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A FINITE CONFORMAL-ELIMINATION FREE ALGORITHM OVER ORIENTED MATROID PROGRAMMING*

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Abstract

This paper presents a new finite pivoting method for oriented programming which works principally on the smallest subscript rule appealing to no process of conformal elimination. In particular, when the oriented matroid programming under consideration is a linear programming, the process of conformal elimination is just the process of minimum ratio test.

An oriented matroid M (1) is a matroid that, for every circuit of it, a partition (C', C'') of $C(C'\cup C''=C, C'\cap C''=\emptyset)$ has been defined, and all these partitions satisfy the following Elimination Axiom. (C', C'') is called the oriented partition of C, and C', C'' are called the oriented parts of C.

Elimination Axiom: Let C_1 and C_2 be any two given circuits of M. Denote (C_1', C_1'') or (C_1'', C_1') by (C_1^+, C_1^-) and denote (C_2', C_2'') or (C_2'', C_2') by (C_2^+, C_2^-). Then, for

$$e\in (C_1^+\cap C_2^-)\cup(C_1^-\cap C_2^+),$$
$$e'\in (C_1\cup C_2)\setminus((C_1^+\cap C_2^-)\cup(C_1^-\cap C_2^+)),$$

there exists a circuit C_3 of M which has the property

1) $e\notin C_3$, $e'\in C_3$,

2) one can denote one of C_3' and C_3'' by C_3^+ and the other by C_3^- such that $C_3^+\subseteq C_1^+\cup C_2^+$, $C_3^-\subseteq C_1^-\cup C_2^-$.

Graphic Matroids induced by digraphs and Matrix-representable Matroids are all examples of oriented matroids.

Any oriented matroid M has a property called Conformal Eliminability, that is: Let C_1 and C_2 be any two given circuits of M, Denote (C_1', C_1'') or (C_1'', C_1') by (C_1^+, C_1^-) and denote (C_2', C_2'') or (C_2'', C_2') by (C_2^+, C_2^-). Then, for

$$e'\in (C_1\cup C_2)\setminus((C_1^+\cap C_2^-)\cup(C_1^-\cap C_2^+)),$$

there exist an element $e\in (C_1^+\cap C_2^-)\cup(C_1^-\cap C_2^+)$ and a circuit C_3 of M such that

1) $e\notin C_3$, $e'\in C_3$,

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2) One can denote one of C'_3 and C''_3 by C_3^+ and the other by C_3^- such that

$$(C_3^+ \cap C_1) \subseteq C_1^+,\ (C_3^- \cap C_1) \subseteq C_1^-,\ C_2^+ \cap (C_2 \backslash C_1) \subseteq C_2^+,\ C_3^- \cap (C_2 \backslash C_1) \subseteq C_2^-.$$

Let M and M^* be a pair of matroids dual to each other. If M is an oriented matroid, then M^* can be uniquely defined as an oriented matroid such that M^* and M are orthogonal, that is: any circuit C of M and any circuit D of M^* are orthogonal. Denote (C', C'') or $(C'', C;)$ by (C^+, C^-), and denote (D', D'') or (D'', D') by (D^+, D^-). Then $(C^+ \cup D^+) \cap (C^- \cap D^-) \neq \emptyset$. if and only if $(C^+ \cap D^-) \cup (C^- \subset D^+) \neq \emptyset$.

Oriented matroid duality is an abstract combinatorial setting of the linear programming duality. Let M and M^* be a pair of oriented matroids dual to each other, E be their ground set, b and b^* be two different elements chosen from E. A circuit C of M is called a (non-negative) feasible circuit of M if

1) $b \in C$,

2) $C \backslash (b^*)$ is contained in one of the oriented parts of C.

A circuit D of M^* is called a (non-negative) feasible circuit of M^* if

1) $b^* \in D$,

2) $D \backslash (b)$ is contained in one of the oriented parts of D.

C and D are called complementary, if

$$(C \backslash (b^*)) \cap (D \backslash (b)) = \emptyset.$$

The oriented matroid duality theorem can be briefly presented as: If both M and M^* have got a feasible circuit of their own, then there are two complementary feasible circuits of M and M^*. The above mentioned duality theorem was first proved non-constructively by Lawrence, and it can also be proved constructively by a number of finite algorithms for oriented matroid programming which were developed by, say, Bland (2), Fukuda (3) and Todd (4). A common feature of these algorithms is that their pivots appeal strongly to the conformal elimination process, that is, the process of realizing the conformal eliminability in the oriented matroid under consideration. (In case that the oriented matroid programming concerned is particularly a linear programming, the conformal elimination process turns out the process of minimum ratio test in linear programming).

This paper aims at providing an oriented matroid duality theorem which is more detailed in content than the one mentioned above, and the algorithm given in this paper for proving constructively this theorem is of primal-dual type. Its pivoting works purely on smallest subscript rule which involves no process of conformal elimination type.

Definition. *Let M and M^* be a pair of oriented matroids dual to each other, E their ground set, b, $b^* \in E (b \neq b^*)$, $E \backslash (b, b^*) = P_1 \cup P_2 \cup P_3 \cup P_4$, $P_i \neq P_j$ $(i \neq j)$.*

A circuitc of M is called a feasible ciruit of M, if

(1) $b \in C$,

(2) $C\cap P_3=\varnothing$,

(3) $C\cap P_1$ *and* b *are in the same orientedpart of* C,

(4) $C\cap P_2$ *and* b *are in the different oriented parts of* C.

An circuit D *of* M^* *is called a feasible circuit of* M^*, *if*

(1) $b^*\in D$,

(2) $D\cap P_4=\varnothing$,

(3) $D\cap P_1$ *and* b^* *are in the same oriented parts of* D,

(4) $D\cap P_2$ *and* b^* *are in the different oriented parts of* D.

A circuit $\widetilde{C}$ *of* M *is called an infinitely -augmenting circuit of* M, *if*

(1) $b\cap\widetilde{C}$, $b^*\in\widetilde{C}$,

(2) $\widetilde{C}\cap P_3=\varnothing$,

(3) $\widetilde{C}\cap P_1$ *and* b^* *are in the same oriented part of* $\widetilde{C}$,

(4) $\widetilde{C}\cap P_2$ *and* b^* *are in the different oriented parts of* $\widetilde{C}$.

A circuit $\widetilde{D}$ *of* M^* *is called an infinitely -augmenting circuit of* M^*, *if*

(1) $b^*\in\widetilde{D}$, $b\in\widetilde{D}$,

(2) $\widetilde{D}\cap P_4=\varnothing$,

(3) $\widetilde{D}\cap P_1$ *and* b^* *are in the same oriented part of* $\widetilde{D}$,

(4) $\widetilde{D}\cap P_2$ *and* b^* *are in the different oriented parts of* $\widetilde{D}$.

Because the circuit family of M and the circuit family of M^* are orthogonal, it is impossible that there can exist, at the same time, a feasible circuit of $M(M^*)$ and an infinitely-augmenting circuit of M^* (M).

Theorem. *the algorithm given below results in making one (and only one) of the following four cases hold.*

(I) *A feasible circuit of* C *of* M *and a feasible circuit* D *of* M *are found simultaneously;* C *and* D *are complementary, that is*

$$C\cap D\cap P_1=\varnothing,\ C\cap D\cap P_2=\varnothing.$$

(II) *An infinitely -augmenting circuit* $\widetilde{C}$ *of* M *and a feasible circuit* C *of* M *are found successively.*

(III) *An infintely -augmenting circuit* $\widetilde{D}$ *of* M^* *and a feasible circuit* D *of* M^* *are found successively.*

(IV) *An infinitely -augmenting circuit* $\widetilde{C}$ *of* M *and an infinitely-augmenting circuit* $\widetilde{D}$ *of* M^* *are found respectively.*

(When $P_2=P_3=P_4=\varnothing$, the above theorem becomes virtuely the original version of the duality theorem over oriented matroids).

Algorithm

Let $E\backslash(b, a^*)=(e_1, e_2, \cdots, e_n)$, and B, B^* denote a pair of complemented bases (that is $B^*=E\backslash B$) of M and M^*. For $e_i\notin B$, $e_j\notin B^*$ let $C_i(B)$ and $D_j(B^*)$ denote respectively tne fundamental circuit of M decided by e_i and B and the fundamental circuit of M^* decided by e_j and B^*.

Step 1

1.1 If (b^*) and (b) are respectively circuits of M and M^*, set $\widetilde{C}:=(b^*)$,$\widetilde{D}:=(b)$ and stop (hence, case IV holds).

1.2 If (b^*) is a circuit of M and (b) is not a circuit of M^*, let b be contained in a base B^* of M^*, set $\widetilde{C}:=(b^*)$ and go to step 2.

1.3 If (b) is a circuit of M^* and (b^*) is not a circuit of M, let b^* be contained in a base B of M, set $\widetilde{D}:=(b)$ and go to step 2.

1.4 If (b^*) and (b^*) are not respectively circuits of M and M^*, let b^* and b be contained resepctively in a base B of M and a base B^* of M^*, and go to step 2.

Step 2

2.1 If $\exists e_j\in B\cap P_3$ such that $\exists e_i\in D_j(B^*)\cap(P_1\cup P_2\cup P_4)$, or, $\exists e_i\in B^*\cap P_4$ such that $\exists\, e_j\in C_i(B)\cap(P_1\cap P_2\cap P_3)$, then set $B:=(B\backslash(e_j))\cup(e_i)$ (or, equivalently, $B^*:=(B^*\backslash(e_i))\cup(e_j))$ and go to 2.1.

2.2 If (b^*) is a circuit of M and (b) is not a circuit of M^*, now let C denote the fundamental circuit of M decided by b and B. If $C\cap P_3\neq\varnothing$, then take $e_j\in C\cap P_3$, set $\widetilde{D}:=D_j(B^*)$ and stop (hence, case IV holds); if $C\cap P_3=\varnothing$, then go to step 3.

2.3 If (b) is a circuit of M^* and (b^*) is not a circuit of M, now let D denote the fundamental circuit of M^* decided by b^* and B^*. If $D\cap P_4\neq\varnothing$, then take $e_i\in D\cap P_4$ and set $\widetilde{C}:=C_i(B)$, stop (hence, case IV holds); if $D\cap P_4=\varnothing$, then go to step 4.

2.4 If (b^*) and (b) are not respectively circuits of M and M^*, now let C denote the fundamental circuit of M decided by b and B, and D denote the fundamental circuit of M^* decided by b^* and B^*.

2.4.1 If $C\cap P_3\neq\varnothing$, $D\cap P_4\neq\varnothing$, then let $e_j\in C\cap P_3$, $e_i\in D\cap P_4$, set $\widetilde{C}:=C_i(B)$, $\widetilde{D}:=D_j(B^*)$, and stop (hence, case IV holds).

2.4.2 If $C\cap P_3=\varnothing$ D, $P_4\neq\varnothing$, then let $e_i\in D\cap P_4$, set $C:=C_i(B)$, and go to step 3.

2.4.3 If $C\cap P_3\neq\varnothing$, $D\cap P_4=\varnothing$, then let $e_j\in C\cap P_3$, set $\widetilde{D}:=D_j(B^*)$, and go to step 4.

2.4.4 If $C\cap P_3=\varnothing$, $D\cap P_4$, then go to step 5.

Step 3 Let C^+ denote the oriented part of C which contains b, and C^- the other oriented part of C. Now, let $K=(i\,|\,e_i\in(P_1\cap C^-)\cup(P_2\cap C^+))$. If $K=\varnothing$, then stop (hence, case II holds).

Otherwise, let $k=\mathrm{Min}(i\,|\,i\in K)$, and let D_k^+ denote the oriented part of $D_k(B^*)$ which contains b, and D_k^- the other oriented part of $D_k(B^*)$. Now let $L'=(i\,|\,e_i\in (P_1\cap D_k^-)\cup(P_2\cap D_k^+))$. If $L'=\varnothing$, then set $\widetilde{D}:=D_k(B^*)$, and stop (hence, case IV holds); Otherwise, let $L=\mathrm{Min}(i\,|\,i\in L')$, set $B^*:=(B^*\backslash(e_L))\cup(e_k)$ (in this case, e_k and

e_L are said respectively to have moved out of B "actively" and B^* "Passively"), and go to step 3.

Step 4 Let D^+ denote the oriented part of D which contains b^*, and D^- the other oriented part of D. Now, let $K=(i|e_i\in(P_1\cap D^-)\cup(P_2\cap D^+))$. If $K=\varnothing$, then stop (hence, case III holds).

Otherwise, let $k=\mathrm{Min}(i|i\in K)$ and let C_k^+ denote the oriented part of $C_k(B)$ which contains b^*, and C_k^- the other oriented part of $C_k(B)$. Now let $L'=(i|e_i\in(P_1\cap C_k^-)\cup(P_2\cap C_k^+))$. If $L'=\varnothing$, then set $\widetilde{C}:=C_k(B)$, and stop (hence, case IV holds); Otherwise, let $L=\mathrm{Min}(i|i\in L')$, set $B:=(B\backslash(e_L))\cup(e_k)$ (in this case, e_k and e_L are said respectively to have moved out of B^* "actively" and B "passively", and go to step 4.

Step 5 Let C^+ denote the oriented part of C which contains b, and C^- the other oriented part of C. Let D^+ denote the oriented part of D which contains b^*, and D^- the other oriented part of D. Now let $K=(i|e_i\in(P_1\cap C^-)\cup(P_2\cap C^+)\cup(P_1\cap D^-)\cup(P_2\cap D^+))$. If $k=\varnothing$, then stop (hence, case I holds); Otherwise, let $k=\mathrm{Min}(i|i\in K)$.

5.1 If $e_k\in C$, let D_k^+ denote the oriented part of $D_k(B^*)$ which contains b, and D_k^- the other oriented part of $D_k(B^*)$. Now let $L'=(i|e_i\in(P_1\cap D_k^-)\cup(P_2\cap D_k^+))$. If $K=\varnothing$, then set $\widetilde{D}:=D_k(B^*)$, and go to step 4; Otherwise, let $L=\mathrm{Min}(i|i\in L')$, set $B^*:=(B^*(e_L))(e_k)$ (in this case, e_k and e_L are said respectively to have move out of B "actively" and B^* "passively"), and go to step 5.

5.2 If $e_k\in D$, let C_k^+ denote the oriented part of $C_k(B)$ which contains b^*, and C_k^- the other oriented part of $C_k(B)$. Now, let $L'=(i|e_i\in(P_1\cap C_k^-)\cup(P_2\cap C_k^+))$. If $L'=\varnothing$, then set $\widetilde{C}:=C_k(B)$ and go to step 3; Otherwise let $L=\mathrm{Min}(i|i\in L')$, set $B:=(B\backslash(e_L))\cup(e_k)$ (in this case, e_k and e_L are said respectively to have moved out of B^* "actively" and B "passively"), and go to step 5.

Proof First, it is easy to see, due to step 2.1, that

(1) When the algorithm operates inside step 3, it always happens that $C\cap P_3=\varnothing$ and, for $e_j\in B\cap(P_1\cap P_2)$, $D_i(B^*)\cap P_4=\varnothing$;

(2) When the algorithm operates inside step 4, it always happens that $D\cap P_4=\varnothing$ and, for $e_i\in B^*\cap(P_1\cup P_2)$, $C_i(B)\cap P_3=\varnothing$;

(3) When the algorithm operates inside step 5, it always happens that

1) $C\cap P_3=\varnothing$ and, for $e_j\in B\cap(P_1\cup P_2)$, $D_j(B^*)\cap P_4=\varnothing$,

2) $D\cap P_4=\varnothing$ and, for $e_i\in B\cap(P_1\cup P_2)$, $C_i(B)\cap P_3=\varnothing$;

Therefore, if the algorithm finally stops, one of those cases I, II, III, IV must hold. So it is only the finiteness of the algorithm which is yet to be proved. It is obvious that the algorithm cannot cycle inside steps 1 and 2.

If the algorithm cycles inside step 3, then those elements of E which take part in pivoting must move not only out of some base of M "actively" but also out of some base of M "passively". Let e_g be one among them which has the largest subscript. Suppose e_g moves out of B_1 "actively" and B_2^* "passively". Now let C_1 be the fundamental circuit of M decided by b and B_1, and assume that when e_g moves out of B_2^* "passively", e_k moves out of B_2 "actively". Let $D_2 = D_k(B^*)$. Then it is not difficult to check up that C_1 and D_2 are not orthogonal. Thus a contradiction happens.

Applying the same approach as the one above, one can prove that the algorithm does not cycle inside step 4.

Now, suppose the algorithm cycles inside step 5. Let e_g be the element which has the largest subscript among those elements of E which take part in pivoting during cycling. Then three cases below may arise.

(1) e_g moves out of some base of $M(M^*)$ "actively", and some base of $M^*(M)$ "passively";

(2) e_g moves out of some base B_1 of M "passively" and some base B_2^* of M^* "passively";

(3) e_g moves out of some base B_1 of M "actively" and some base B_2 of M^* "actively".

If case (1) happens, using the same approach taken when proving non-cycling inside step 3, one can get a contradiction deduced.

If case (2) happens, assume that when e_g moves out of B_1 "passively", e_i moves out of B_1^* "actively", and when e_g moves out of B_2^* "passively", e_j moves out of B_2 "actively". Then it is not difficult to verify that $C_i(B_1)$ and $D_j(B_2)$ are not orthogonal. Thus, a contradiction happens.

Finally, if case (3) arises, let

C_1 be the fundamental circuit of M decided by b and B_1,

C_2 be the fundamental circuit of M decided by b and B_2,

D_1 be the fundamental circuit of M^* decided by b^* and B_1^*,

D_2 be the fundamental circuit of M^* decided by b^* and B_2^*.

By applying the elimination axiom of oriented matroids to C_1 and C_2, and let $e = b$, $e' = e_g$, a circuit C_3 of M can be obtained which contains e_g but not b. Because C_3 and D_1 are orthogonal, it can be derived that b^* and e_g must locate in the different oriented parts of C_3, this would then lead to a contradiction that C_3 and D_2 are not orthogonal.

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