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CERTAIN PROPERTIES OF REALIZABLE MODULES OVER THE STEE NROD ALGEBRA

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Abstract

Let Q_i, P^R be Milnor base elements of the mod p Steenrod algebra, $p>2$. $P_t^s=P^{(0,\cdots 0,p^s,0,\cdots)}$ with p^s in the t-th position, $s<t$. The present paper obtains a construction of killing P_1^0 homology group of A module M: (1) For bounded below A module M, there exists an A module $\overline{M}$ and monomorphism $f:M\to\overline{M}$ such that $H(\overline{M}, P_1^0)=0$ and $H(f, Q_i)$, $H(f, P_t^s)$, $H(f, (P_t^s)^{p-1})$ are isomorphisms for all $i\geqslant 0$, $s<t\neq 1$. (2) For bounded below spectrum X there exists a spectrum Y and a map $f:Y\to X$ such that $H(Y^*, P_1^0)=0$ and $H(f^*, Q_i)$, $H(f^*, P_t^s)$, $H(f^*, (P_t^s)^{p-1})$ are isomorphisms for all $i\geqslant 0$, $s<t\neq 1$, where Y^* is the Z_p cohomology of Y and $f^*:X^*\to Y^*$ is the A module morphism of Z_p cohomology induced by f.

Let A be mod p Steenrod algebra. An A module M is called realizable if there exists a spectrum X such that $H^*(X, Z_p)\cong M$. In the case $p=2$, [5] obtained a construction of killing the P_t^s homology groups of M in the category of A modules. This construction can also be carried out in the category of realizable A modules, hence if M is realizable then the constructed new module is also realizable, this gives a necessary condition of realizable modules.

The present paper considers the case $p>2$ and obtains partial results corresponding to the results in [5]: For P_1^0, there is a construction in the category of A modules and of realizable A modules such that P_1^0 homology groups of the constructed new module vanishes, but other P_t^s or Q_i homology groups of it remains isomorphic with homology groups of the original module.

The proofs of all results listed in the present paper had been given in details. Unless there is an outline of proof of one theorem in [5], the author had not seen any literature which states the proof of results in [5] whenever the author established the original manuscript. After the original manuscript had been established, in the end of 1984, the author saw the book[8] (published in 1983) which states the proofs of results in [5] ([8] doesn't concern the case $p>2$ in this

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problem). But unless a little revision in the proof of Theorem 1.4. this revised manuscript remains all contents of the original manuscript.

§ 1. The Main Results

In the present paper, A denotes the mod p Steenrod algebra, $p>2$. Let Q_i, P_t^s $(i\geqslant 0)$ be Milnor base elements of A, $P_t^s=P^{(0,\cdots,0,p^s,0,\cdots)}$ with p^s in the t-th position, then $(P_t^s)^p=0$, $Q_i^2=0$ for $i\geqslant 0$, $s<t$, hence there are homology groups $H\ (M, Q_i)=\ker Q_i/\operatorname{im} Q_i$ and $H(M, (P_t^s)^j)=\ker (P_t^s)^j/\operatorname{im}(P_t^s)^{p-j}$ (see [7]). These are functorial invariants, if $f:M\to N$ is an A morphism, then f induces homomorphism $H\ (f, \alpha)$: $H(M, \alpha)\to H(N, \alpha)$, where $\alpha=P_t^s$ or Q_i.

Theorem 1.1. *Let $f:M\to N$ be A morphism of bounded below A modules such that $H(f, Q_i)$ is an isomorphism for all $i\geqslant 0$, $H(f, P_t^s)$ and $H(f, (P_t^s)^{p-1})$ are isomorphisms for all $s<t$, then M, N are stably isomorphic, i. e. isomorphic up to free A modules factors.*

Theorem 1.2. *Let E be any subalgebra of the exterior algebra $E(Q_0, Q_1, Q_2, \cdots)$ or E be $Z_p[P_t^s]/((P_t^s)^p)$ and N be E module, $M=A\otimes_E N$, then $H(M, \alpha)=0$ for $\alpha\notin E$, where $\alpha=P_{t_1}^{s_1}$ or Q_i.*

Theorem 1.3. *Let M be bounded below A module, then for $\alpha=Q_0$ or P_1^0:*

(1) *There exists an A module $\overline{M}$ and monomorphism $f:M\to\overline{M}$ such that $H(\overline{M}, \alpha)=0$ and $H(f, Q_i)$, $H(f, P_t^s)$, $H(f,(P_t^s)^{p-1})$ are all isomorphisms for all Q_i or P_t^s other than α.*

(2) *If there exist A modules N_1, N_2 and monomorphisms $f_1:M\to N_1$, $f_2:M\to N_2$ satisfied the conditions of (1), then N_1, N_2 are stably isomorphic.*

We write $M(\alpha)$ for $\overline{M}$ obtained in the above theorem and $\alpha=Q_0$ or P_1^0. For spectrum X, let X^* denote the Z_p cohomology $H^*(X, Z_p)$ of X, hence X^* is an A module. For map $f:Y\to X$, $f^*:X^*\to Y^*$ is an A module morphism induced by f.

Theorem 1.4. *Let X be bounded below spectrum, then there exists a spectrum Y and a map $f:Y\to X$ such that $H(Y^*, P_1^0)=0$ and $H(f^*, Q_i)$, $H(f^*, P_t^s)$, $H(f^*, (P_t^s)^{p-1})$ are all isomorphisms for Q_i or P_t^s other than P_1^0.*

Theorem 1.5. *If M is a realizable bounded below A module, then $M(P_1^0)$ is also realizable up to stable isomorphism.*

§ 2. Preliminaries

In this paragraph, we prove some results on P_t^s homology groups of A modules. First, we consider the relation of P_t^s homology groups and Ext functor.

Theorem 2.1. *Let $E=E(\alpha)$, $\alpha=Q_i(i\geqslant 0)$ or $E=Z_p[\alpha]/(\alpha^p)$, $\alpha=P_t^s$ $(s<t)$ and*

M be E module, then

$$\mathrm{Ext}_E^{1,t}(Z_p, M) \cong H_{t+\deg\alpha}(M, \alpha^k),$$

where $k=1$ if $\alpha=Q_i$ and $k=p-1$ if $\alpha=P_t^s$. Hence E injective modules are free.

Proof If $\alpha=P_t^s$, $E=Z_p[P_t^s]/((P_t^s)^p)$, let

$$\cdots\to B_2 \xrightarrow{d_2} B_1 \xrightarrow{d_1} B_0 \xrightarrow{e} Z_p \to 0$$

be E bar resolution of Z_p (see [1]). Any cocycle $f\in \mathrm{Hom}_E^t(B_1, M)$, then

$$\delta f[\alpha^i|\alpha^j] = \alpha^i f[\alpha^j] - f[\alpha^{i+j}] = 0$$

and $f[\alpha^{i+j}] = \alpha^i f[\alpha^j]$, this means that f is uniquely determined by $f[\alpha]$ and $\alpha^{p-1} f[\alpha]=0$. Define a function

$$\mu: \mathrm{Ext}_E^{1,t}(Z_p, M) \to H_{t+\deg\alpha}(M, \alpha^{p-1})$$

such that $\mu\{f\}=\{f[\alpha]\}$. If $\{f\}=0$, then $f=\delta g$ and $f[\alpha]=\delta g[\alpha]=g\alpha[\ ]=\alpha g[\ ]\in \alpha M_t$, i. e. $\{f[\alpha]\}=0$, hence μ is well defined. μ is clearly epic. If $\{f[\alpha]\}=0$, then there is $m\in M_t$ such that $f[\alpha]=\alpha m$, let $g: B_0\to M$ be $g[\ ]=m$, then $f=\delta g$ and $\{f\}=0$, hence μ is monic. If $\alpha=Q_i$, $E=E(Q_i)$, the proof is similar.

In fact, it can be proved that E free module is E injective. But we do not state this in detail, since it is useless in this paper.

Secondly, we consider some calculations of the homology groups of A modules such as A/AQ_i.

Proposition 2.2. (1) *Let $\alpha=Q_i (i\geq 0)$, then $H(A/A\alpha, \alpha_1)=0$ for $\alpha_1\neq\alpha$, where $\alpha_1=Q_{i_1}$ or P_t^s.*

(2) *Let $\alpha=P_t^s$ $(s<t)$, then $H(A/A\alpha^j, \alpha_1)=0$ $(1\leq j\leq p-1)$ for $\alpha_1\neq\alpha$ where $\alpha_1=Q_i$ or $P_{t_1}^{s_1}$.*

Proof (1) We give proof for $\alpha_1=P_t^s$ and the proof for $\alpha_1=Q_i$ is similar. $A/A\alpha$ has a character like near simple module in [3], i.e. there is a short exact sequence

$$0\to A/A\alpha \xrightarrow{f} A \xrightarrow{g} A/A\alpha \to 0,$$

where $f(\alpha^*)=a\alpha$, $g(a)=a^*$ and $\deg f=\deg\alpha=m$, $\deg g=0$. Hence it induces a long exact sequence ([7] Prop. 3.2.)

$$\cdots\to H_{i-m}(A/A\alpha, \alpha_1) \xrightarrow{f_*} H_i(A, \alpha_1) \xrightarrow{g_*} H_i(A/A\alpha, \alpha_1)$$
$$\xrightarrow{\delta} H_{i+m_1-m}(A/A\alpha, \alpha_1^{p-1}) \to H_{i+m_1}(A, \alpha_1^{p-1}) \to \cdots,$$

where $m_1=\deg\alpha_1$. From [7], $H(A, \alpha_1)=H(A, \alpha_1^{p-1})=0$, then

$$H_i(A/A\alpha, \alpha_1) \cong H_{i+m_1-m}(A/A\alpha, \alpha_1^{p-1})$$

and the following long exact sequence

$$\cdots\to H_{i-m}(A/A\alpha, \alpha_1^{p-1}) \xrightarrow{f_*} H_i(A, \alpha_1^{p-1}) \xrightarrow{g_*} H_i(A/A\alpha, \alpha_1^{p-1})$$
$$\xrightarrow{\delta} H_{i+(p-1)m_1-m}(A/A\alpha, \alpha_1) \xrightarrow{f_*} H_{i+(p-1)m_1}(A, \alpha_1) \to \cdots$$

induces isomorphism

$$H_i(A/A\alpha, \alpha_1^{p-1}) \cong H_{i+(p-1)m_1-m}(A/A\alpha, \alpha_1),$$

hence there is an isomorphism

$$H_{i+m-m_1}(A/A\alpha,\ \alpha_1)\cong H_{i+(p-1)m_1-m}(A/A\alpha,\ \alpha_1),$$

it is easily seen that $m-m_1\neq(p-1)m_1-m$, then $H(A/A\alpha,\ \alpha_1)=0$.

(2) The short exact sequence $0\to A/A\alpha^{p-j}\xrightarrow{f}A\xrightarrow{g}A/A\alpha^j\to 0$, where $f(a^*)=a\alpha^j$, $g(a)=a^*$, $\deg f=\deg\alpha^j=jm$, $\deg g=0$, induces a long exact sequence

$$\cdots\to H_{i-jm}(A/A\alpha^{p-j},\ \alpha_1)\to H_i(A,\ \alpha_1)\to H_i(A/A\alpha^j,\ \alpha_1)$$

$$\xrightarrow{\delta}H_{i+m_1-jm}(A/A\alpha^{p-j},\ \alpha_1^{p-1})\to H_{i+m_1}(A,\ \alpha_1^{p-1})\to\cdots$$

and obtain an isomorphism

$$H_i(A/A\alpha^{p-j},\ \alpha_1^{p-1})\cong H_{i+jm-m_1}(A/A\alpha^j,\ \alpha_1)$$

also from $0\to A/A\alpha^j\to A\to A/A\alpha^{p-j}\to 0$, we have

$$H_i(A/A\alpha^{p-j},\ \alpha_1^{p-1})\cong H_{i+(p-1)m_1-(p-j)m}(A/A\alpha^j,\ \alpha_1)$$

since $jm-m_1\neq(p-1)m_1-(p-j)m$, it follows that $H(A/A\alpha^j,\ \alpha_1)=0$.

Proposition 2.3. *Let* $\alpha=Q_i$ *or* P_t^s $(i\geqslant 0,\ s<t)$, *then if* $\alpha\neq Q_{i_r}$ $(1\leqslant r\leqslant k)$ $H(A/AQ_{i_1}+AQ_{i_2}+\cdots+AQ_{i_k},\ \alpha)=0$.

Proof From Proposition 2.2., this holds for $k=1$. Suppose that this holds for $k-1$. Observe the following short exact sequence

$$0\to A/AQ_{i_1}+\cdots+AQ_{i_k}\xrightarrow{f}A/AQ_{i_1}+\cdots+AQ_{i_{k-1}}\xrightarrow{g}A/AQ_{i_1}+\cdots+AQ_{i_k}\to 0,$$

where $f(a^{**})=(aQ_{i_k})^*$, $g(a^*)=a^{**}$. Clearly g is epic and $gf=0$. If $g(a^*)=0$, then

$$a=a_1Q_{i_1}+\cdots+a_kQ_{i_k}$$

$f(a_k^{**})=(a_kQ_{i_k})^*=a^*$, hence im $f=\ker g$. Now prove that f is monic. If $f(a^{**})=0$, then

$$a\ Q_{i_k}=a_1Q_{i_1}+\cdots+a_{k-1}Q_{i_{k-1}}.$$

Since A is $E(Q_{i_1},\ \cdots,Q_{i_k})$ free, then

$$a=\lambda y+\Sigma\, y_je_j,$$

where $\lambda\in Z_p$, $e_j\in E(Q_{i_1},\cdots,\ Q_{i_k})$ and $\deg e_j\neq 0$, y, y_j is $E(Q_{i_1},\ \cdots,\ Q_{i_n})$ free base elements of A. It follows from $aQ_{i_1}\cdots Q_{i_k}=0$ that $\lambda yQ_{i_1}\cdots Q_{i_k}=0$ and then $\lambda=0$, $a=\Sigma\, y_je_j\in AQ_{i_1}+\cdots+AQ_{i_k}$, i.e. $a^*=0$. Thus the above sequence is short exact and $\deg f=\deg Q_{i_k}\neq\deg\alpha$. From inductive hypothesis, $H(A/AQ_{i_1}+\cdots+AQ_{i_{k-1}},\ \alpha)=0$, then following by the similar argument in Proposition 2.2., the proposition is proved.

Proposition 2.4. $H(A/AQ_{i_1}\cdots Q_{i_k},\ \alpha)=0$ *if* $\alpha\neq Q_{i_r}$ $(1\leqslant r\leqslant k)$ *and* $\alpha=Q_i$ *or* P_t^s.

Proof Holds for $k=1$. By induction on k. Consider

$$0\to A/AQ_{i_1}+\cdots+AQ_{i_k}\xrightarrow{f}A/AQ_{i_1}\cdots Q_{i_k}\xrightarrow{g}A/AQ_{i_1}\cdots Q_{i_{k-1}}\to 0,$$

where $f(a^*)=(aQ_{i_1}\cdots Q_{i_{k-1}})^{**}$, $g(a^{**})=a^{***}$. Clearly g is epic and $gf=0$. If $g(a^{**})=0$ then $a=a'Q_{i_1}\cdots Q_{i_{k-1}}$ and $f(a'^*)=a^{**}$, im $f=\ker g$. Now prove that f is monic. If $f(a^*)=0$, then

$$aQ_{i_1}\cdots Q_{i_{k-1}}=a'Q_{i_1}\cdots Q_{i_k},$$

then $(a\pm a'Q_{i_k})\ Q_{i_1}\cdots Q_{i_{k-1}}=0$. Similar to the argument in 2.3, we have

$$a \pm a' Q_{i_k} \in AQ_{i_1} + \cdots + AQ_{i_{k-1}}$$

and $a^* = 0$, f is monic, the above sequence is short exact. Following by inductive hypothesis and Proposition 2.3, the proposition is proved.

Proposition 2.5. *Let e_j ($1 \leqslant j \leqslant k$) are elements of the form $Q_{i_1} Q_{i_2} \cdots Q_{i_r}$, two of which are distinct, then*

$$H(A/Ae_1 + \cdots + Ae_k, \alpha) = 0,$$

where $\alpha = P_t^s$ or Q_i which does not appear in e_j ($1 \leqslant j \leqslant k$).

Proof By induction on k. From the following short exact sequence

$$0 \to A/AQ_{r_1} + \cdots + AQ_{r_s} \to A/Ae_1 + \cdots + Ae_{k-1} \to A/Ae_1 + \cdots + Ae_k \to 0,$$

the conclusion follows directly, where $Q_{r_1}, \cdots, Q_{r_s}$ are all Q_i appear in $e_1, e_2, \cdots, e_k$, $f(a^*) = (ae_k)^{**}$ and $g(a^{**}) = a^{***}$.

§ 3. Proofs of the Main Theorems

Proof of Theorem 1.1. First let $f: M \to N$ be an A epimorphism and $\ker f = F$, then there is an A module short exact sequence

$$0 \to F \to M \xrightarrow{f} N \to 0$$

which induces a long exact sequence (see [7])

$$\cdots \to H_i(M, (P_t^s)^{p-1}) \xrightarrow{H(f, (P_t^s)^{p-1})} H_i(N, (P_t^s)^{p-1})$$

$$\xrightarrow{\delta} H_{i+(p-1)m}(F, P_t^s) \to H_{i+(p-1)m}(M, P_t^s) \xrightarrow{H(f, P_t^s)} \cdots.$$

It follows from $H(f, P_t^s)$ and $H(f, (P_t^s)^{p-1})$ are isomorphisms that $H(F, P_t^s) = 0$ for all $s < t$. Similarly $H(F, Q_i) = 0$ for all $i \geqslant 0$. Then F is free A module (see [7]). But free A module is injective (see [4]), then the sequence splits, and $M \cong N \oplus F$.

If $f: M \to N$ isn't epic, find the free resolution of N, $F_0 \xrightarrow{g} N \to 0$, then $f \oplus g$: $M \oplus F_0 \to N$ is epic, let $\ker(f \oplus g) = F_1$, then there is a short exact sequence $0 \to F_1 \to M \oplus F_0 \xrightarrow{f \oplus g} N \to 0$ and $H(f \oplus g, Q_i)$ $H(f \oplus g, P_t^s)$, $H(f \oplus g, (P_t^s)^{p-1})$ are also isomorphisms, then it follows from the above argument that $M \oplus F_0 \cong N \oplus F_1$.

Proof of Theorem 1.2. Let E be any subalgebra of $E(Q_0, Q_1, Q_2, \cdots)$ and $\alpha = P_t^s$. Any element of $A \otimes_E N$, we may assume it be $a \otimes n$ for simplicity. If $\alpha(a \otimes n) = 0$ then either $\alpha a = 0$ or $\alpha a = \Sigma a_j' e_j$ and $e_j n = 0$ ($e_j \in E$, a_j' is E free base elements of A). In the former, it is easily seen that $\{a \otimes n\} = 0 \in H(A \otimes_E N, \alpha)$ and in the latter, since e_j is of the form $Q_{i_1} Q_{i_2} \cdots Q_{i_k}$, the two of e_j are distinct and $\alpha = P_t^s$, then from 2.5. we have

$$a = \alpha^{p-1} b + \Sigma b_j e_j,$$

where b, $b_j \in A$. Since $a \otimes n = \alpha^{p-1}(b \otimes n)$ then $\{a \otimes n\} = 0 \in H(A \otimes_E N, \alpha)$. In the case $\alpha = Q_i \notin E$, the proof is similar.

If $E=Z_p[P_t^s]/((P_t^s)^p)$, $\alpha=P_{t_1}^{s_1}\neq P_t^s$, any $a\otimes n\in A\otimes_E N$ such that $\alpha(a\otimes n)=0$, then either $\alpha a=0$ or $\alpha a=a'(P_t^s)^j$ and $(P_t^s)^j n=0$. Then from Proposition 2.2 we have $a=\alpha^{p-1}b+c(P_t^s)^j$, thus $\{a\otimes n\}=0\in H(A\otimes_E N,\alpha)$. In the case $\alpha=Q_i$, the proof is similar.

Proof of Theorem 1.3. (1) If we can prove the following lemma, then by using this lemma repeatedly and using limit, Theorem 1.3. (1) will be proved. Now we prove the following

Lemma 3.1. *Let $\alpha=Q_0$ (or P_1^0), if $H_i(M, Q_0)=0$ for $i<w$ (or $H_i(M, P_1^0)=H_i(M, (P_1^0)^{p-1})=0$ for $i<w$), then there exists an A module $\overline{M}$ and monomorphism $h: M\to\overline{M}$ such that $H(h, Q_i)$, $H(h, P_t^s)$, $H(h, (P_t^s)^{p-1})$ are isomorphisms for all Q_i, P_t^s other than α and $H_i(\overline{M},\alpha)=0$ for $i\leqslant w$ (or $H_i(\overline{M}, P_1^0)=H_i(\overline{M}, (P_1^0)^{p-1})=0$ for $i\leqslant w$).*

Proof If $\alpha=P_1^0$, $E=Z_p[P_1^0]/((P_1^0)^p)$. Regard M as E module, then there exists an E injective extention of M, i.e. there exists an E injective module N and E monomorphism $f: M\to N$. From 2.1, we know that N is E free. Let $L=\operatorname{cok} f$, in the free resolution $F\xrightarrow{d}N\xrightarrow{e}L\to 0$ of L, there is g such that the following diagram commutes

$$\begin{array}{ccccccc} 0\to & M & \xrightarrow{f} & N & \xrightarrow{e} & L & \to 0.\\ & g\uparrow & & \| & \nearrow e & & \\ & F & \xrightarrow{d} & N & & & \end{array}$$

Let $\bar g: A\otimes_E F\to M$ be $\bar g(a\otimes e)=ag(e)$, then $\bar g$ is an A morphism, in the free resolution of $A\otimes_E L$

$$A\otimes_E F\xrightarrow{1\otimes d}A\otimes_E N\xrightarrow{1\otimes e}A\otimes_E L\to 0,$$

we have $\bar g\,(\ker(1\otimes d))=0$, hence $\bar g$ determines an extention

$$0\to M\xrightarrow{h}\overline{M}\to A\otimes_E L\to 0.$$

From Theorem 1.2

$$H(A\otimes_E L, Q_i)=H(A\otimes_E L, P_t^s)=0$$

for all Q_i and P_t^s other than α. Hence we have also $H(A\otimes_E L, (P_t^s)^{p-1})=0$ (see [7]). It follows from the induced long exact sequence that $H(h, Q_i)$, $H(h, P_t^s)$, $H(h, (P_t^s)^{p-1})$ are isomorphisms. Now we prove that $H(\overline{M},\alpha)=H(\overline{M},\alpha^{p-1})=0$ for $i\leqslant w$.

Suppose that $\beta: L\to A\otimes_E L$ is defined by $\beta(l)=1\otimes l$, then β is E monomorphism, and there is E morphism $\gamma: N\to\overline{M}$ such that the following diagram commutes

$$\begin{array}{cccccc} 0\to M & \xrightarrow{h} & \overline{M} & \to & A\otimes_E L & \to 0\\ & f\searrow & \uparrow\gamma & & \uparrow\beta & \\ & & N & \longrightarrow & L & \longrightarrow 0 \end{array}$$

In fact, $\overline{M}$ is the extention determined by $\bar g$, thus $\overline{M}$ is a quotient group

$M\oplus A\otimes_E N/P$, where P is submodule generated by elements $(g(1\otimes e),\ 1\otimes d(e))$ and $h(m)=(m,\ 0)+P$, $k[(m,\ a\otimes n)+P]=a\otimes sn$. Hence, if we let $\gamma(n)=(a,\ 1\otimes n)+P$, the above diagram commutes. Thus, it induces the following commutative diagram of exact sequence

$$\begin{array}{ccccccc}
\cdots\to H_{i-(p-1)m}(A\otimes_E L,\ \alpha^{p-1}) & \xrightarrow{\delta_1} & H_i(M,\ \alpha)\to H_i(\overline{M},\alpha)\to & H_i(A\otimes_E L,\ \alpha) & \xrightarrow{\delta_3} & H_{i+m}(M,\ \alpha^{p-1}), \\
H(\beta,\ \alpha^{p-1})\ \nwarrow & & \nearrow\ \delta_2\ \cong & H(\beta,\ \alpha)\ \nwarrow & & \nearrow\ \delta_4\ \cong \\
& H_{i-(p-1)m}(L,\ \alpha^{p-1}) & & & H_i(L,\ \alpha) &
\end{array}$$

where $m=\deg\alpha$. Since $H(N,\ \alpha)=H(N,\ \alpha^{p-1})=0$, then δ_2, δ_4 are isomorphisms and δ_1, δ_3 are epic, $H(\beta,\ \alpha^{p-1})$, $H(\beta,\ \alpha)$ are monic. See the following diagram

$$\begin{array}{ccccccc}
\cdots\to H_{i-m}(A\otimes_E L,\ \alpha) & \xrightarrow{\delta_5} & H_i(M,\alpha^{p-1})\to H_i(\overline{M},\alpha^{p-1})\to & H_i(A\otimes_e L,\ \alpha^{p-1}) & \xrightarrow{\delta_7} & H_{i+(p-1)m}(M,\ \alpha) \\
H(\beta,\alpha)\ \nwarrow & & \nearrow\ \delta_6\ \cong & H(\beta,\ \alpha^{p-1})\ \nwarrow & & \nearrow\ \delta_8\ \cong \\
& H_{i-m}(L,\ \alpha) & & & H_i(L,\ \alpha^{p-1}) &
\end{array}$$

If we can prove δ_1, δ_3, δ_5, δ_7 are isomorphisms for $i\leqslant w$, then $H_i(\overline{M},\ \alpha)=H_i(\overline{M},\ \alpha^{p-1})=0$ for $i\leqslant w$ and it suffices to prove that

$$H(\beta,\ \alpha):H_i(L,\ \alpha)\to H_i(A\otimes_E L,\ \alpha),$$
$$H(\beta,\ \alpha^{p-1}):H_i(L,\ \alpha^{p-1})\to H_i(A\otimes_E L,\ \alpha^{p-1})$$

are epic for $i\leqslant w$. We prove the latter first.

For any elements of $A\otimes_E L$, let it be $a\otimes l$. Suppose that $\deg(a\otimes l)=i\leqslant w$ and $\alpha^{p-1}(a\otimes l)=0$, divided into two cases:

(i) $\deg a\geqslant pm$, $m=\deg\alpha$.

Then either $\alpha^{p-1}a=0$ or $\alpha^{p-1}a=a'\alpha^j$ $(1\leqslant j\leqslant p-1)$ and $\alpha^j l=0$. In the former it is easily shown that $\{a\otimes l\}=0\in H_i(A\otimes_E L,\ \alpha^{p-1})$. In the latter, since $\deg a\geqslant pm$ then $\deg l\leqslant w-pm<w-jm$. It follows from $H_i(M,\ \alpha^{p-1})=0$ $(i<w)$ that $H_i(L,\ \alpha)=0$ $(i<w-m)$. From [7] Proposition 3.1, we have $H_i(L,\ \alpha^j)=0$ $(i<w-jm)$, then $l=\alpha^{p-j}l'$ and $a\otimes l=a\alpha^{p-j}\otimes l'$. But

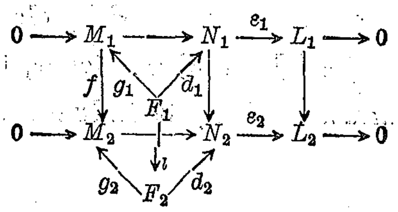
$$\alpha^{p-1}a\,\alpha^{p-j}=a'\alpha^j\alpha^{p-j}=0,$$

hence $a\,\alpha^{p-j}=\alpha c$ for some $c\in A$. Thus $\{a\otimes l\}=0\in H_i(A\otimes_E L,\ \alpha^{p-1})$ and $H(\beta,\ \alpha^{p-1})$ is epic for $i\leqslant w$.

(ii) $\deg a<pm$.

Now $a=\alpha^j$ $(1\leqslant j\leqslant p-1)$, since a cannot equal to Q_0 in this case (For $\alpha=P_1^0$ and $\alpha^{p-1}Q_0\neq Q_0\alpha^j$). Then $a\in E$ and $a\otimes l=1\otimes al$, $H(\beta,\ \alpha^{p-1})$is epic for $i\leqslant w$. Similarly, we can prove $H(\beta,\ \alpha)$ is epic for $i\leqslant w$, and the proof is complete if $\alpha=P_1^0$. If $\alpha=Q_0$, the proof is similar.

Proof of Theorem 1.3. (2) Firstly, we prove that the construction of 1.3 (1) is natural. Let $f:M_1\to M_2$ be A morphism, and there are $\overline{M}_1$, $\overline{M}_2$ constructed in 1.3(1) and monomorphisms $h_1:M_1\to\overline{M}_1$, $h_2:M_2\to\overline{M}_2$. According to the construction in 1.3(1), we have a commutative diagram



Then we have commutative diagram

$$\begin{array}{ccccc} M_1 & \xleftarrow{\bar{g}_1} & A\otimes_E F_1 & \xrightarrow{1\otimes d_1} & A\otimes_E N_1 \\ \downarrow f & & \downarrow 1\otimes l & & \downarrow 1\otimes h \\ M_2 & \xleftarrow{\bar{g}_2} & A\otimes_E F_2 & \xrightarrow{1\otimes d_2} & A\otimes_E N_2 \end{array}$$

and $\overline{M}_i$ is the quotient group $M_i\oplus A\otimes_E N_i/P_i$ as stated in the proof of 1.3.(1), then there exists an A morphism $\bar{f}:\overline{M}_1\to\overline{M}_2$ such that the following diagram commutes

$$\begin{array}{ccc} M_1 & \xrightarrow{h_1} & \overline{M}_1 \\ \downarrow f & & \downarrow \bar{f} \\ M_2 & \xrightarrow{h_2} & \overline{M}_2 \end{array}$$

i.e. the construction in 1.3.(1) is natural. If $f_i:M\to N_i$ is a monomorphism satisfying the condition of 1.3.(1) then for $\overline{M}$ in 1.3.(1) there is an A morphism $\bar{f}_1:M_i\to\overline{N}_i$ such that the following diagram commutes ($i=1, 2$)

$$\begin{array}{ccc} M & \xrightarrow{h} & \overline{M} \\ \downarrow f_i & & \downarrow \bar{f}_i \\ N_i & \xrightarrow{h_i} & \overline{N}_i \end{array}$$

and $H(\overline{M}, \alpha)=H(\overline{N}_1, \alpha)=0$, $H(h, Q_j)$, $H(h_i, Q_j)$, $H(h, P^s_t)$, $H(h_i, (P^s_t))$, $H(h, (P^s_t)^{p-1})$, $H(h_i, (P^s_t)^{p-1})$ are isomorphisms for all Q_j or P^s_t other than α. But $H(\bar{f}_i, Q_j)$, $H(\bar{f}_1, P^s_t)$, $H(\bar{f}_1, (P^s_t)^{p-1})$ are isomorphisms. It follows from Theorem 1.1 that $\overline{M}$, $\overline{N}_i$ stably isomorphic and so N_i, $\overline{N}_i$ and N_i, $\overline{M}$.

Before proving Theorem 1.4 we prove the following two lemmas first.

Lemma 3.2. *Let $E=Z_p[P^0_1]/((P^0_1)^p)$, M be E module such that $M^i=0$ $(i<r)$ and $H_i(M, P^0_1)=0$ $(i<n)$, then there exists an extention N of M such that N is E free module and $N^i=M^i$ for $i<n-4(p-1)^2$.*

Proof Since $H_i(M, P^0_1)=0$ $(i<n)$, then there is an E free module F such that $F^i=M^i$ for $i<n-(p-1)\cdot\deg P^0_1=n-2(p-1)^2$. Let $\overline{M}=AM^r+AM^{r+1}+\cdots+AM^k$ $(k=n-2(p-1)^2)$, then $\overline{M}$ is free submodule of M and $(M/\overline{M})^i=0 (i<k)$. Let $\overline{N}$ be smallest E injective extention (i.e. injective envelope) of $M/\overline{M}$. Then it follows from Theorem 2.1 that $\overline{N}$ is free E module and $\overline{N}^i=0$ $(i<k-(p-1)\deg P^0_1$ $=n-4(p-1)^2)$, Since $\overline{N}$ is the smallest one. Now, since M is free and hence injective: the following short exact sequence

$$0\to\overline{M}\to M\to M/\overline{M}\to 0$$

splits, thus $M\cong\overline{M}\oplus M/\overline{M}$. Then $N=\overline{M}\oplus\overline{N}$ is an extention of M such that $N^i=M^i$ for $i<n-4(p-1)^2$.

Lemma 3.3. *Let $\alpha=P_1^0$, X be bounded below spectrum such that $H_i(X^*, \alpha)=H_i(X^*, \alpha^{p-1})=0$ $(i<w)$, then there exists a spectrum Y and map $f:Y\to X$ such that $H(f^*, Q_i)$, $H(f^*, P_t^s)$, $H(f^*, (P_t^s)^{p-1})$ are isomorphisms for all Q_i or P_t^s other than α and $H_i(Y^*, \alpha)=H_i(Y^*, \alpha^{p-1})=0$ for $i<w+1, f_*:\pi_i(Y)\to\pi_i(X)$ is isomorphism for $i<w-4(p-1)^2$.*

Proof Let $E=Z_p[P_1^0]/((P_1^0)^p)$, regard X^* as E module, then there exists an extention N of X^* in Lemma 3.2. such that N free and $X^i=N^i(i<w-4(p-1)^2)$. Let $\gamma:X^*\to N$ be injection and $L=\operatorname{cok}\gamma$. In the free resolution $F_1\xrightarrow{d}F_0\xrightarrow{\varepsilon}L\to 0$ of L, there are E morphism η_0, η_1 such that the following diagram commutes

$$\begin{array}{ccccccc} 0\to X^* & \xrightarrow{\gamma} & N & \to & L & \to & 0 \\ \uparrow{\scriptstyle\eta_1} & & \uparrow{\scriptstyle\eta_0} & \nearrow{\scriptstyle\varepsilon} & & & \\ F_1 & \xrightarrow{d} & F_0 & & & & \end{array}$$

Let $\bar\eta:A\otimes_E F\to X^*$ be $\bar\eta(a\otimes e)=a\cdot\eta(e)$, then $\bar\eta$ is A morphism and $\bar\eta(\ker(1\otimes d))=0$. Since $A\otimes_E F_1$, $A\otimes_E F_0$ are free A modules, from [4], there are maps $g:X\to KV_{F_1}$, $h:KV_{F_0}\to KV_{F_1}$ such that $KV_{F_1}^*=A\otimes_E F_1$, $KV_{F_0}^*=A\otimes_E F_0$, $g^*=\bar\eta$, $h^*=1\otimes d$, where V_{F_1}, V_{F_0} are Z_p vector spaces $Z_p\otimes_E F_1$, $Z_p\otimes_E F_0$ and KV_{F_1}, KV_{F_0} are Eilenberg-Maclane spectra. Let Y be pullback

$$\begin{array}{ccccc} X & \xleftarrow{f} & Y & \leftarrow\cdots\cdots & Z \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle k} & & \| \\ KV_{F_1} & \xleftarrow{h} & KV_{F_0} & \leftarrow & Z \end{array}$$

then f and h have the same cofibre Z. In the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc} X^* & \xrightarrow{f^*} & Y^* & \longrightarrow & Z^* & \longrightarrow & X^* & \xrightarrow{f^*} & Y^* \\ \uparrow{\scriptstyle g^*} & & \uparrow{\scriptstyle k^*} & & \| & & \uparrow{\scriptstyle g^*} & & \uparrow{\scriptstyle k^*} \\ KV_{F_1}^* & \xrightarrow{h^*} & KV_{F_0}^* & \to & Z^* & \to & KV_{F_1}^* & \xrightarrow{h^*} & KV_{F_0}^* \end{array}$$

Since $g^*(\ker h^*)=0$, then we have a short exact sequence

$$0\to X^*\xrightarrow{f^*}Y^*\to Z^*\to 0$$

and also easily obtain the following exact sequences

$$\begin{array}{c} 0\to\operatorname{cok}h^*\xrightarrow{\beta}Z^*\xrightarrow{\mu}\ker h^*\to 0 \\ 0\to\operatorname{im}h^*\to KV_{F_0}^*\to\operatorname{cok}h^*\to 0 \\ 0\to\ker h^*\to KV_{F_1}^*\to\operatorname{im}h^*\to 0 \end{array} \qquad (*)$$

Since $\operatorname{cok}h^*=A\otimes_E L$, from Theorem 1.2.we have $H(\operatorname{cok}h^*, Q_i)=H(\operatorname{cok}h^*, P_t^s)=H(\operatorname{cok}h^*, (P_t^s)^{p-1})=0$ for all Q_i, P_t^s other than α. It follows from the second and third sequences in $(*)$ that

$$H(\ker h^*, Q_i)=H(\ker h^*, P_t^s)=H(\ker h^*, (P_t^s)^{p-1})=0,$$

thus we have

$$H(Z^*, Q_i)=H(Z^*,P_t^s)=H(Z^*, (P_t^s)^{p-1})=0$$

and $H(f^*, Q_i)$, $H(f^*, P_t^s)$, $H(f^*, (P_t^s)^{p-1})$ are isomorphisms.

Now we prove $H_i(Y^*, \alpha)=H_i(Y^*, \alpha^{p-1})=0$ for $i<w+1$. Let $\bar{\beta}: L\to Z^*$ be $\bar{\beta}(l)=\beta(1\otimes l)$, then $\bar{\beta}$ is E morphism and there is E morphism ξ such that the following diagram commutes up to sign

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^* & \longrightarrow & Y^* & \longrightarrow & Z^* & \longrightarrow & 0 \\ & & & {\scriptstyle (-1)}\searrow & \uparrow{\scriptstyle \xi} & & \uparrow{\scriptstyle \bar{\beta}} & & \\ & & & & N & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

In fact, since there are commutative diagrams of E modules exact sequence

$$\begin{array}{ccccccc} 0\to & X^* & \xrightarrow{f^*} & Y^* & \to & Z^* & \to 0 \\ & \uparrow{\scriptstyle \eta_1} & d & \uparrow{\scriptstyle \phi} & & \uparrow{\scriptstyle \beta} & \\ & F_1 & \longrightarrow & F_0 & \to & L & \to 0 \end{array} \qquad \begin{array}{ccccccc} 0\to & X^* & \longrightarrow & N & \to & L & \to 0 \\ & \downarrow{\scriptstyle \eta_1} & d & \downarrow{\scriptstyle \eta_0} & & \| & \\ & F_1 & \longrightarrow & F_0 & \to & L & \to 0 \end{array}$$

and N is isomorphic to $X^*\oplus F_0/P$, P is a submodule generated by elements of the form $(\eta_1(e), d(e))$, then if ξ is defined by $\xi((x, y)+P)=\phi(y)-f^*(x)$ the commutativity follows. Then they induce commutative diagrams ($m=\deg\alpha$)

$$\begin{array}{ccccccc} H_{i-(p-1)m}(Z^*, \alpha^{p-1}) & \xrightarrow{\delta_1} & H_i(X^*, \alpha)\to H_i(Y^*, \alpha)\to & H_i(Z^*, \alpha) & \xrightarrow{\delta_3} & H_{i+m}(X^*, \alpha^{p-1}) \\ {\scriptstyle H(\bar{\beta}, \alpha^{p-1})}\nwarrow & (-1) & \nearrow{\scriptstyle \delta_2} & {\scriptstyle H(\bar{\beta}, \alpha)}\nwarrow & (-1) & \nearrow{\scriptstyle \delta_4} \\ & H_{i-(p-1)m}(L, \alpha^{p-1}) & & & H_i(L, \alpha) & \end{array}$$

$$\begin{array}{ccccccc} H_{i-m}(Z^*, \alpha) & \xrightarrow{\delta_5} & H_i(X^*, \alpha^{p-1})\to H_i(Y^*, \alpha^{p-1})\to & H_i(Z^*, \alpha^{p-1}) & \xrightarrow{\delta_7} & H_{i+(p-1)m}(X^*, \alpha) \\ {\scriptstyle H(\bar{\beta}, \alpha)}\nwarrow & (-1) & \nearrow{\scriptstyle \delta_6} & {\scriptstyle H(\bar{\beta}, \alpha^{p-1})}\nwarrow & (-1) & \nearrow{\scriptstyle \delta_8} \\ & H_{i-m}(L, \alpha) & & & H_i(L, \alpha^{p-1}) & \end{array}$$

Since $H(N, \alpha)=H(N, \alpha^{p-1})=0$, then δ_2, δ_4, δ_6, δ_8 are isomorphisms and so δ_1, δ_3, δ_5, δ_7 are epic, $H(\bar{\beta}, \alpha)$, $H(\bar{\beta}, \alpha^{p-1})$ are monic. Now we prove that δ_1, δ_3, δ_5, δ_7 are monic for $i<w+1$, i.e. $H(\bar{\beta}, \alpha)$, $H(\bar{\beta}, \alpha^{p-1})$ are epic for $i<w+1$. But $\bar{\beta}$ is the composition $L \xrightarrow{j} A\otimes_E L \xrightarrow{\beta} Z^*$ and it has been proved in 1.3.(1) that $H(j,\alpha)$, $H(j, \alpha^{p-1})$is epic. Then it suffices to prove that $H(\beta, \alpha)$, $H(\beta, \alpha^{p-1})$ are epic for $i<w+1$ and can be reduced to that $H_i(\ker h^*, \alpha)=H_i(\ker h^*, \alpha^{p-1})=0$ for $i<w+2$.

It follows from exact sequences (*) that

$$H_i(\ker h^*, \alpha^{p-1})\cong H_{i-m}(\operatorname{im} h^*, \alpha)\cong H_{i-pm}(\operatorname{cok} h^*, \alpha^{p-1}),$$

but for $i<w+2$, we have $i-pm<w+2-pm\leqslant w-(p-1)m$, then

$$H_{i-pm}(\operatorname{cok} h^*, \alpha^{p-1})\cong H_{i-pm}(L, \alpha^{p-1})\cong H_{i-m}(X^*, \alpha)=0.$$

Then $H_i(\ker h^*, \alpha^{p-1})=0$ $(i<w+2)$ and so $H_i(\ker h^*, \alpha)=0$ $(i<w+2)$.

Lastly, since $X^i=N^i(i<w-4(p-1)^2)$ then $L^i=0$ and $F_1^i=F_0^i=0$ $(i<w-4(p-1)^2)$. Then $\pi_i(KV_{F_1})=\pi_i(KV_{F_0})=0$ and so $f_*: \pi_i(Y)\to\pi_i(X)$ is isomorphism for $i<w-4(p-1)^2$.

Proof of Theorem 1.4. Let $X^i=0(i<r)$. From Lemma 3.3. there is a sequence of spectra and maps

$$\cdots\to X_n \xrightarrow{f_n} X_{n-1}\to\cdots\xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0=X$$

such that (1) $H_i(X_n^*, \alpha)=H_i(X_n^*, \alpha^{p-1})=0 \quad (i<r+n)$.

(2) If Q_i, $P_t^s\neq\alpha$, $H(f_n^*, Q_i)$, $H(f_n^*, P_t^s)$, $H(f_n^*, (P_t^s)^{p-1})$ are isomorphisms.

(3) $f_{n*}:\pi_i(X_n)\to\pi_i(X_{n-1})$ is isomorphism for $i<r+n-1-4(p-1)^2$. Let $Y=\varprojlim X_n$, there is a map $h:Y\to X_n$ such that commutes with f_n, and it follows from the condition (3) that $h_{n*}:\pi_i(Y)\to\pi_i(X_n)$ is isomorphism for $i<r+n-1-4(p-1)^2$ and $h_n^*:X_n^i\to Y^i$ is isomorphism for $i<r+n-1-4(p-1)^2$. Then $Y^*=\varinjlim X_n^*$, $H(Y^*, \alpha)=\varinjlim H(X_n^*, \alpha)=0$, $h_0:Y\to X_0=X$ is the required map.

Proof of Theorem 1.5. If M is realizable, then there is a spectrum X and an isomorphism $\eta:X^*\cong M$. From Theorem 1.4., there exists a spectrum Y and map $f:Y\to X$ such that $H(f^*, Q_i)$, $H(f^*, P_t^s)$, $H(f^*, (P_t^s)^{p-1})$ are isomorphisms for all Q_i, P_t^s other than P_1^0 and $H(Y^*, P_1^0)=0$. Let $h:M\to M(P_1^0)$ be monomorphism in 1.3.(1), then $h\eta:X^*\to M(P_1^0)$ is also monic. But $H(M(P_1^0), P_1^0)=0$, $H(h\eta, Q_i)$, $H(h\eta, P_t^s)$, $H(h\eta, (P_t^s)^{p-1})$ are isomorphisms, it follows from 1.3(2) that Y^*, $M(P_1^0)$ are stably isomorphic, then M is realizable.

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