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# OUTER-$\Sigma$ GROUPS OF FINITE ORDER

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**Abstract**

Suppose that $\Sigma$ is a group-theoretic property. A group whose every proper subgroup but itself is a $\Sigma$ group is called an outer-$\Sigma$ group.

The paper gives a series of results to groups which possess trivial Frattini subgroup and only one solvable minimal normal subgroup. The outer groups are such groups when the classe of $\Sigma$ groups is a saturated formation.

By use of aforementioned results, the $c(k)$ groups (group with classes less than $k$), $\Gamma_k$-$pn$ groups (groups whose $k$-th term of lower central series are $p$-nilpotenit) and $p$-supersolvable groups are discussed.

## § 1. General Results

All groups which are discussed in this paper are of finite order.

**Definition 1.1.** *Groups with property $\Sigma$ are called $\Sigma$ groups. Groups whose every proper subgroup is a $\Sigma$ group but itself is not are called inner-$\Sigma$ groups. Groups whose every proper factor group is a $\Sigma$ group but itself is not are called outer-$\Sigma$ groups. Groups whose every proper subgroup and every factor group are $\Sigma$ groups but itself is not are called minimal non-$\Sigma$ groups.*

**Lemma 1.1.** *If property $\sigma$ is preserved for subgroups and all inner-$\Sigma$ groups are not $\sigma$ groups, then the property $\sigma$ is a sufficient condition of $\Sigma$ groups. If $\sigma$ is preserved for factor groups and all outer-$\Sigma$ groups are not $\sigma$ groups, then $\sigma$ is a sufficient condition of $\Sigma$ groups.*

*Proof* The lemma follows from the minimal counterexample.

From Lemma 1.1, outer-$\Sigma$ and inner-$\Sigma$ groups are of importance for studying $\Sigma$ groups. This had been showed by the research of supersolvable groups in the paper [1]. There are more papers regarding the inner-$\Sigma$ groups about some most common properties $\Sigma$, while less research of the outer-$\Sigma$ groups appears. Klein[2] and Baartmans[3] studied the outerabelian groups; Doerk[4] and the paper[1] studied the outersupersolvable groups.

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The following lemma shows that outer-$\Sigma$ groups are more important than inner-$\Sigma$ groups for studying $\Sigma$ groups.

**Lemma 1.2.** *Suppose that property $\Sigma$ is preserved for factor groups. Then inner-$\Sigma$ groups $G$ with $\Phi(G)=1$ is an outer-$\Sigma$ group, i.e. $G$ is a minimal non-$\Sigma$ group. Conversely, if $\Sigma$ satisfies the condition:*

(*) *Suppose that $G/\Phi(G)$ is a $\Sigma$ group implies that $G$ is a $\Sigma$ group, then minimal non-$\Sigma$ group $G$ is an inner-$\Sigma$ group with $\Phi(G)=1$.*

*Proof* Suppose that there exists $N$, $1<N\lhd G$, such that $G/N$ is not a $\Sigma$ group. Let $H$ be any subgroup of $G$ such that $G=HN$. Then $H$ is not a $\Sigma$ group, since $H/H\cap N\simeq HN/N=G/N$. We have $H=G$, since every proper subgroup of $G$ is a $\Sigma$ group. From this, $M\leqslant\Phi(G)=1$, and it is a contradiction. Therefore $G$ is an outer-$\Sigma$ group. The last conclusion of this lemma is clear.

We note that nilpotency, supersolvability, $\sigma$-sylow tower and $p$-supersolvability ([15], Th. IV, 8.6.a) are all satisfying the condition (*).

In this paper we discuss the outer-$\Sigma$ groups for some common properties $\Sigma$ (nilpotency, metanilpotency supersolvability etc.) and minimal non-$\Sigma$ groups by using results of outer-$\Sigma$ groups.

The main result of this paper is the following principal lemma. Its hypotheses are few and its conclusions are rich. It shows that various conclusions of this paper may be developed, because these conclusions are based upon the principal lemma and the hypotheses of principal lemma is quite general.

**Principal Lemma.** *Suppose that $G$ has a unique minimal normal subgroup $N$ which is an elementary abelian $p$-group of order $p^\alpha$ and $\Phi(G)=1$, where $\Phi(G)$ is the Frattini subgroup of $G$. Then*

1) *$G=AN$, $A\cap N=1$, and $A$ is a maximal subgroup of $G$.*

2) *$C_G(N)=N$.*

3) *$A$ has no non-trivial normal $p$-subgroups, that is, $O_p(A)=1$.*

4) *The Fitting subgroup $F(G)$ of $G$ is equal to $N$ and $N$ is a maximal abelian subgroup of any Sylow $p$-subgroup of $G$.*

5) *If $O_{p'}(A)\neq 1$ (this holds when $A$ is $p$-solvable), then $H$ conjugate to $A$ whenever $H<G$ with $G=HN$.*

6) *Every noninedtity element of the center $Z(A)$ of $A$ acts $N$ fixed-point-free and so $Z(A)$ is cyclic.*

7) *If $1\neq H\trianglelefteq G$, then*

i) *$H=BN$, $B\cap N=1$, $B=A\cap H$.*

ii) *either $\Gamma_k(H)=1, H=N$ or $\Gamma_k(H)=\Gamma_k(B)N$. If $\Gamma_k(H)$ is $p$-nilpotent, then $B$ is a nilpotent group of class less than $k$ and $p\nmid |B|$, where $\Gamma_k(H)$ is the $k$-th term of the lower central series of $H$ $(\Gamma_1(H)=H)$.*

8) *The following four propositions are equivalent:*

i) $G'$ *is* $p$-*nilpotent.*

ii) $G'$ *is nilpotent.*

iii) $A$ *is abelian.*

iv) $A$ *is cyclic.*

*In case* 8), *the exponent of* $p$ $(\mathrm{mod}\,|A|)$ *is* $\alpha$.

9) *If* $A$ *is* $p$-*solvable and non-abelian, then* $(|A|, \alpha)\neq 1$.

*Proof* 1) From $\Phi(G)=1$, there exists a maximal subgroup $A$ of $G$ which does not contain $N$. Hence $G=AN$. Let $A\cap N=D$. $D\trianglelefteq\langle A, N\rangle=G$. By minimality of $N$ and $A\not\geqslant N$, $D=1$.

2) Let $D=C_G(N)\cap A$. Then $D\trianglelefteq A$, $D\trianglelefteq\langle A, N\rangle=G$. By uniqueness of $N$, $D=1$. Hence $C_G(N)=N$.

3) If $A$ has a non-trivial normal $p$-subgroup $B$, then $N_P(B)>B$, where $P=BN$. Therefore $N_G(B)$ contains non-identity elements of $N$. Hence $N_G(B)>A$, $N_G(B)=G$, $B\trianglelefteq G$, contrary to uniqueness of $N$.

4) Clearly $F(G)\geqslant N$. If $F(G)>N$, then $F(G)$ must be a $p$-group by uniqueness of $N$. It is trivial that $F(G)\cap A$ is a non-trivial normal $p$-subgroup of $A$, contrary to 3).

5) In analogy to proof of 1), $H<G$, $HN=G$ implies $H\cap N=1$. Therefore
$$H=H/H\cap N\simeq HN/N=G/N=AN/N\simeq A.$$
$O_{p'}(H)\neq 1$, since $O_{p'}(A)\neq 1$. Hence $O_{p'}(G/N)\simeq O_{p'}(A)N/N=O_{p'}(H)N/N$, $O_{p'}(A)N=O_{p'}(H)N$. By Schur-Zassenhaus' theorem, there exists an element $g$ such that $O_{p'}(H)=O_{p'}(A)^g$. Thus $O_{p'}(H)\triangleleft\langle A^g. H\rangle$. If $H\neq A^g$, then $O_{p'}(H)\triangleleft G$, contrary to uniqueness of $N$.

6) If $1\neq d\in Z(A)$, then $\langle d\rangle\trianglelefteq A$. If there exists an element $1\neq a\in N$, such that $ad=da$, then $N_G(\langle d\rangle)\geqslant\langle A, a\rangle=G$, contrary to uniqueness of $N$. Hence the non-identity elements of $Z(A)$ act $N$ fixed-point-free. By [5]. 7.24, $Z(A)$ is cyclic.

7) i) $H\cap N\trianglelefteq G$, since $H\trianglelefteq G$. By minimality of $N$, $H\cap N=1$ or $N$. From $1\neq H\trianglelefteq G$ and uniqueness of $N$, $H\cap N=N$. Thus $|G|=|AN|=|AH|=|A|\cdot|H|/|A\cap H|$, $|H|=|A\cap H|.\,|N|$, since $|AN|=|A|\cdot|N|$. Clearly $H\geqslant(H\cap A)N$, whence $H=(H\cap A)N$. Put $B=A\cap H$, we derive that $H=BN$, $B\cap N=1$.

ii) From $H/\Gamma_k(B)N=(B)N/\Gamma_k(B)N\simeq B/\Gamma_k(B)$, $\Gamma_k(H)\leqslant\Gamma_k(B)N(B/\Gamma_k(B)$ is a nilpotent group of class less than $k$). $\Gamma_k(H)\trianglelefteq G$, since $\Gamma_k(H)$ is a characteristic subgroup of $H$. If $\Gamma_k(H)=1$, then $H$ is a nilpotent normal subgroup. Hence $H\leqslant F(G)=N$ and so $H=N$. If $\Gamma_k(H)\neq 1$, then $\Gamma_k(H)\geqslant N$ by i). Thus $\Gamma_k(H)/N=\Gamma_k(H/N)\simeq\Gamma_k(B)$.

If $\Gamma_k(H)$ is $p$-nilpotent, then the normal $p$-complement $K$ of $\Gamma_k(H)$ is a characteristic subgroup of $\Gamma_k(H)$, $K\triangleleft G$. Therefore $K=1$ and $\Gamma_k(H)$ is a $p$-group.

Thus $\Gamma_k(H) \leqslant F(G) = N$. If $\Gamma_k(H) \neq 1$, then $\Gamma_k(H) \geqslant N$. Thus $\Gamma_k(H) = N$ and so $\Gamma_k(B) = 1$. If $\Gamma_k(H) = 1$, then $\Gamma_k(B) = 1$ also. Hence $B$ is a nilpotent group of class less than $k$.

If $p \mid |B|$, then the Sylow $p$-subgroups of $B$ are distinct to 1. They are characteristic in $B$ and so normal in $A$, since $B = H \cap A \trianglelefteq A$. This is contrary to 3).

8) $G'$ is $p$-nilpotent, that is $\Gamma_2(G)$ being $p$-nilpotent. By 7) ii), $A$ is abelian. From 6), $A$ is cyclic. By [6], 9.4.3, the exponent of $p$ (mod $|A|$) is $\alpha$.

9) Derived from [15] VI. 8.1.

**Lemma 1.3.** *Suppose that property $\Sigma$ satisfies the condition*

($\triangle$) *If $G/N_1$, $G/N_2$ are $\Sigma$ groups, then $G/N_1 \cap N_2$ is also a $\Sigma$ group.*

*If $G$ is an outer-$\Sigma$ group and $G$ possesses a non-trivial normal $p$-subgroup, then $G$ has a unique normal $p$-subgroup $N$. Thus $G$ is $p$ group described in the principal lemma when $\Phi(G) = 1$.*

*Proof* $G$ has non-trivial normal $p$-subgroup. $G$ must have minimal normal $p$-subgroup $N$. If $M$ were a minimal normal subgroup of $G$ distinct to $N$, then by minimality $M \cap N = 1$. By condition ($\triangle$), $G/M \cap N = G$ is a $\Sigma$ group, This is a contradiction. Hence $N$ is the unique minimal normal subgroup of $G$.

We note that if property $\Sigma$ satisfies the condition (*), then outer-$\Sigma$ group must have $\Phi(G) = 1$. By Lemma 1.3, we derive

**Theorem 1.1.** *If the class which consists of all $\Sigma$ groups is a saturated formation, then solvable outer-$\Sigma$ groups are groups described in the principal lamma for some minimal normal subgroup.*

**Lemma 1.4.** *If the property $\Sigma$ implies $\Pi$-solvable, then the minimal non-$\Sigma$ group $G$ is $\Pi$-solvable, or a simple group.*

*Proof* If $G$ is not simple, then $G$ has a proper normal subgroup $N$.$N$ and $G/N$ are $\Sigma$ groups and so $G$ is $\Pi$-solvable.

## § 2. $c(k)$ Groups

**Definition 2.1.** *Nilpotent groups of class less than $k$ are called $c(k)$ groups. $c(1)$ group means the identity group.*

**Theorem 2.1.** *Solvable outer-nilpotent group $G$ is a group described in the Principal Lemma. Furthermore*

1) *$F(G) = N$ is the Sylow $p$-subgroup of $G$.*

2) *$A$ is nilpotent and the center of every Sylow subgroup of $A$ is cyclic.*

3) *If $A$ is generated by $k$ elements, then $G$ is generated by two elements when $k = 1$ and $G$ is generated by $k$ elements when $k > 1$.*

*Proof* The nilpotent group class is a saturated formation. By Lemma 1.1. $G$ is a group described in the Principal Lemma and $A$ is nilpotent. 2) comes from Principal Lemma 6).

Now prove 3). If $k=1$, then $A$ is cyclic, $A=\langle a\rangle$. Then $G=\langle a, b\rangle$ evidently, where $1\neq b\in N$. Suppose that $k\geqslant 2$, $A=\langle a_1, a_2, \cdots, a_k\rangle$. If there is a non-identity element $b\in N$ which commutes with $a_1$, then since $A$ is nilpotent, $p\nmid |a_1|$, $\langle a_1b, a_2, \cdots, a_k\rangle=\langle (a_1b)^p, (a_1b)^{|a_1|}, a_2, \cdots, a_k\rangle=\langle a_1^p, b^{|a_1|}, a_2, \cdots, a_k\rangle$ contains $A$ properly. Whence $\langle a_1b, a_2, \cdots, a_k\rangle=G$.

Suppose that all non-identity elements of $N$ cannot commute with $a_1$. Take $1\neq b\in N$. Consider the group $H=\langle a_1, ba_2, \cdots, a_k\rangle$. Clearly $HN\geqslant AN$, so $NH=G$. If $H<G$, tnen by Principal Lemma 5), $H$ is conjugate to $A$. $H$ must be nilpotent. Let the class of $H$ be $c$. We must have

$$[ba_2, \overbrace{a_1, a_1, \cdots, a_1}^{c}]=1.$$

Using the formula $[ab, c]=[a, c]^b[b, c]$, we have

$$[[\cdots[[b, a_1]^{a_2}, a_1]^{[a_2, a_1]}\cdots], a_1]^{[\overbrace{a_2, a_1, \cdots, a_1}^{c}}, [a_1, \overbrace{a_1, \cdots, a_1}^{c}]=1.$$

The last commutator is 1, since the class of $A$ is $c$ too. Hence

$$[[\cdots[[b, a_1]^{a_2}, a_1]^{[a_2, a_1[\cdots}], a_1]=1.$$

We note that the commutators in successive order of the left hand side are all in $N$ and the elements of $N$ which commuts with $a_1$ are only 1. We have $[b, a_1]=1$ ultimately. This is a contradiction.

**Theorem 2.2.** *A solvable outer-$c(k)$ group $G$ is either an outer-nilpotent group or a $p$-group. If $G$ is a $p$-group, then $\Gamma_k(G)$ is of order $p$ and $Z(G)$ is cyclic.*

*Proof* If $G$ is not nilpotent, then $G$ is an outer-nilpotent group. Now suppose that $G$ is nilpotent and $G=P_1\times\cdots\times P_r$ to be the direct product of its Sylow subgroups. If $r>1$, then the class of each $P_i$ is less than $k$, since $P_i\simeq G/\prod_{j=1} P_j$. Hence the class of $G$ is also less than $k$, which is contrary to that the class of $G$ is not less than $k$. Thus $G$ is a $p$-group. Suppose that $M$ is any normal subgroup of $G$. Then the class of $G/M$ is less than $k$. Hence $M\geqslant\Gamma_k(G)$. Thus we have shown that any non-trivial normal subgroup of $G$ contains $\Gamma_k(G)$ and so $|\Gamma_k(G)|=p$. From this, $Z(G)$ has a unique subgroup of order $p$(that is $\Gamma_k(G)$). Therefore $Z(G)$ is cyclic.

Set $k=2$, the conclusions to solvable outer-abelian groups are derived (see [2, 3]).

**Theorem 2.3.** *A solvable outer-abelian group $G$ is either a $p$-group with cyclic center and $|G'|=p$, or a Frobenius group with cyclic complement $A$ and elementary abelian kernel $N$ and if $|N|=p^\alpha$, then the exponent of $p$* (mod $|A|$) *is $\alpha$.*

*Proof* The preceding case of this theorem is a special case of Theorem 2.2. Here $G=AN$. Let $g\in G\backslash A$, we prove $A\cap A^g=1$. Let $A\cap A^g=D$. Then $D\lhd\langle A, A^g\rangle$. By the maximality of $A$ and the uniqueness of $N$, $D=1$ or $A$. If $D=A$, then $A=A^g$. Thus $g\in N_G(A)=A$. This is a contradiction. Hence $D=1$.

The last conclusion comes from Principal Lemma, 8).

**Theorem 2.4.** *Suppose that $G$ is solvable and every Sylow subgroup of $G$ is generated by $k(>1)$ elements. Then $G$ is nilpotent if and only if each $k$-generator factor group of $G$ is nilpotent.*

*Proof* Our theorem follows from Theorem 2.1, 3).

**Theorem 2.5.** *Inner-$c(k)$ group $G$ is generated by $k$ elements.*

*Proof* There exists elements $a_1, \cdots, a_k\in G$, such that $[a_1, \cdots, a_k]\neq 1$, since the class of $G$ is not less than $k$. Then the class of group $\langle a_1, \cdots, a_k\rangle$ is not less than $k$. It can not be a proper subgroup of $G$, whence $G=\langle a_1, \cdots, a_k\rangle$.

## § 3. $\Gamma_k$-$pn$ Groups and $p$-Metanilpotent Groups

From Lemma 1.2, the inner-$\Sigma$ groups with $\Phi(G)=1$ are the minimal non-$\Sigma$ groups when $\Sigma$ is nilpotency, supersolvability, $p$-nilpotency or $p$-supersolvability. They were studied by predecessors (see [7, 8, 9, 10]). Now we are discussing broader group classes.

**Definition 3.1.** *Group $G$ is called $\Gamma_k$-$pn$ group, if $\Gamma_k(G)$ is $p$-nilpotent. Let $\Gamma(G)=\bigcap_k \Gamma_k(G)$. $G$ is called $p$-metanilpolent, if $\Gamma(G)$ is $p$-nilpotent.*

Evidently, Properties "$\Gamma_k$-$pn$" and "$p$-metanilpotent" are preserved for subgroups and factor groups and satisfy the condition ($\triangle$).

$p$-supersolvable groups are $\Gamma_2$-$pn$ groups; $\Gamma_k$-$pn$ groups are $p$-solvable groups.

**Theorem 3.1.** *$G$ is a $\Gamma_k$-$pn$ group if and only if $G/\Phi(G)$ is a $\Gamma_k$-$pn$ group.*

*Proof* The necessity is clear. We only need to prove the sufficiency.

From $\Gamma_k(G/\Phi(G)) = \Gamma_k(G)\Phi(G)/\Phi(G)$, we have $\Phi(G)\leqslant \Gamma_k(G)\Phi(G)\lhd G$. $\Gamma_k(G)\Phi(G)/\Phi(G)$ is $p$-nilpotent by the hypotheses of the theorem. From [15] VI, 6.3, $\Gamma_k(G)\Phi(G)$ is $p$-nilpotent and so is $\Gamma_k(G)$.

**Theorem 3.2.** *Suppose that $G$ is a $p$-solvable outer-$\Gamma_k$-$pn$ group. Then $G$ is a group described in the principal lemma. Furthermore, $A$ is a $\Gamma_k$-$pn$ group and $\alpha>1$. If $G$ is a $p$-solvable minimal non-$\Gamma_k$-$pn$ group, $k\geqslant 2$, then $A$ is an inner-$c(k)$ group; If $p\,|\,|A|$, then $p\,|\,|A|$, $A$ is an inner-$c(k)$ group of order $pq^\beta$ and the Sylow $q$-subgroup of $A$ is normal.*

*Proof* Suppose $G$ is an outer-$\Gamma_k$-$pn$ group. By Theorem 3.1, property "$\Gamma_k$-$pn$" satisfies condition (*), whence $\Phi(G)=1$. If $O_{p'}(G)\neq 1$, then $G/O_{p'}(G)$ is a $\Gamma_k$-$pn$ group. Thus $\Gamma_k(G/O_{p'}(G))=\Gamma_k(G)O_{p'}(G)/O_{p'}(G)$ is $p$-nilpotent. Hence $\Gamma_k(G)O_{p'}(G)$ is $p$-nilpotent and so is $\Gamma_k(G)$. This is contrary to that $G$ is not a $\Gamma_k$-$pn$ group. Then $O_p(G)\neq 1$, since $G$ is $p$-solvable. $G$ has non-trivial normal

$p$-subgroups. By Lemma 1.3, $G$ is a group described in the principal lemma. By Principal Lemma 7), $\Gamma_k(G)=\Gamma_k(A)N$. $\Gamma_k(A)$ is not a $p$-group, since $\Gamma_k(G)$ is not $p$-nilpotent.

If $\alpha=1$, then $N$ is cyclic group of order $p$. From $C_G(N)=N$, we have

$N_G(N)/C_G(N)=G/N\simeq A\simeq a$ subgroup of Aut$(N)$. Hence $G/N$ is abelian, $\Gamma_k(G)\leqslant N(k\geqslant 2)$. Whence $\Gamma_k(G)$ is $p$-nilpotent, and it is a contradiction. Hence $\alpha>1$.

Suppose that $G$ is a minimal non-$\Gamma_k$-$pn$ group. Let $B$ be any proper subgroup of $A$. Then $H=BN<G$, whence $\Gamma_k(H)$ is $p$-nilpotent. Let $K$ be the normal $p$-complement of $\Gamma_k(H)$, then $K\trianglelefteq H$, whence $KN=K\times N$. Hence $K=1$, and so $\Gamma_k(H)$ is a $p$-group. Here

$H/\Gamma_k(H)\geqslant B\Gamma_k(H)/\Gamma_k(H)\simeq B/\Gamma_k(H)\cap B$. If $p\nmid |B|$, then $\Gamma_k(H)\cap B=1$ and $B$ is a $c(k)$ group. If $p\mid |B|$, then $p\mid |A|$, Let $C$ be a $p$-complement of $A$. $C$ is a $c(k)$ group, since $C<A$. By [11], IX. 2.e, $A$ is solvable and so is $G$. Hence $\Gamma_k(G)<G$, when $k\geqslant 2$. By hypothesis, $\Gamma_k(G)$ is a $\Gamma_k$-$pn$ group. $\Gamma_k(\Gamma_k(G))$ is $p$-nilpotent. By Principal Lemma 7), $\Gamma_k(G)=\Gamma_k(A)N$, $\Gamma_k(A)$ is a $c(k)$ group and $p\nmid|\Gamma_k(A)|$.From $A/\Gamma_k(A)$ being a $c(k)$ group, $A$ is $p$-nilpotent. Thus $A$ has a normal subgroup $A_1$ such that $|A: A_1|=p$, $A_1N\triangleleft G$. By Principal Lemma 7) again, $A_1$ is a $c(k)$ group and $p\nmid|A_1|$. Therefore $p\| |A|$ and so $p\| |B|$. If $\Gamma_k(H)\cap B=P\neq 1$, then $P$ is of order $p$ and $P$ is a Sylow $p$-subgroup of $B$. Let $B_1$ be the normal $p$-complement of $B$. Then $B=P\times B_1$. We have proved that $B_1$ is a $c(k)$ group. Thus $B$ is a $c(k)$ group in any case.

Now prove that $A$ is not a $c(k)$ group. If A is a $c(k)$ group then by Principal Lemma 7), $\Gamma_k(G)=\Gamma_k(A)N=N$. Whence $G$ is *a* $\Gamma_k$-$pn$ group, and it is a contradiction. Thus we have shown that $A$ is an inner-$c(k)$ group and $p\mid |A|$, when $p\mid |A|$. By Principal Lemma 3), $A$ is an inner-$c(k)$ group of order $pq^\beta$ and the Sylow $q$-subgroup of $A$ is normal.

**Corollary 3.1.** *$p$-solvable minimal non-$\Gamma_k$-$pn$ groups are solvable.*

**Corollary 3.2.** *The orner of a $p$-solvable inner-$\Gamma_k$-$pn$ group contains at most three distinct prime factors.*

*Proof* The distinct prime factors of the orders of $G$ and $G/\Phi(G)$ are the same. By Theorem 3.1 and Lemma 1.2, $G/\Phi(G)$ is a minimal non-$\Gamma_k$-$pn$ group. An inner-$c(k)$ group contains at most two distinct prime factors (Theorem 2.4). By Theorem 3.3, the order of $G/\Phi\ (G)$ contains at most three distinct prime factors.

**Theorem 3.3.** *$p$-solvable inner-$\Gamma_k$-$pn$, $k\geqslant 2$, group $G=AN$ is generated by $k$ elements. When $A$ is not nilpotent, $G$ is generated by two elements.*

*Proof* We may suppose that $G$ is a minimal non-$\Gamma_k$-$pn$ group, since $G$ and $G/\Phi(G)$ have the same number of generators.

If $A$ is nilpotent, then by Theorem 2.1, 3) and Theorem 2.5, $G$ is generated by $k$ elements. If $A$ is not nilpotent, then $A$ is an inner-nilpotent group.

1) $|A|=r^\gamma q^\beta$, $|N|=p^\alpha$, $p$, $q$, $r$ are distinct primes.

We have known that $A=\langle c,b\rangle$ and $\langle c\rangle=R$ is a Sylow $r$-subgroup of $A$; $b$ is any generator of the Sylow $q$-subgroup $Q$ of $A$. If $C_G(c)$ contains an element $a\neq 1$ of $N$, then $\langle ca, b\rangle=\langle c, a, b\rangle$ contains $Q$ properly and so $\langle ca, b\rangle=G$. Theorem is proved. Suppose $C_G(c)\cap N=1$. Let $H=\langle c, ba\rangle$, where $b$ is any generator of $Q$, $1\neq a\in N$. Clearly $HN=G$. Let $H\cap N=D$, then $D\lhd\langle H, N\rangle=G$. By the minimality of $N$, $D=N$ or 1. If $D=N$, then $H\geqslant N$ and $H=HN=G$. If $D=1$, then $H$ is conjugate to $A=RQ$. Hence there exists $g\in G$, such that $H^g=\langle c^g, (ba)^g\rangle=RQ$. From $(ba)^{q^\beta}=b^{q^\beta}a'=a'\in N$ and $H\cap N=1$, $a'=1$. The order of $ba$ is a power of $q$, whence $(ba)^g\in Q$. Here $c$, $c^g\in A$, $c$ conjugate to $c^g$ in $A$. Hence there exists $b_1\in Q$, such that $c^{b_1}=c^g$. Then $gb_1^{-1}\in C_G(c)$. From $C_G(N)\cap N=1$, $C_G(c)\leqslant A$. We have $gb_1^{-1}\in A$, whence $g\in Ab_1=A$. Here $(ba)^g\in Q$. Since $Q\lhd A$, $ba\in Q$, this is contrary to $a\overline{\in} Q$.

2) $A=pq^\beta$, $A=RQ$, $G=RQN$, $P=RN$ is a Sylow $p$-subgroup of $G$. If the exponent of $P$ is $p$, then $P$ is a regular $p$-group. $N_G(P)=P$, since $C_G(N)=N$. From Wielandt Theorem [15, IV, 8.1], $G$ has a normal $p$-complement $Q$, and it is a contradiction. Therefore $P$ must have an element $ca$ of order $p^\alpha$, where $c\in R$, $a\in N$ and so $1\neq(ca)^p=a'\in N$. Let $H=\langle ca, b\rangle$, $b$ is a generator of $Q$. Evidently $HN=G$. $1\neq H\cap N=D\lhd\langle H, N\rangle=G$, since $a'\in H$. From the minimality of $N$, $D=N$. Then $G=HN=H$. This proves the theoaem.

**Corollary 3.3.** *$p$-solvable inner-$\Gamma_2$-pn group is generated by two elements.*

**Corollary 3.4.** *$p$-solvable inner-$p$-metanilpotent group is generated by two elemets.*

**Theorem 3.4.** *If $\Gamma_k(H)$ of every $k(k\geqslant 2)$ generator subgroup $H$ of a $p$-solvable group $G$ is $p$-nilpotent, then $\Gamma_k(G)$ is $p$-nilpotent. If the derived subgroup of every two generator subgroup of a group $G$ is nilpotent, then $G'$ is nilpotent.*

*Proof* The first result is an immediate consequence of Corollary 3.3. In order to prove the second result, we need only to prove that $G$ is solvable. By induction, we have the derived snbgroup of every proper subgroup $H$ is nilpotent and so $H$ is solvable. If $G$ is not solvable, then $G$ is inner-solvable and $G/\Phi(G)$ is a minimal simple group. All minimal simple groups are generated by two elements. Hence $G$ is generated by two elements and so $G'$ is nilpotent by the hypotheses. It is a contradiction. Therefore $G$ is solvable, which proves the theorem.

Analogy to the proof of Theorem 3.4, we can prove

**Theorem 3.5.** *If every two generator subgroup of a $p$-solvable group $G$ is $p$-metanilpolent, then so is $G$.*

[13, Th. 1.47] shows that every simple group can be generated by two elements. Using this result and Lemma 1.4, the hypotheses "$p$-solvable" of Corollaay 3.3, 3.4

and Theorem 3.4, 3.5 may be omitted.

**Theorem 3.6.** *$G$ is a $\Gamma_k$-pn group if and only if $G$ is p-solvable and $\Gamma_k(N_G(P)/C_G(P))$ is a p-group for every p-subgroup $P \trianglelefteq S_p$, where $S_p$ is a Sylow p-subgroup of $G$.*

*Proof* We have

$$\Gamma_k\left(\frac{N_G(P)}{C_G(P)}\right)=\frac{\Gamma_k(N_G(P))C_G(P)}{C_G(P)}\simeq\frac{\Gamma_k(N_G(P))}{C_G(P)\cap\Gamma_k(N_G(P))}.$$

If $G$ is a $\Gamma_k$-pn group, then so is $N_G(P)$. Whence $\Gamma_k(N_G(P))$ is p-nilpotent. Let $K$ be its normal p-complement. $K \trianglelefteq N_G(P)$, since $K$ is characteristic in $\Gamma_k(N_G(P))$. Hence $PK=P\times K$ and so $K\leqslant C_G(P)$. $K\leqslant C_G(P)\cap\Gamma_k(N_G(P)$. Thus $\Gamma_k(N_G(P))/C_G(P)\cap\Gamma_k(N_G(P)$ is a p-group and so is $\Gamma_k(N_G(P)/C_G(P))$. The necessity is proved.

To prove the sufficiency. The minimal normal subgroup $N$ of $G$ is a $p'$-group, or a p-group, since $G$ is p-solvable. Let $\bar{G}=G/N$. It is easy to show that $N_{\bar{G}}(\bar{P})=N_G(P)N/N$ for any p-subgroup $P$, where $\bar{P}=PN/N$, when $N$ is a $p'$-group, $\bar{P}=P/N$ when $N$ is a p-group. Clearly, $C_{\bar{G}}(\bar{P})\geqslant C_G(P)N/N$.Thus.

$$\frac{N_G(P)}{C_G(P)}\sim\frac{N_G(P)}{C_G(P)(N_G(P)\cap N)}\simeq\frac{N_G(P)N/N}{C_G(P)N/N}\sim\frac{N_{\bar{G}}(\bar{P})}{C_{\bar{G}}(\bar{P})}.$$

Suppose $\bar{p}\trianglelefteq\bar{S}_p$. Then $P\trianglelefteq S_p$, when $N$ is a p-group; $PN\trianglelefteq S_pN$, when $N$ is a $p'$-group. By Frattini argument $S_pN=N_{S_pN}(P)\cdot N$. Hence $N_{S_pN}N(P)$ contains a conjugate of $S_p$. Thus $P$ is normal in certain Sylow p-subgroup. We may suppose $P\trianglelefteq S_p$. By hypothesis, $\Gamma_k(N_G(P)/C_G(P))$ is a p-group, and so is its homomorphic image $\Gamma_k(N_{\bar{G}}(\bar{P})/C_{\bar{G}}(\bar{P}))$.By induction, $G/N$ is *a* $\Gamma_k$-pn group. Thus we have shown that the factor groups of $G$ for any minimal normal subgroups are $\Gamma_k$-pn groups. Every proper factor group of $G$ is a $\Gamma_k$-pn group, since property "$\Gamma_k$-pn" is preserved for factor groups. If $G$ is not *a* $\Gamma_k$-pn group, then $G$ is an outer-$\Gamma_k$-pn group. By Theorem 3.2, $G=AN$, $A\cap N=1$; $N$ is a normal subgroup; $\Gamma_k(A)$ is not a p-group. Since $N_G(N)=G$, $C_G(N)=N$, then $N_G(N)/C_G(N)\simeq A$. But $\Gamma_k(A)$ is not a p-group, and it is contrary to the hypothesis.

The theorems mentioned above may be stated for p-metanilpotent groups.

## § 4. p-Supersolvable Groups

**Theorem 4.1.** *p-solvable outer-supersolvable group $G$ is a group described in the prinlipal lemma. Furthermore $A$ is p-supersolvable and $\alpha>1$. When $G'$ is p-nilpotent, $A$ can not decomposed into $A=A_1A_2$ such that $A_i$ $N$ is p-supersolvable, $i=1, 2$.*

*Proof* Similar to the proof of Theorem 3.2, we may prove that $G$ is a group described in the Principal Lemma. $A$ is p-supersolvable and $\alpha>1$. Then $G'$ is p-nilpotent, $A$ is cyclic by Principal Lemma 8). If $A_iN$ is p-supersolvable, then $A_iN$

has a normal subgroup $P$ of order $p$. By Principal Lemma 8), the non-identity elements of $A$ act $N$ fixed-point-free and so does $A_i$, $i=1, 2$. Hence $C_{A_iN}(P)=N$, $i=1, 2$. Whence $N_{A_iN}(P)/C_{A_iN}(P)=A_iN/N\simeq A_i\simeq$ a subgroup of Aut$(P)$ and so $|A_i|\,|p-1$, $i=1, 2$. $|A|=[|A_1|, |A_2|]$, since $A$ is cyclic. Thus $|A|\,|p-1$ and the exponent of $p$ (mod $|A|$) is 1. This is contrary to $\alpha>1$.

**Theorem 4.2.** *$p$-solvable inner-$p$-supersolvable groups are generated by two elements.*

*Proof* We may suppose that $G$ is minimal non-supersolvable, $G=AN$. If $G'$ is $p$-nilpotent, then by Theoaem 4.1 and Principal Lemma 8), $A$ is cyclic, while $G$ is generated by two elements. If $G'$ is not $p$-nilpotent, then $G$ is an inner-$\Gamma_2$-$pn$ group by [15, VI. 9.1]. From Corollary 3.3, $G$ is generated by two elements.

**Theorem 4.3.** *If every two generator subgroup of a $p$-solvable group $G$ is $p$-supersolvable, then so is $G$.*

*Proof* The theorem is corollary of Theorem 4.2.

**Theorem 4.4**[14, 15]. *Suppose that $G=G_1G_2$, where $G_i$ is a normal $p$-supersolvable subgroup of $G$, $i=1, 2$. Then $G$ is $p$-supersolvable if and only if the commutator group $[G_1, G_2]$ is $p$-nilpotent.*

*Proof* The necessity of this theorem is clear. To prove the sufficiency. Here $G'=[G_1G_2, G_1G_2]=G'[G_1, G_2]\ G_2'$([11]VI, 1.1). $G'$ is a product of mormal $p$-nilpotent groups and so $G$ is $p$-nilpotent. The conditions of this theorem are preserved for factor groups. If $G$ is not $p$-supersolvable, then $G$ is an outer-$p$-supersolvable group. By Theorem 4.1 and Principal Lemma 8), $G=AN$, $A$ is cyclic. By Principal Lemma 7), $G=(G\cap A)N$, $G_i\cap A=A_i$, $i=1, 2$. Evidently $A=A_1A_2$, contrary to Theorem 4.1. Hence $G$ is $p$-supersolvable.

**Theorem 4.5**[16]. *Suppose that $G$ is $p$-solvable, or $p$ is the smallest prime factor of $|G|$. If each maximal subgroup of every Sylow $p$-subgroup is normal in $G$, then $G$ is $p$-supersolvable.*

*Proof* Let $P$ be the maximal subgroup of a Sylow $p$-subgroups $S_p$ of $G$. Then $p\|\,|G/P|$. $G/P$ possesses cyclic $p$-subgroup. When $p$ is the smallest prime factor of $|G|$, $G/P$ has a normal $p$-complement and so $G$ is $p$-solvable. Thus $G$ is $p$-solvable in any case. Evidently the conditions of this theorem are preserved for factor groups. If $G$ is not $p$-supersolvable, then $G$ is a $p$-solvable outer-$p$-supersolvable group. By Theorem 4.1, $G=AN$, $O_p(G)=N$. Hence $N=P$. Therefore $P$ is the unique maximal subgroup of $S_p$ and so $\Phi(S_p)=P$. Moreover $|S_p:P|=p$, $S_p/\Phi(S_p)$ is cyclic and so is $S_p$. Thus $N$ is cyclic, $|N|=p$, contrary to $\alpha>1$. Hence $G$ is $p$-supersolvable.

**Theorem 4.6.** *Suppose that $G$ is a $p$-solvable group. If the numbers of indices divided by pin the index series of all maximal chains of subgroups of $G$ are coincident, then $G$ is $p$-supersolvable.*

*Proof* By induction, every proper subgroup of $G$ is $p$-supersolvable. If $G$ is not $p$-supersolvable, the $G$ is inner-$p$-supersolvable. $G/\Phi(G)=AN$ is minimal non-$p$-supersolvable, $|N|=p^{\alpha}$, $\alpha>1$. $A$ is a maximal subgroup of $G/\Phi(G)$. Then the inverse image of $A$ is a maximal subgroup of $G$. Its index in $G$ is $p^{\alpha}$, $\alpha>1$. But $G$ is solvable. The composition series of $G$ is a maximal chain of subgroups. Hence $G$ have two maximal chains. The numbers of indices divided by $p$ of them are distinct. It is contrary to hypotheses.

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