in\langle 0\rangle$.*

After simplification, φ may be written as

$$\varphi(t)=g_0(t)-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k_0(t-\tau)g_0(\tau)d\tau, \tag{2.4}$$

where $g_0=\mathbf{V}^{-1}G_0$, $k_0=\mathbf{V}^{-1}K_0$, in which

$$G_0(s)=\begin{cases}\dfrac{G(s)}{a-b}, & s\geqslant 0,\\ \dfrac{G(s)}{a+b}, & s\leqslant 0;\end{cases}\qquad K_0(s)=\begin{cases}\dfrac{K(s)}{a-b+K(s)}, & s>0,\\ \dfrac{K(s)}{a+b+K(s)}, & s<0.\end{cases} \tag{2.5}$$

Noting that, although $K_0(s)$ is discontinuous at $s=0$, it would not influence the property $k_0*g_0\in\{0\}$ since $G(0)=0$.

§ 3. Singular Integral Equations with Two Convolution Kernels

Let us solve the equation

$$a\varphi(t)+\frac{b}{\pi i}\int_{-\infty}^{+\infty}\frac{\varphi(\tau)}{\tau-t}d\tau+\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}k_1(t-\tau)\varphi(\tau)d\tau+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}k_2(t-\tau)\varphi(\tau)d\tau$$
$$=g(t),\quad b\neq 0,\ -\infty<t<+\infty, \tag{3.1}$$

where a, b are again constants, k_1, k_2, $g\in\{0\}$ and the unknown functions $\varphi_\pm$ and hence φ are required to be in $\{0\}$. Here we have denoted

$$\varphi_+(t)=\begin{cases}\varphi(t), & t\geqslant 0,\\ 0, & t<0;\end{cases}\qquad \varphi_-(t)=\begin{cases}0, & t\geqslant 0,\\ -\varphi(t) & t<0.\end{cases}$$

Assume that equation (3.1) has a solution. Taking Fourier transform, by Lemma 1, we get

$$(a-b\operatorname{sgn}s)\Phi(s)+K_1(s)\Phi^+(s)-K_2(s)\Phi^-(s)=G(s), \tag{3.2}$$

where $\Phi^\pm(s)$ are respectively the Fourier transforms of $\varphi_\pm(t)$, which are the boundary values of the (sectionally) holomorphic function $\Phi(z)$ in the upper and the lower half-planes respectively, and $\Phi(s)=\Phi^+(s)-\Phi^-(s)$[2]. In order that $\Phi^\pm(s)$ and then $\Phi(s)$ are continuous at $s=0$, it is necessary that $\Phi(0)=0$, i.e., $\Phi^+(0)=\Phi^-(0)$.

(3.2) is the Riemann boundary value problem with discontinuous coefficients:

$$\Phi^+(s)=D(s)\Phi^-(s)+F(s), \quad -\infty<s<+\infty, \tag{3.3}$$

in which we have put

$$D(s)=\frac{a-b\operatorname{sgn}s+K_2(s)}{a-b\operatorname{sgn}s+K_1(s)}, \quad F(s)=\frac{G(s)}{a-b\operatorname{sgn}s+K_1(s)}, \tag{3.4}$$

and restricted ourselves to the normal type case

$$K_j(s)\neq\begin{cases}-(a-b), & 0\leqslant s\leqslant+\infty,\\ -(a+b), & -\infty\leqslant s\leqslant 0,\end{cases} \quad j=1,\ 2. \tag{3.5}$$

Noting that $K_j(\infty)=0$ which implies $D(\infty)=1$ and $F(\infty)=0$ since $G(\infty)=0$, we know that $s=\infty$ is not a nodal point of the problem. Its unique nodal point is $s=0$. We require that the solutions of (3.3) should be at least continuous along the whole real axis and $\Phi(\infty)=0$.

According to the method used in [1], take a continuous branch of $\log D(s)$ such that it is continuous at $s=\infty$, e.g., $\log D(\infty)=0$, and denote

$$\gamma_0=\alpha_0+i\beta_0=\frac{1}{2\pi i}\{\log D(-0)-\log D(+0)\}. \tag{3.6}$$

Then choose an integer $\varkappa$, the index of the problem, such that $0\leqslant\alpha=\alpha_0-\varkappa<1$. Denote $\gamma=\gamma_0-\varkappa=\alpha+i\beta_0$. Since we require $\Phi(\infty)=0$, so we get: when $\varkappa\geqslant0$, the general solution of (3.3) (without considering the behavior of $\Phi^\pm(s)$ at $s=0$ for the time being) is

$$\Phi(z)=X(z)\left\{\Psi(z)+\frac{Q_\varkappa(z)}{(z+i)^\varkappa}\right\}, \quad \Psi(z)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{F(t)dt}{X^+(t)(t-z)}, \tag{3.7}$$

where

$$Q_{\varkappa-1}(z)=C_0+C_1z+\cdots+C_{\varkappa-1}z^{\varkappa-1} \tag{3.8}$$

is an arbitrary polynomial of degree $\varkappa-1$ (in the sequel we understand $Q_k\equiv0$ when $k<0$); when $\varkappa\leqslant-1$, the problem is solvable if and only if the conditions (3.9)

$$\int_{-\infty}^{+\infty}\frac{F(t)dt}{X^+(t)(t+i)^j}=0, \quad j=1, \cdots, -\varkappa \tag{3.9}$$

are satisfied, and then the problem has the unique solution (3.7). Here

$$X(z)=\begin{cases}e^{\Gamma(z)}, & \operatorname{Re}z>0,\\ \left(\dfrac{z+i}{z-i}\right)^\varkappa e^{\Gamma(z)}, & \operatorname{Re}z<0,\end{cases} \tag{3.10}$$

where

$$\Gamma(z)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\log D_0(t)}{t-z}\,dt, \quad D_0(t)=\left(\frac{t+i}{t-i}\right)^\varkappa D(z), \tag{3.11}$$

in which we have taken the definite branch of

$$\log D_0(t)=\varkappa\log\frac{t+i}{t-i}+\log D(t), \tag{3.12}$$

provided we have chosen $\log\left.\dfrac{t+i}{t-i}\right|_{t=\infty}=0$, what is the same, $\log\left.\dfrac{t+i}{t-i}\right|_{t=\pm0}=\pm i\pi$. It is easy to prove

$$X^+(t)=\sqrt{D_0(t)}e^{\Gamma(t)},\quad X^-(t)=e^{\Gamma(t)}/\sqrt{D_0(t)},\tag{3.13}$$

where $\sqrt{D_0(t)}=\exp\left\{\frac{1}{2}\log D_0(t)\right\}$ has definite value. By (3.7), we get

$$\left.\begin{aligned}\Phi^+(s)&=\frac{1}{2}F(s)+X^+(s)\left\{\Psi(s)+\frac{Q_{\varkappa-1}(s)}{(s+i)^\varkappa}\right\},\\ \Phi^-(s)&=-\frac{1}{2}\frac{F(s)}{D(s)}+X^-(s)\left\{\Psi(s)+\frac{Q_{\varkappa-1}(s)}{(s+i)^\varkappa}\right\},\end{aligned}\right\}\tag{3.14}$$

and thereby

$$\Phi(s)=\frac{F(s)}{2}\left[1+\frac{1}{D(s)}\right]+[X^+(s)-X^-(s)]\left\{\Psi(s)+\frac{Q_{\varkappa-1}(s)}{(s+i)^\varkappa}\right\}.\tag{3.15}$$

Since $X^+(s)$ are bounded and $\neq 0$, it is easy to verify $\Phi^\pm(s)$, $\Phi(s)\in L_2(-\infty,+\infty)$ and $\in H$ on any closed interval exterior to $s=0$.

The only thing required to be considered is their behavior near $s=0$. In order that they belong to $\{\{0\}\}$, they ought to be continuous at $s=0$. We prove that it is then necessary for $G(0)=0$.

First, let $s=0$ be an ordinary node, i.e., $0<\alpha<1$. Then $\gamma\neq 0$. It is known that, in the neighborhood of $s=0$,

$$X^+(s)=\sqrt{D_0(s)}s^\gamma e^{\Gamma_0(s)},\quad \Gamma_0(s)\in H,$$

where, by (3.12)

$$\sqrt{D_0(\pm 0)}=\exp\frac{1}{2}\{\pm\varkappa\pi i+\log D(\pm 0)\},\ \frac{\sqrt{D_0(+0)}}{\sqrt{D_0(-0)}}=e^{-\gamma\pi i}.\tag{3.16}$$

On the other hand, from [1], § 26, 4°, when $s>0$,

$$\Psi(s)=\frac{e^{-\Gamma_0(s)}}{s^\gamma}\left\{\frac{\operatorname{ctg}\gamma\pi}{2i}\frac{F(+0)}{\sqrt{D_0(+0)}}-\frac{e^{-\gamma\pi i}}{2i\sin\gamma\pi}\frac{F(-0)}{\sqrt{D_0(-0)}}\right\}+\Psi^*(s),\tag{3.17}$$

where $\Psi^*(s)=\Psi^{**}(s)/|s|^{\alpha'}$, $0<\alpha'<\alpha$ and $\Psi^{**}(s)\in H$. Then, by (3.16), after simplifying (3.14), we may get

$$\Phi^+(+0)=\frac{e^{\gamma\pi i}}{2i\sin\gamma\pi}[F(+0)-e^{-3\gamma\pi i}F(-0)].\tag{3.18}$$

When $s<0$, instead of (3.17), we have

$$\Psi(s)=\frac{e^{-\Gamma_0(s)}}{s^\gamma}\left\{\frac{e^{\gamma\pi i}}{2i\sin\gamma\pi}\frac{F(+0)}{\sqrt{D_0(+0)}}-\frac{\operatorname{ctg}\gamma\pi}{2i}\frac{F(-0)}{\sqrt{D_0(-0)}}\right\}+\Psi^*(s),\tag{3.17$'$}$$

and then

$$\Phi^+(-0)=\frac{e^{2\gamma\pi i}}{2i\sin\gamma\pi}[F(+0)-e^{-3\gamma\pi i}F(-0)].\tag{3.18$'$}$$

On comparing (3.18) with (3.18)$'$, we know that $\Phi(s)$ is continuous at $s=0$ if and only if (regarding $e^{\gamma\pi i}\neq 1$)

$$F(+0)=e^{-3\gamma\pi i}F(-0).\tag{3.19}$$

And then we have $\Phi^+(0)=0$. Since we require $\Phi(0)=0$, we must also require $\Phi^-(0)=0$. Returning to (3.2), we know that we must have $F(0)=0$ and hence $G(0)=0$. But once $G(0)=0$, we really have $\Phi^\pm(0)=\Phi(0)=0$ and $\Phi^\pm(s)$, $\Phi(s)\in H$ in the neighborhood of $s=0$, and therefore $\Phi^\pm(s)$, $\Phi(s)$ belong to $\{\{0\}\}$.

Now, let $s=0$ be a special node, i.e., $\alpha=0$. Then $\gamma=i\beta_0$. If $\beta_0\neq0$, then (3.17) and (3.17)′ remain valid, with $\Psi^*(s)\in H_0$, i.e., $\Psi^*(\pm0)$ exist but do not equal to each other possibly. In place of (3.18), we have

$$\Phi^+(+0)=\frac{e^{\gamma\pi i}}{2i\sin\gamma\pi}[F(+0)-e^{-3\gamma\pi i}F(-0)]$$
$$+\sqrt{D_0(+0)}\,e^{\Gamma_0(0)}\lim_{s\to+0}s^{i\beta_0}[\Psi^*(s)+A_0],$$

where

$$A_0=\begin{cases}C_0/i^\varkappa, & \varkappa>0,\\ 0, & \varkappa\leqslant0,\end{cases}$$

and a similar formula for $\Phi^+(-0)$. In order that $\Phi^+(\pm0)$ exist, we should have $\Psi^*(\pm0)=-A_0$. And then we are back to (3.18) and (3.18)′ and hence again to (3.19). Thus, we have $G(0)=0$ again.

Once $G(0)=0$ is fulfilled, then $F(0)=0$ and therefore $\Psi(s)\in H$ near $s=0$. Thus, in order that $\Phi^+(s)$ is continuous at $s=0$, the constant term of $Q_{\varkappa-1}(z)$ should take the value

$$C_0=\frac{i^{\varkappa+1}}{2\pi}\int_{-\infty}^{+\infty}\frac{F(t)dt}{X^+(t)t} \tag{3.20}$$

if $\varkappa\geqslant1$; and an additional condition of solvability

$$\int_{-\infty}^{+\infty}\frac{F(t)dt}{X^+(t)t}=0 \tag{3.21}$$

should be supplemented if $\varkappa\leqslant0$. When C_0 is taken to be (3.20) or (3.21) is fulfilled, it is readily seen $\Phi^\pm(0)=\Phi(0)=0$.

If $\beta_0=0$, i.e., $\gamma=0$, then $D(s)$ is continuous at $s=0$. Since $b\neq0$, we know at once $K_1(0)=K_2(0)=0$. So $D(0)=1$ and hence again we must have $F(0)=0$ and then $G(0)=0$ in order that $\Phi(0)=0$. Thus, $s=0$ is not a nodal point at all. There is no problem in this case.

In all of the above cases, $\Phi^\pm(s)$, $\Phi(s)\in\hat{H}$ undoubtedly.

Thus, we have

Theorem 2. *Under supposition, in the normal type case, equation* (3.1) *is possibly solvable in class* $\{0\}$ *only when* $G(0)=0$. *Assume that this is fulfilled. If* $s=0$ *is an ordinary node, then, when the index* $\varkappa\geqslant0$, *it always has the solution* $\varphi=\mathbf{V}^{-1}\Phi$, *where* Φ *is given by* (3.15); *when* $\varkappa\leqslant-1$, *it has the* (*unique*) *solution as above, provided the conditions of solvability* (3.9) *are fulfilled. If* $s=0$ *is a special node and* $K_1(0)=K_2(0)$, *the above statements remain true; in case* $K_1(0)\neq K_2(0)$, *then, it has the solution as above with* C_0 *to be taken as* (3.20) *if* $\varkappa\geqslant1$, *and it is solvable as above if and only if the conditions of solvability* (3.21) *and* (3.9) *are fulfilled if* $\varkappa\leqslant0$ (*the latter disappear when* $\varkappa=0$).

Remark. In applications, we often have real equation (3.1), in which $a=A$

is real, $b=Bi$ is purely imaginary ($\neq 0$) and k_1, k_2 are real functions. In this case

$$D(\pm 0)=\frac{A+k_2(0)\mp Bi}{A+k_1(0)\mp Bi}$$

are conjugate to each other, and $\alpha_0=\frac{1}{2\pi}\{\arg D(-0)-\arg D(+0)\}$ so that α_0 is not an integer. Thus, $s=0$ is an ordinary node for the real equation (3.1).

§ 4. Singular Integral Equation of Wiener–Hopf Type

In this section we consider the method of solution for the equation

$$a\varphi(t)+\frac{b}{\pi i}\int_0^{+\infty}\frac{\varphi(\tau)}{\tau-t}\,d\tau+\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}k(t-\tau)\varphi(\tau)d\tau=g(t),$$
$$b\neq 0,\quad 0<t<+\infty, \tag{4.1}$$

where a, b are constants, k, $g=g_+\in\langle 0\rangle$ and $\varphi=\varphi_+$ is required to be in $\{0\}$.

Let k, $g\in\langle 0\rangle^\mu$ $\left(\frac{1}{2}<\mu<1\right)$.

On extending (4.1) to $-\infty<t<0$, the right-hand side of (4.1) is augmented with an unknown function $\varphi_-(t)$. Taking the Fourier transforms of both of its sides, by Lemma 1, we get

$$[a-b\operatorname{sgn}s+K(s)]\Phi^+(s)=G(s)+\Phi^-(s).$$

Restricted to the normal type case, i.e., $K(s)$ satisfying (2.2), it can be written as (3.3) with

$$D(s)=\frac{1}{a-b\operatorname{sgn}s+K(s)},\quad F(s)=D(s)G(s). \tag{4.2}$$

Denote

$$\gamma_\infty=\alpha_\infty+i\beta_\infty=\frac{1}{2\pi i}\{\log D(+\infty)-\log D(-\infty)\}=\frac{1}{2\pi i}\log\frac{a+b}{a-b},$$

where $\log D(s)$ is taken to be continuous for $s>0$ and $s<0$ respectively such that $0\leqslant\alpha_\infty<1$. Note that $\gamma_\infty\neq 0$ since $b\neq 0$.

Then take $\gamma_0=\alpha_0+i\beta_0$, $0\leqslant\alpha=\alpha_0-\varkappa<1$, $\gamma=\alpha+i\beta_0$ as in § 3, $\varkappa$ being the index of the problem. We also have $\gamma\neq 0$ since $b\neq 0$.

Therefore both $s=0$ and $s=\infty$ are nodes. Note that $\Phi^-(\infty)=\Phi(\infty)=0$ since we require $\Phi^+(\infty)=0$.

Let $s=\infty$ be an ordinary node at first. From [1], we know that, the general solution of (3.3) is

$$\Phi(z)=X(z)\left\{\Psi_1(z)+\frac{Q_\varkappa(z)}{(z+i)^\varkappa}\right\},\quad \Psi_1(z)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{z+i}{t+i}\,\frac{F(t)dt}{X^+(t)(t-z)}, \tag{4.3}$$

when $\varkappa\geqslant -1$, where $Q_\varkappa(z)=C_0+C_1z+\cdots+C_\varkappa z^\varkappa$ is an arbitrary polynomial of degree $\varkappa$ and $X(z)$ is still given by (3.10)①. Since then $X^-(s)=\chi^+(s)/s^{\gamma_\infty}$, $\chi^+(s)\in H_0(N_\infty)$,

① However, a factor $(z+i)/(t+i)$ should be multiplied in the integrand of (3.11).

hence we write $Q_\varkappa(z)$ in (4.3) instead of $Q_{\varkappa-1}(z)$, which is sufficient for $\Phi(\infty)=0$. Moreover, it should be noted that $\Psi_1(z)$ cannot be separated as, in general

$$\Psi_1(z)=\Psi(z)-\Psi(-i)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{F(t)dt}{X^+(t)(t-z)}-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{F(t)dt}{X^+(t)(t+i)}, \tag{4.4}$$

because the integrals in the right-hand member may be divergent. When $\varkappa\leqslant-2$, the conditions of solvability read

$$\int_{-\infty}^{+\infty}\frac{F(t)dt}{X^+(t)(t+i)^j}=0, \quad j=2, \cdots, -\varkappa. \tag{4.5}$$

In this case

$$\Phi^+(s)=\frac{1}{2}F(s)+X^+(s)\left\{\Psi_1(s)+\frac{Q_\varkappa(s)}{(s+i)^\varkappa}\right\}. \tag{4.6}$$

Since $F(\infty)=0$, we know that $\Psi_1(s)=\Psi_1^*(s)/|s|)^{\alpha'}$, $\alpha'<\alpha$ and $\Psi_1^*(s)\in H$ in the neighborhood of $s=0$. Hence it is sure that $\Phi^+(\infty)=0$. We consider the following two cases.

1° Let $\alpha_\infty>\frac{1}{2}$. If $\mu>\alpha_\infty$, $\Psi_1(s)$ is bounded and so $X^+(s)\Psi_1(s)=O(1/|s|^{\alpha_\infty})$ near $s=\infty$. And since $F(s)=O(1(|s|^\mu)$, $\Phi^+(s)=O(1/|s|^\mu)$. If $\mu\leqslant\alpha_\infty$, again by [1], we know $X^+(s)\Psi_1(s)=O(1/|s|^{\alpha_\infty-\varepsilon})$ with $\varepsilon>0$ arbitrarily small. Take ε such that $\alpha_\infty-\varepsilon>\frac{1}{2}$. Therefore, in any case, $\Phi^+(s)=O(1/|s|^\nu)$, $\nu=\min(\mu, \alpha_\infty-\varepsilon)>\frac{1}{2}$.

2° Let $\alpha_\infty\leqslant\frac{1}{2}$. Since $F(t)\in H^\mu(N_\infty)$, so, by [1], § 6, we know that $F(t)/X^+(t)\in H^{\mu-\alpha_\infty}(N_\infty)$. In this case, (4.4) becomes valid, the integrals in which are convergent now, and thereby $X^+(s)\Psi(s)\in H(N_\infty)$, being of $O(1/|s|^\mu)$. In order to guarantee $\Phi^+(s)\in L_2(-\infty, +\infty)$, we are obliged to take

$$C_\varkappa=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{F(t)dt}{X^+(t)(t+i)}$$

in (4.6) when $\varkappa\geqslant0$, that is to say, (4.6) may be written as (3.14) in this case. When $\varkappa<0$, there is an additional condition of solvability

$$\int_{-\infty}^{+\infty}\frac{F(t)dt}{X^+(t)(t+i)}=0$$

which should be supplemented to (4.5) if $\varkappa<-1$. Thus, the conditions of solvability return back to (3.9).

Next, let $s=\infty$ be a special node: $\alpha_\infty=0$, $\gamma_\infty=i\beta_\infty\neq0$. In this case, (4.4) remains valid and the discussion is the same as 2° above.

Thus, $\Phi^+(s)$ satisfies our requirement near $s=\infty$: $\Phi^+(s)\in H$ and $\in L_2(-\infty,+\infty)$ $\left(\text{actually being of } O(1/|s|^\nu),\ \nu>\frac{1}{2}\right)$.

Now we consider the situation near $s=0$. The discussion is similar to that in § 3. By the requirement $\Phi^+(+0)=\Phi^+(-0)$, we may get (3.19) again. But since we have $F(\pm0)=G(0)D(\pm0)$ now, it may be rewritten as

$$G(0)\,[D(+0)-e^{3\gamma\pi i}D(-0)]=0.$$

By noting that $D(-0)/D(+0)=e^{2\gamma_0\pi i}$ and $\gamma=\gamma_0-\varkappa\neq 0$, so

$$D(+0)-e^{3\gamma\pi i}D(-0)=D(+0)(1-e^{-\gamma\pi i})\neq 0$$

and then again we get $G(0)=0$.

The remaining discussions are the same as in § 3. But we should note that, in case $\alpha_\infty>\frac{1}{2}$, if $\varkappa\geqslant 0$, since $\Phi^+(s)$ is then given by (4.6), the constant term in $Q_\varkappa$ must be taken as

$$C_0=\frac{i^{\varkappa-1}}{2\pi}\int_{-\infty}^{+\infty}\frac{F(t)dt}{X^+(t)(t+i)t} \tag{4.7}$$

instead of (3.20), and if $\varkappa\leqslant -1$, there is an additional condition of solvability

$$\int_{-\infty}^{+\infty}\frac{F(t)dt}{X^+(t)(t+i)t}=0. \tag{4.8}$$

Finally we get $\Phi^+(0)=0$, $\Phi^+(s)\in\hat{H}$.

Futhermore, it is seen that actually $\Phi^+(s)\in((0))$.

Thus, we obtain

Theorem 3. *Under supposition, in case of normal type, the necessary condition for the equation* (4.1) *to be solvable in* $\{0\}$ (*actually in* (0)) *is* $G(0)=0$. *Assume that this is fulfilled. In case* $s=0$ *is an ordinary node, if* $\alpha_\infty>\frac{1}{2}$, *then it has the general solution* $\varphi=\mathbf{V}^{-1}\Phi^+$ *when* $\varkappa\geqslant -1$, *where* $\Phi^+(s)$ *is given by* (4.6), *and* (4.5) *is the condition of solvability when* $\varkappa\leqslant -2$; *if* $\alpha_\infty\leqslant\frac{1}{2}$, *it is solvable as above when* $\varkappa\geqslant 0$ *with* Φ^+ *given by* (3.14), *and the condition of solvability is* (3.9) *when* $\varkappa\leqslant -1$. *In case* $s=0$ *is a special node, if* $\alpha_\infty>\frac{1}{2}$, Φ^+ *is again given by* (4.6) *when* $\varkappa\geqslant 0$ *with the constant term of* $Q_\varkappa$ *taken as* (4.8), *and when* $\varkappa\leqslant -1$, *besides* (4.5), *the condition of solvability* (4.8) *should be supplemented; if* $\alpha_\infty\leqslant\frac{1}{2}$, *then the constant term of* $Q_{\varkappa-1}$ *in* (3.14) *should be taken as* (3.20) *when* $\varkappa\geqslant 1$, *and when* $\varkappa\leqslant 0$, *besides* (3.9), *the condition of solvability* (3.21) *should be supplemented.*

Remark 1. It is seen from the above discussions, when $\alpha_\infty>\frac{1}{2}$, in fact, the obtained $\Phi^+\in\langle\langle 0\rangle\rangle$ and hence $\varphi\in\langle 0\rangle$.

Remark 2. When (4.1) is a real equation, as shown at the end of § 3, $s=0$ must be an ordinary node. It is also easily seen that the characteristic feature for $\alpha_\infty>\frac{1}{2}$ or $\leqslant\frac{1}{2}$. Denote $a=A$ and $b=Bi$ as before. By definition of α_∞, it is obvious that

$$0\leqslant\alpha_\infty=\frac{1}{2\pi}\arg\frac{A+Bi}{A-Bi}<1.$$

Hence $\alpha_\infty > \frac{1}{2}$ means $\pi < \arg \frac{A+Bi}{A-Bi} < 2\pi$, i.e., $A+Bi$ lies in the quadrant II or IV; $\alpha_\infty \leqslant \frac{1}{2}$ means $A+Bi$ lies in the quadrant I or III (including the case $A=0$).

§ 5. Dual Singular Integral Equations

The above method is applicable to solving the dual singular integral equations

$$\begin{cases} a_1\omega(t) + \dfrac{b_1}{\pi i}\displaystyle\int_{-\infty}^{+\infty} \dfrac{\omega(\tau)}{\tau - t}\, d\tau + \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{+\infty} k_1(t-\tau)\omega(\tau)d\tau = g(t), & 0<t<+\infty, \\ a_2\omega(t) + \dfrac{b_2}{\pi i}\displaystyle\int_{-\infty}^{+\infty} \dfrac{\omega(\tau)}{\tau - t}\, d\tau + \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{+\infty} k_2(t-\tau)\omega(\tau)d\tau = g(t), & -\infty<t<0, \end{cases} \tag{5.1}$$

where a_j, b_j are constants, k_j, $g \in \langle 0 \rangle$ $(j=1, 2)$. Find its solution $\varphi \in \{0\}$. Assume b_1, b_2 are not equal to zero simultaneously.

Rewrite (5.1) as

$$\left.\begin{aligned} a_1\omega + b_1\mathbf{T}\omega + k_1 * \omega = g - \varphi_-, \\ a_2\omega + b_2\mathbf{T}\omega + k_2 * \omega = g + \varphi_+, \end{aligned}\right\}$$

where $\varphi_\pm$ are unknown functions, to be required belonging to $\{0\}$ too. Taking Fourier transforms, we get

$$\left.\begin{aligned} (a_1 - b_1 \operatorname{sgn} s + K_1)\Omega = G + \Phi^-, \\ (a_2 - b_2 \operatorname{sgn} s + K_2)\Omega = G + \Phi^+. \end{aligned}\right\}$$

Since we requrire Ω is continuous at $s=0$, we must have $\Omega(0)=0$. Restricted to the normal type case:

$$K_j(s) = \begin{cases} -(a_j - b_j), & 0 \leqslant s \leqslant +\infty, \\ -(a_j + b_j), & -\infty \leqslant s \leqslant 0, \end{cases} \quad j=1, 2, \tag{5.2}$$

we then have

$$\Omega(s) = \frac{G(s) + \Phi^-(s)}{a_1 - b_1 \operatorname{sgn} s + K_1(s)} = \frac{G(s) + \Phi^+(s)}{a_2 - b_2 \operatorname{sgn} s + K_2(s)}. \tag{5.3}$$

Therefore, we should solve the Riemann boundary value problem (3.3) again, in which

$$\Omega(s) = \frac{a_2 - b_2 \operatorname{sgn} s + K_2(s)}{a_1 - b_1 \operatorname{sgn} s + K_1(s)}, \quad F(s) = [D(s) - 1]G(s). \tag{5.4}$$

In order that $\Omega(s)$ is continuous at $s=0$, it is necessary that $\Phi^\pm(s)$ are continuous at $s=0$ and $\Phi^\pm(0) = -G(0)$. Discussions may be made fully analogous to thosein § 4. Since we also require $\Phi^+(+0) = \Phi^+(-0)$,we get $G(0)=0$ again. Hence all the results as stated in Theorem 3 remain true and $\omega = \mathbf{V}^{-1}\Omega$ in which Ω is given by (5.3). The only difference lies in that γ_∞ or (and) γ may be zero, for instance, $\gamma_\infty = 0$ if $a_1/a_2 = b_1/b_2$, in which cases the analysis will be even simpler. It is also obvious that the solution ω in $\{0\}$ belongs also to (0).

Finally we remark that the methods of this paper may be used to solve the equations mentioned above in the exceptional cases.

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