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# GLOBAL SMOOTH SOLUTIONS TO THE CAUCHY PROBLEM OF NONLINEAR THERMOELASTIC EQUATIONS WITH DISSIPATION

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**Abstract**

This paper considers the Cauchy problem for the nonlinear thermoelastic equations with dissipation. By means of energy methods, under the reasonable assumptions in mechanics the global existence, uniqueness and the decay rates of smooth solutions are proved.

## § 1. Introduction

In this paper we are concerned with the global existence of solutions to the Cauchy problem of equations governed by one-dimensional nonlinear thermoelasticity with viscous damping which are of the following form:

$$\begin{cases} u_{tt}-a(u_x, \theta)u_{xx}+b(u_x, \theta)\theta_x+\alpha u_t=0, \\ c(u_x, \theta)\theta_t+b(u_x, \theta)u_{xt}=d(\theta, \theta_x)\theta_{xx} \end{cases} \tag{1.1}$$

(refer to, [1, 2]).

In the above equations the term $\alpha u_t$ with constant $\alpha>0$ stands for the viscous damping which is important for the global existence of solutions. Instead of the above equations, by introducing new unknown functions

$$u_1=u_x,\ u_2=u_t,\ v=\theta, \tag{1.2}$$

we consider the following system

$$\begin{cases} \dfrac{\partial u_1}{\partial t}-\dfrac{\partial u_2}{\partial x}=0, \\ \dfrac{\partial u_2}{\partial t}-a(u_1, v)\dfrac{\partial u_1}{\partial x}+b(u_1, v)v_x+\alpha u_2=0, \\ c(u_1, v)\dfrac{\partial v}{\partial t}+b(u_1, v)\dfrac{\partial u_2}{\partial x}-d(v, v_x)v_{xx}=0 \end{cases} \tag{1.3}$$

with

$$t=0:\ u_1=u_{10}(x),\ u_2=u_{20}(x),\ v=v_0(x). \tag{1.4}$$

We make the following assumptions on (1.3), (1.4) which are reasonable in mechanics:

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(i) $a, b, c, d \in C^3$, and there exist positive constants $\gamma_0, \delta_0, \delta_1, A_0$ such that when

$$|u|, |v|, |v_x| \leqslant \gamma_0,$$
$$a \geqslant \delta_0 > 0, \ c \geqslant \delta_1 > 0, \ d \geqslant A_0 > 0, \ b \neq 0, \tag{1.5}$$

(ii) constant $\alpha > 0$,

(iii) $(u_{10}, u_{20}, v_0) \in H^3(R)$.

We now have

**Main Theorem.** *Under the above assumptions* (i)—(iii) *problem* (1.3), (1.4) *admits a unique global smooth solution* $(u_1, u_2, v)$, $(u_1, u_2) \in C(0, +\infty; H^3) \cap C^1(0, +\infty; H^2)$, $v \in C(0, +\infty; H^3) \cap C^1(0, +\infty; H^1)$ *provided* $\|(u_{10}, u_{20}, v_0)\|_{H^3} = (\|u_{10}\|^2_{H^3} + \|u_{20}\|^2_{H^3} + \|v_0\|^2_{H^3})^{\frac{1}{2}}$ *is sufficiently small. Moreover, the solution has the following decay rates as* $t \to +\infty$:

$$|(u_1, u_2, v)(t)|_{L^\infty} = O(t^{-\frac{1}{4}}), \tag{1.6}$$

$$\|D(u_1, u_2, v)(t)\|_{L^2} = O(t^{-\frac{1}{2}}), \tag{1.7}$$

$$|D(u_1, u_2, v)|_{L^\infty} = O(t^{-\frac{3}{4}}), \tag{1.8}$$

$$\|D^2(u_1, u_2, v)(t)\|_{H^1} = O(t^{-1}), \tag{1.9}$$

$$|D^2(u_1, u_2, v)(t)|_{L^\infty} = O(t^{-1}), \tag{1.10}$$

*where the notation* $D = \dfrac{\partial}{\partial x}$ *has been used.*

Before giving the detailed proof of Main Theorem let us first recall some related work. In [1] the global existence of solutions to various initial boundary value problem of one-dimensional nonlinear thermoelasticity has been proved. But, it seems that there are not any results concerning the Cauchy problem. On the other hand, the system (1.3) is a special form of general nonlinear hyperbolic-parabolic coupled system which also arises in the motion of viscous compressible and heat-conductive fluids and other mechanical problems. In [3, 4, 6] the global existence of solutions to the Cauchy problem of the equations of viscous compressible and heat-conductive fluids was proved. Later in [5] Matsumura improved the proof in the three dimensional case. We refer to [7–15] for other equations.

We emphasize that the main difficulty in proving the global existence for the Cauchy problem in one-dimensional case is that the solutions of linearized equations usually have not enough decay. In this paper by taking advantage of damping term $\alpha u_2$ and by using the energy estimate method and the technique given in [5] we are able to prove our Main Theorem.

Throughout this paper we denote by $C$ the constant independent of $u_1, u_2, v$ and $t$, and denote by $\|\cdot\|$ the $L^2(R)$ norm.

## § 2. The Uniform A Priori Estimates

It is well known (see [5, 15]) that by continuation argument the proof of the global existence is usually divided into three parts:

(i) To prove the local existence of smooth solutions.

(ii) To get the uniform a priori estimates of solutions in $R\times[0, T]$.

(iii) To prove the global existence by combining the above two steps when the initial data is sufficiently small.

We will proceed along this line.

Let $E_0$ be a positive constant such that for $\|f\|_{H^1}\leqslant E_0$, by Sobolev's lemma we have $|f|_{L^\infty}$, $|Df|_{L^\infty}\leqslant C$, $\|f\|_{H^2}\leqslant\gamma_0$. For some $E\leqslant E_0$ and $0\leqslant t_1<t_2\leqslant+\infty$ we now define the set of functions $x(t_1, t_2; E)$ as follows.

$$\begin{aligned}X(t_1, t_2; E)=\{&(u_1, u_2, v)\mid(u_1, u_2)\in C(t_1, t_2; H^3)\cap C^1(t_1, t_2; H^2),\\ &Du_1\in L^2(t_1, t_2; H^2),\ u_2\in L^2(t_1,t_2; H^3),\\ &(1+t)^{\frac{1}{2}}D^2u_1\in L^2(t_1, t_2; L^2),\ (1+t)D^3u_1\in L^2(t_1, t_2; L^2),\\ &(1+t)^{\frac{1}{2}}Du_2\in L^2(t_1, t_2; L^2),\ (1+t)D^2u_2\in L^2(t_1,t_2; H^1),\\ &v\in C(t_1, t_2; H^3)\cap C^1(t_1, t_2; H^1),\ Dv\in L^2(t_1, t_2; H^3),\\ &(1+t)^{\frac{1}{2}}D^2v\in L^2(t_1, t_2; L^2),\ (1+t)D^3v\in L^2(t_1, t_2; H^1),\\ &\text{and } N^2(t_1, t_2)\leqslant E\},\end{aligned}\tag{2.1}$$

where $N(t_1, t_2)$ is defined by

$$\begin{aligned}N^2(t_1, t_2)=\sup_{t_1<t<t_2}&(\|(u_1, u_2, v)(t)\|^2_{H^3}+t\|D(u_1, u_2, v)(t)\|^2\\ &+t^2\|D^2(u_1, u_2, v)(t)\|^2_{H^1})+\int_{t_1}^{t_2}(\|u_2(\tau)\|^2_{H^3}+\|Du_1(\tau)\|^2_{H^2}\\ &+\|Dv\|^2_{H^3}+\tau(\|Du_2\|^2+\|D^2(u_1, v)\|^2)+\tau^2(\|D^3u_1\|^2\\ &+\|D^2u_2\|^2_{H^1}+\|D^3v\|^2_{H^1}))d\tau.\end{aligned}\tag{2.2}$$

We now have

**Theorem 1** (Local Existence). *Consider the initial value problem* (1.3) *for $t\geqslant t_1$ with the initial data at $t=t_1$ as*

$$(u_1, u_2, v)\in H^3(R).\tag{2.3}$$

*Then there exist positive constants $\varepsilon_1$, $C_1$ ($\varepsilon_1C_1\leqslant E_0$) and $\delta$ depending only on $\varepsilon_1$ but independent of $t_1$ such that if $N(t_1, t_1)\leqslant\varepsilon_1$, the Cauchy problem* (1.3)(2.3) *admits a unique solution $(u_1, u_2, v)$ in $R\times[t_1, t_1+\delta]$. Moreover*

$$(u_1, u_2, v)\in x(t_1, t_1+\delta, C_1N(t_1, t_1)).\tag{2.4}$$

*Proof* The local existence theorem for the more general quasilinear hyperbolic parabolic coupled systems has been given in [16] (see also [4, 5]). Although the norm $N$ used here is slightly different from those in [16, 4], the proof can be given in the same way. So we omit the details here.

The key step for the continuation argument is to get the uniform a priori estimates of solutions for nonlinear system (1.3).

We now have

**Theorem 2** (A Priori Estimates). *Suppose that the Cauchy Problem* (1.3), (1.4) *has a solution*

$$(u_1, u_2, v) \in X(0, T; E) \tag{2.5}$$

*for some* $T>0$ *and some* $E\leqslant E_0$. *Then there exist positive constants* $\varepsilon_2$ $(\varepsilon_2\leqslant\varepsilon_1)$ *and* $C_2$ *independent of* $T$ *such that if* $E\leqslant\varepsilon_2$, *then the solution* $(u_1, u_2, v)$ *satisfies*

$$(u_1, u_2, v) \in X(0, T; C_2\|(u_{10}, u_{20}, v_0)\|_{H^3}). \tag{2.6}$$

*Proof* We first rewrite the system (1.3) as follows:

$$\left.\begin{aligned}&\frac{a_0}{c_0}\frac{\partial u_1}{\partial t}-\frac{a_0}{c_0}\frac{\partial u_2}{\partial x}=0,\\&\frac{1}{c_0}\frac{\partial u_2}{\partial t}-\frac{a_0}{c_0}\frac{\partial u_1}{\partial x}+\frac{b_0}{c_0}\frac{\partial v}{\partial x}+\frac{a}{c_0}u_2=\frac{a-a_0}{c_0}\cdot\frac{\partial u_1}{\partial x}+\frac{(b_0-b)}{c_0}\frac{\partial v}{\partial x}\triangleq f_1,\\&\frac{\partial v}{\partial t}+\frac{b_0}{c_0}\frac{\partial u_2}{\partial x}-\frac{d_0}{c_0}\frac{\partial^2 v}{\partial x^2}=\left(\frac{b_0}{c_0}-\frac{b}{c}\right)\frac{\partial u_2}{\partial x}+\left(\frac{d}{c}-\frac{d_0}{c_0}\right)\frac{\partial^2 v}{\partial x^2}\triangleq f_2,\end{aligned}\right\} \tag{2.7}$$

where $a_0=a(0, 0)\geqslant\delta_0>0$, $b_0=b(0, 0)\neq 0$, $c_0=c(0, 0)\geqslant\delta_1>0$, $d_0=d(0, 0)\geqslant A_0>0$.

Multiplying both sides of the first equation of (2.7) by $u_1$, the second equation by $u_2$, the third equation of (2.7) by $v$, and then integrating them with respect to $x$, respectively, for $t\in[0, T]$ we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\frac{a_0}{c_0}\|u_1(t)\|^2+\frac{1}{c_0}\|u_2(t)\|^2+\|v(t)\|^2\right)\\&\quad+\frac{d_0}{c_0}\|Dv(t)\|^2+\frac{a}{c_0}\|u_2(t)\|^2=\int_R(f_1u_2+f_2v)\,dx.\end{aligned} \tag{2.8}$$

Differentiating (2.7) once and twice with respect to $x$ respectively and then using the same procedure as above, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\frac{a_0}{c_0}\|Du_1(t)\|^2+\frac{1}{c_0}\|Du_2(t)\|^2+\|Dv(t)\|^2\right)\\&\quad+\frac{d_0}{c_0}\|D^2v(t)\|^2+\frac{a}{c_0}\|Du_2(t)\|^2=\int_R(Df_1\cdot Du_2+Df_2\cdot Dv)\,dx,\end{aligned} \tag{2.9}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\frac{a_0}{c_0}\|D^2u_1(t)\|^2+\frac{1}{c_0}\|D^2u_2(t)\|^2+\|D^2v(t)\|^2\right)\\&\quad+\frac{d_0}{c_0}\|D^3v(t)\|^2+\frac{a}{c_0}\|D^2u_2'(t)\|^2=\int_R(D^2f_1D^2u_2+D^2f_2D^2v)\,dx.\end{aligned} \tag{2.10}$$

Now by integrating (2.8), (2.9), (2.10) with respect to $t$, we obtain

$$\begin{aligned}&\frac{1}{2}\left(\frac{a_0}{c_0}\|u_1(t)\|^2+\frac{1}{c_0}\|u_2(t)\|^2+\|v(t)\|^2\right)\\&\quad+\frac{d_0}{c_0}\int_0^t\|Dv\|^2\,d\tau+\frac{a}{c_0}\int_0^t\|u_2\|^2\,d\tau=\int_0^t\int_R(f_1u_2+f_2v)\,dx\,d\tau\end{aligned}$$

$$+\frac{1}{2}\left(\frac{a_0}{c_0}\|u_{10}\|^2+\frac{1}{c_0}\|u_{20}\|^2+\|v_0\|^2\right), \tag{2.11}$$

$$\frac{1}{2}\left(\frac{a_0}{c_0}\|Du_1(t)\|^2+\frac{1}{c_0}\|Du_2(t)\|^2+\|Dv(t)\|^2\right)$$
$$+\frac{d_0}{c_0}\int_0^t\|D^2v\|^2d\tau+\frac{a}{c_0}\int_0^t\|Du_2\|^2d\tau=\int_0^t\int_R(Df_1Du_2+Df_2Dv)\,dx\,d\tau$$
$$+\frac{1}{2}\left(\frac{a_0}{c_0}\|Du_{10}\|^2+\frac{1}{c_0}\|Du_{20}\|^2+\|Dv_0\|^2\right), \tag{2.12}$$

$$\frac{1}{2}\left(\frac{a_0}{c_0}\|D^2u_1(t)\|^2+\frac{1}{c_0}\|D^2u_2(t)\|^2+\|D^2v(t)\|^2\right)$$
$$+\frac{d_0}{c_0}\int_0^t\|D^3v\|^2d\tau+\frac{a}{c_0}\int_0^t\|D^2u_2\|^2d\tau=\int_0^t\int_R(D^2f_1D^2u_2+D^2f_2D^2v)\,dx\,d\tau$$
$$+\frac{1}{2}\left(\frac{a_0}{c_0}\|D^2u_{10}\|^2+\frac{1}{c_0}\|D^2u_{20}\|^2+\|D^2v_0\|^2\right). \tag{2.13}$$

On the other hard, by differentiating (1.3) three times with respect to $x$, we obtain

$$\left.\begin{aligned}
&a\frac{\partial D^3u_1}{\partial t}-a\frac{\partial D^3u_2}{\partial t}=0.\\
&\frac{\partial D^3u_2}{\partial t}-a\frac{\partial D^3u_1}{\partial x}+\alpha D^3u_2+b\frac{\partial D^3v}{\partial x}=D^3\left(a\frac{\partial u_1}{\partial x}\right)\\
&\quad-a\frac{\partial D^3u_1}{\partial x}-D^3\left(b\frac{\partial v}{\partial x}\right)+b\frac{\partial D^3v}{\partial x},\\
&\frac{\partial D^3v}{\partial t}+\frac{b}{c}\frac{\partial D^3u_2}{\partial x}-\frac{\partial}{\partial x}\left(\frac{d}{c}D^4v\right)=D^3\left(\frac{d}{c}D^2v\right)-D\left(\frac{d}{c}D^4v\right)\\
&\quad+\frac{b}{c}\frac{\partial D^3u_2}{\partial x}-D^3\left(\frac{b}{c}\frac{\partial u_2}{\partial x}\right).
\end{aligned}\right\} \tag{2.14}$$

Thus in the same way as above we obtain

$$\frac{1}{2}\left(\int_R a(D^3u_1)^2dx+\|D^3u_2(t)\|^2+\int_R c(D^3v)^2dx\right)$$
$$+\int_0^t\int_R d(D^4v)^2dx\,d\tau+\alpha\int_0^t\|D^3u_2\|^2d\tau=\frac{1}{2}\int_0^t\int_R\frac{\partial a}{\partial t}(D^3u_1)^2dx\,d\tau$$
$$-\int_0^t\int_R\frac{\partial a}{\partial x}D^3u_1D^3u_2\,dx\,d\tau+\frac{1}{2}\int_0^t\int_R\frac{\partial c}{\partial t}(D^3v)^2dx\,d\tau$$
$$+\int_0^t\int_R\frac{\partial b}{\partial x}D^3vD^3u_2\,dx\,d\tau+\int_0^t\int_R(F_1D^3u_2+F_2D^3v)\,dx\,d\tau+R_3(u_{10},u_{20},v_0), \tag{2.15}$$

where

$$F_1=D^3\left(a\frac{\partial u_1}{\partial x}\right)-aD^4u_1-D^3(bDv)+bD^4v,$$
$$F_2=-\frac{d}{c}\frac{\partial c}{\partial x}D^4v+c\left(D^3\left(\frac{d}{c}D^2v\right)-D\left(\frac{d}{c}D^4v\right)\right)+c\left(\frac{b}{c}D^4u_2-D^3\left(\frac{b}{c}Du_2\right)\right), \tag{2.16}$$

$$R_3(u_{10},u_{20},v_0)=\frac{1}{2}\int_R a(u_{10},v_0)(D^3u_{10})^2dx+\|D^3u_{20}\|^2+\int_R c(u_{10},v_0)(D^3v_0)^2dx.$$

We would like to point out that because of the limited smoothness of solution

$(u_1, u_2, v)$, the argument from (1.3) to (2.15) is formal. But by using the mollofier and reasoning in the same way as in [15, 4], we can conclude that (2.15) still holds. Since the procedure is standard (see [15, 4]), we omit the details here.

In order to get the estimates for the double integral of the derivatives of $u_1$, by differentiating the first equation of (1.3) with respect to $x$, then multiplying the first two equations of (1.3) by $-u_2$ and $-\frac{\partial u_1}{\partial x}$ respectively, and integrating them with respect to $x$, we obtain

$$-\frac{d}{dt}\int_R u_2 \frac{\partial u_1}{\partial x}\,dx+\int_R a\left(\frac{\partial u_1}{\partial x}\right)^2 dx-\alpha\int_R u_2\frac{\partial u_1}{\partial x}\,dx$$
$$-\int_R b\,\frac{\partial u_1}{\partial x}\,\frac{\partial v}{\partial x}\,dx-\int_R\left(\frac{\partial u_2}{\partial x}\right)^2 dx=0. \tag{2.17}$$

Therefore

$$-\int_R u_2\frac{\partial u_1}{\partial x}\,dx+\int_0^t\int_R a\left(\frac{\partial u_1}{\partial x}\right)^2 dx\,d\tau-\alpha\int_0^t\int_R u_2\frac{\partial u_1}{\partial x}\,dx d\tau$$
$$-\int_0^t\int_R b\,\frac{\partial u_1}{\partial x}\,\frac{\partial v}{\partial x}\,dx\,d\tau-\int_0^t\|u_2(\tau)\|^2 d\tau=-\int_R u_{20}\frac{\partial u_{10}}{\partial x}\,dx. \tag{2.18}$$

Similarly, we have

$$-\frac{d}{dt}\int_R\frac{\partial u_2}{\partial x}\,\frac{\partial^2 u_1}{\partial x^2}\,dx+\int_R a\left(\frac{\partial^2 u_1}{\partial x^2}\right)dx-\|D^2u_2(t)\|^2$$
$$-\alpha\int_R\frac{\partial u_2}{\partial x}\,\frac{\partial^2 u_1}{\partial x^2}\,dx-\int_R b\,\frac{\partial^2 u_1}{\partial x^2}\,\frac{\partial^2 v}{\partial x^2}\,dx$$
$$=\int_R\frac{\partial b}{\partial x}\,\frac{\partial v}{\partial x}\,\frac{\partial^2 u_1}{\partial x^2}\,dx-\int_R\frac{\partial a}{\partial x}\,\frac{\partial u_1}{\partial x}\,\frac{\partial^2 u_1}{\partial x^2}\,dx. \tag{2.19}$$

Hence

$$-\int_R\frac{\partial u_2}{\partial x}\,\frac{\partial^2 u_1}{\partial x^2}\,dx+\int_0^t\int_R a\left(\frac{\partial^2 u_1}{\partial x^2}\right)^2 dx\,d\tau$$
$$-\int_0^t\|D^2u_2(\tau)\|^2 d\tau-\int_0^t\int_R b\,\frac{\partial^2 u_1}{\partial x^2}\,\frac{\partial^2 v}{\partial x^2}\,dx\,d\tau$$
$$-\alpha\int_0^t\int_R\frac{\partial u_2}{\partial x}\,\frac{\partial^2 u_1}{\partial x^2}\,dx\,d\tau=\int_0^t\int_R\frac{\partial b}{\partial x}\,\frac{\partial v}{\partial x}\,\frac{\partial^2 u_1}{\partial x^2}\,dx\,d\tau$$
$$-\int_0^t\int_R\frac{\partial a}{\partial x}\,\frac{\partial u_1}{\partial x}\,\frac{\partial^2 u_1}{\partial x^2}\,dx\,d\tau-\int_R\frac{\partial u_{20}}{\partial x}\,\frac{\partial^2 u_{10}}{\partial x^2}\,dx, \tag{2.20}$$

$$-\frac{d}{dt}\int_R\left(\frac{\partial^2 u_2}{\partial x^2}\,\frac{\partial^3 u_1}{\partial x^3}\right)dx+\int_R a\left(\frac{\partial^3 u_1}{\partial x^3}\right)^2 dx-\|D^3u_2(t)\|^2$$
$$-\alpha\int_R\frac{\partial^2 u_2}{\partial x^2}\,\frac{\partial^3 u_1}{\partial x^3}\,dx-\int_R b\,\frac{\partial^3 u_1}{\partial x^3}\,\frac{\partial^3 v}{\partial x^3}\,dx$$
$$=-\int_R\frac{\partial a}{\partial x}\,\frac{\partial^2 u_1}{\partial x^2}\,\frac{\partial^3 u_1}{\partial x^3}\,dx-\int_R\frac{\partial^2 a}{\partial x^2}\,\frac{\partial u_1}{\partial x}\,\frac{\partial^3 u_1}{\partial x^3}\,dx$$
$$+\int_R\frac{\partial^2 b}{\partial x^2}\,\frac{\partial^3 u_1}{\partial x^3}\,\frac{\partial v}{\partial x}\,dx+\int_R\frac{\partial b}{\partial x}\,\frac{\partial^3 u_1}{\partial x^3}\,\frac{\partial^2 v}{\partial x^2}\,dx. \tag{2.21}$$

$$-\int_R\left(\frac{\partial^2 u_2}{\partial x^2}\frac{\partial^3 u_1}{\partial x^3}\right)dx+\int_0^t\int_R a\left(\frac{\partial^3 u_1}{\partial x^3}\right)^2 dx\,d\tau-\int_0^t\|D^3u_2(\tau)\|^2 d\tau$$

$$-\alpha\int_0^t\int_R\frac{\partial^2 u_2}{\partial x^2}\frac{\partial^3 u_1}{\partial x^3}\,dx\,d\tau-\int_0^t\int_R b\frac{\partial^3 u_1}{\partial x^3}\frac{\partial^3 v}{\partial x^3}\,dx\,d\tau$$

$$=-\int_0^t\int_R\frac{\partial a}{\partial x}\frac{\partial^2 u_1}{\partial x^2}\frac{\partial^3 u_1}{\partial x^3}\,dx\,d\tau-\int_0^t\int_R\frac{\partial^2 a}{\partial x^2}\frac{\partial u_1}{\partial x}\frac{\partial^3 u_1}{\partial x^3}\,dx\,d\tau$$

$$+\int_0^t\int_R\frac{\partial^2 b}{\partial x^2}\frac{\partial^3 u_1}{\partial x^3}\frac{\partial v}{\partial x}\,dx\,d\tau+\int_0^t\int_R\frac{\partial b}{\partial x}\frac{\partial^3 u_1}{\partial x^3}\frac{\partial^2 v}{\partial x^2}\,dx\,d\tau$$

$$-\int_R\frac{\partial^2 u_{20}}{\partial x^2}\frac{\partial^3 u_{10}}{\partial x^3}\,dx. \tag{2.22}$$

Furthermore, in order to get the estimates of the other terms appearing in $N(0, T)$, we multiply both sides of (2.9) by $t$ and obtain

$$\frac{1}{2}\frac{d}{dt}\left(\frac{a_0}{c_0}t\|Du_1(t)\|^2+\frac{t}{c_0}\|Du_2(t)\|^2+t\|Dv(t)\|^2\right)$$

$$+\frac{d_0}{c_0}t\|D^2v(t)\|^2+\frac{\alpha}{c_0}t\|Du_2(t)\|^2-\frac{1}{2}\left(\frac{a_0}{c_0}\|Du_1(t)\|^2\right.$$

$$\left.+\frac{1}{c_0}\|Du_2(t)\|^2+\|Dv(t)\|^2\right)=t\int_R(Df_1Du_2+Df_2Dv)\,dx. \tag{2.23}$$

Therefore

$$\frac{t}{2}\left(\frac{a_0}{c_0}\|Du_1(t)\|^2+\frac{1}{c_0}\|Du_2(t)\|^2+\|Dv(t)\|^2\right)$$

$$+\frac{d_0}{c_0}\int_0^t\tau\|D^2v(\tau)\|^2d\tau+\frac{\alpha}{c_0}\int_0^t\tau\|Du_2(\tau)\|^2d\tau$$

$$-\frac{1}{2}\left(\frac{a_0}{c_0}\int_0^t\|Du_1(\tau)\|^2d\tau+\frac{1}{c_0}\int_0^t\|Du_2(\tau)\|^2d\tau\right.$$

$$\left.+\int_0^t\|Dv(\tau)\|^2d\tau\right)=\int_0^t\tau\int_R(Df_1Du_2+Df_2Dv)\,dx\,d\tau. \tag{2.24}$$

Similarly, it follows from (2.10), (2.14) that

$$\frac{t^2}{2}\left(\frac{a_0}{c_0}\|D^2u_1(t)\|^2+\frac{1}{c_0}\|D^2u_2(t)\|^2+\|D^2v(t)\|^2\right)$$

$$+\frac{d_0}{c_0}\int_0^t\tau^2\|D^3v\|^2d\tau+\frac{\alpha}{c_0}\int_0^t\tau^2\|D^2u_2\|^2d\tau$$

$$-\frac{a_0}{c_0}\int_0^t\tau\|D^2u_1\|^2d\tau-\frac{1}{c_0}\int_0^t\tau\|D^2u_2(\tau)\|^2d\tau-\int_0^t\tau\|D^2v\|^2d\tau$$

$$=\int_0^t\tau^2\int_R(D^2f_1D^2u_2+D^2f_2D^2v)\,dx\,d\tau. \tag{2.25}$$

$$\frac{t^2}{2}\left(\int_R a(D^3u_1)^2dx+\|D^3u_2(t)\|^2+\int_R c(D^3v)^2dx\right)$$

$$+\int_0^t\tau^2\int_R d(D^4v)^2dx\,d\tau+\alpha\int_0^t\tau^2\|D^3u_2(\tau)\|^2d\tau$$

$$-\int_0^t\tau\int_R a(D^3u_1)^2dx\,d\tau-\int_0^t\tau\|D^3u_2\|^2d\tau$$

$$-\int_0^t\tau\int_R c(D^3v)^2dx\,d\tau=\frac{1}{2}\int_0^t\tau^2\int_R\frac{\partial a}{\partial t}(D^3u_1)^2dx\,d\tau$$

$$-\int_0^t \tau^2 \int_R \frac{\partial a}{\partial x} D^3u_1 D^3u_2\, dx\, d\tau + \frac{1}{2}\int_0^t \tau^2 \int_R \frac{\partial c}{\partial \tau}(D^3 v)^2\, dx\, d\tau$$
$$+\int_0^t \tau^2 \int_R \frac{\partial b}{\partial x} D^3 v D^3 u_2\, dx\, d\tau + \int_0^t \tau^2 \int_R (F_1 D^3 u_2 + F_2 D^3 v)\, dx\, d\tau. \tag{2.26}$$

From (2.19), (2.21) we have

$$-t\int_R \frac{\partial u_2}{\partial x}\frac{\partial^2 u_1}{\partial x^2}\,dx + \int_0^t\int_R \frac{\partial u_2}{\partial x}\frac{\partial^2 u_1}{\partial x^2}\,dx\,d\tau + \int_0^t \tau\int_R a\left(\frac{\partial^2 u_1}{\partial x^2}\right)^2 dx\,d\tau$$
$$-\int_0^t \tau \|D^2 u_2(\tau)\|^2\, d\tau - \alpha\int_0^t \tau\int_R \frac{\partial u_2}{\partial x}\frac{\partial^2 u_1}{\partial x^2}\,dx\,d\tau$$
$$-\int_0^t \tau\int_R b\,\frac{\partial^2 u_1}{\partial x^2}\frac{\partial^2 v}{\partial x^2}\,dx\,d\tau = \int_0^t \tau\int_R \frac{\partial b}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2 u_1}{\partial x^2}\,dx\,d\tau$$
$$-\int_0^t \tau\int_R \frac{\partial a}{\partial x}\frac{\partial u_1}{\partial x}\frac{\partial^2 u_1}{\partial x^2}\,dx\,d\tau, \tag{2.27}$$

$$-t^2\int_R \frac{\partial^2 u_2}{\partial x^2}\frac{\partial^3 u_1}{\partial x^3}\,dx + \int_0^t \tau^2\int_R a\left(\frac{\partial^3 u_1}{\partial x^3}\right)^2 dx\,d\tau$$
$$-\int_0^t \tau^2\int_R b\,\frac{\partial^3 u_1}{\partial x^3}\frac{\partial^3 v}{\partial x^3}\,dx\,d\tau + 2\int_0^t \tau\int_R \frac{\partial^2 u_2}{\partial x^2}\frac{\partial^3 u_1}{\partial x^3}\,dx\,d\tau$$
$$-\int_0^t \tau^2\|D^3 u_2(\tau)\|^2\,d\tau - \alpha\int_0^t \tau^2\int_R \frac{\partial^2 u_2}{\partial x^2}\frac{\partial^3 u_1}{\partial x^2}\,dx\,d\tau$$
$$= -\int_0^t \tau^2\int_R \frac{\partial a}{\partial x}\frac{\partial^2 u_1}{\partial x^2}\frac{\partial^2 u_1}{\partial x^3}\,dx\,d\tau$$
$$-\int_0^t \tau^2\int_R \frac{\partial^2 a}{\partial x^2}\frac{\partial u_1}{\partial x}\frac{\partial^3 u_1}{\partial x^3}\,dx\,d\tau + \int_0^t \tau^2\int_R \frac{\partial^2 b}{\partial x^2}\frac{\partial^3 u_1}{\partial x^3}\frac{\partial v}{\partial x}\,dx\,d\tau$$
$$+\int_0^t \tau^2\int_R \frac{\partial b}{\partial x}\frac{\partial^2 v}{\partial x^2}\frac{\partial^3 u_1}{\partial x^3}\,dx\,d\tau. \tag{2.28}$$

Set

$$(2.11)+(2.12)+(2.13)+(2.15)+\mu((2.18)+(2.20)+(2.22))$$
$$+\mu^2(2.24)+\mu^3(2.27)+\mu^4(2.25)+\mu^5(2.26)+\mu^6(2.28), \tag{2.29}$$

where $\mu$ is a small positive constant specified later.

Now (2.29) can be rewritten as follows:

$$I_1(t)+I_2(t)=H(t)+\tilde{R}_3(u_{10}, u_{20}, v_0), \tag{2.30}$$

where

$$I_1(t)=\frac{1}{2}\left(\frac{a_0}{c_0}\|u_1(t)\|^2_{H^1}+\frac{1}{c_0}\|u_2(t)\|^2_{H^1}+\|v(t)\|^2_{H^1}\right)$$
$$+\frac{1}{2}\left(\int_R a\left(\frac{\partial^3 u_1}{\partial x^3}\right)^2 dx+\|D^3 u_2(t)\|^2\right.$$
$$\left.+\int_R c(D^3 v)^2 dx\right)-\mu\left(\int_R u_2\frac{\partial u_1}{\partial x}\,dx+\int_R \frac{\partial u_2}{\partial x}\frac{\partial^2 u_1}{\partial x^2}\,dx\right.$$
$$\left.+\int_R \frac{\partial^2 u_2}{\partial x^2}\frac{\partial^3 u_1}{\partial x^3}\,dx\right)+\frac{\mu^2 t}{2}\left(\frac{a_0}{c_0}\|Du_1(t)\|^2+\frac{1}{c_0}\|Du_2(t)\|^2\right.$$
$$\left.+\|Dv(t)\|^2\right)-\mu^3 t\int_R \frac{\partial u_2}{\partial x}\frac{\partial^2 u_1}{\partial x^2}\,dx+\frac{\mu^4 t^2}{2}\left(\frac{a_0}{c_0}\|D^2 u_1(t)\|^2\right.$$
$$\left.+\frac{1}{c_0}\|D^2 u_2(t)\|^2+\|D^2 v(t)\|^2\right)+\frac{\mu^5 t^2}{2}\left(\int_R a(D^3 u_1)^2\,dx\right.$$

$$+\|D^3u_2(t)\|^2+\int_R c(D^3v)^2dx\Big)-\mu^6t^2\int_R\frac{\partial^2u_2}{\partial x^2}\frac{\partial^3u_1}{\partial x^3}dx, \tag{2.31}$$

$$I_2(t)=\frac{d_0}{c_0}\int_0^t\|Dv(\tau)\|^2_{H^2}d\tau+\frac{\alpha}{c_0}\int_0^t\|u_2(\tau)\|^2_{H^2}d\tau$$
$$+\int_0^t\int_R d(D^4v)^2dx\,d\tau+\alpha\int_0^t\|D^3u_2(\tau)\|^2d\tau$$
$$+\mu\Big(\int_0^t\int_R a\Big(\Big(\frac{\partial u_1}{\partial x}\Big)^2+\Big(\frac{\partial^2u_1}{\partial x^2}\Big)^2+\Big(\frac{\partial^3u_1}{\partial x^3}\Big)^2\Big)dx\,d\tau$$
$$-\alpha\int_0^t\int_R\Big(u_2\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial x}\frac{\partial^2u_1}{\partial x^2}+\frac{\partial^2u_2}{\partial x^2}\frac{\partial^3u_1}{\partial x^3}\Big)dx\,d\tau$$
$$-\int_0^t\int_R b\Big(\frac{\partial u_1}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial^2u_1}{\partial x^2}\frac{\partial^2v}{\partial x^2}+\frac{\partial^3u_1}{\partial x^3}\frac{\partial^3v}{\partial x^3}\Big)dx\,d\tau$$
$$-\int_0^t\|Du_2(\tau)\|^2_{H^2}d\tau\Big)+\mu^2\frac{d_0}{c_0}\int_0^t\tau\|D^2v\|^2d\tau$$
$$+\frac{\alpha}{c_0}\mu^2\int_0^t\tau\|Du_2\|^2d\tau-\frac{\mu^2}{2}\Big(\frac{a_0}{c_0}\int_0^t\|Du_1\|^2d\tau$$
$$+\frac{1}{c_0}\int_0^t\|Du_2\|^2d\tau+\int_0^t\|Dv\|^2d\tau\Big)+\mu^3\Big(\int_0^t\tau\int_R a\Big(\frac{\partial^2u_1}{\partial x^2}\Big)^2dx\,d\tau$$
$$-\int_0^t\tau\|D^2u_2(\tau)\|^2d\tau+\int_0^t\int_R\frac{\partial u_2}{\partial x}\frac{\partial^2u_1}{\partial x^2}dx\,d\tau$$
$$-\alpha\int_0^t\tau\int_R\frac{\partial u_2}{\partial x}\frac{\partial^2u_1}{\partial x^2}dx\,d\tau-\int_0^t\tau\int_R b\frac{\partial^2u_1}{\partial x^2}\frac{\partial^2v}{\partial x^2}dx\,d\tau\Big)$$
$$+\mu^4\Big(\frac{d_0}{c_0}\int_0^t\tau^2\|D^3v\|^2d\tau+\frac{\alpha}{c_0}\int_0^t\tau^2\|D^2u_2\|^2d\tau$$
$$-\frac{a_0}{c_0}\int_0^t\tau\|D^2u_1\|^2d\tau-\frac{1}{c_0}\int_0^t\tau\|D^2u_2\|^2d\tau$$
$$-\int_0^t\tau\|D^2v\|^2d\tau\Big)+\mu^5\Big(\int_0^t\tau^2\int_R d\Big(\frac{\partial^4v}{\partial x^4}\Big)^2dx\,d\tau$$
$$+\alpha\int_0^t\tau^2\|D^3u_2\|^2d\tau\Big)-\mu^5\Big(\int_0^t\tau\int_R a\Big(\frac{\partial^3u_1}{\partial x^3}\Big)^2dx\,d\tau$$
$$+\int_0^t\tau\|D^3u_2\|^2d\tau+\int_0^t\tau\int_R c(D^3v)^2dx\,d\tau\Big)$$
$$+\mu^6\Big(\int_0^t\tau^2\int_R a\Big(\frac{\partial^3u_1}{\partial x^3}\Big)^2dx\,d\tau-2\int_0^t\tau\int_R\frac{\partial^2u_2}{\partial x^2}\frac{\partial^3u_1}{\partial x^3}dx\,d\tau$$
$$-\int_0^t\tau^2\|D^3u_2\|^2d\tau-\int_0^t\tau^2\int_R\frac{\partial^2u_2}{\partial x^2}\frac{\partial^3u_1}{\partial x^3}dx\,d\tau$$
$$-\int_0^t\tau^2\int_R b\frac{\partial^3u_1}{\partial x^3}\frac{\partial^3v}{\partial x^3}dx\,d\tau\Big), \tag{2.32}$$

$$H(t)=\int_0^t\int_R(f_1u_2+f_2v)dx\,d\tau+\int_0^t\int_R(Df_1Du_2+Df_2Dv)dx\,d\tau$$
$$+\int_0^t\int_R(D^2f_1D^2u_2+D^2f_2D^2v)dx\,d\tau+\frac{1}{2}\int_0^t\int_R\frac{\partial a}{\partial t}(D^3u_1)^2dx\,d\tau$$
$$-\int_0^t\int_R\frac{\partial a}{\partial x}D^3u_1D^3u_2dx\,d\tau+\frac{1}{2}\int_0^t\int_R\frac{\partial c}{\partial t}(D^3v)^2dx\,d\tau$$

$$+\int_0^t\int_R \frac{\partial b}{\partial x} D^3vD^3u_2\,dx\,d\tau+\int_0^t\int_R (F_1D^3u_2+F_2D^3v)\,dx\,d\tau$$

$$+\mu\left(\int_0^t\int_R \frac{\partial b}{\partial x}\frac{\partial^2 u_1}{\partial x^2}-\frac{\partial v}{\partial x}\,dx\,d\tau-\int_0^t\int_R \frac{\partial a}{\partial x}\frac{\partial u_1}{\partial x}\frac{\partial^2 u_1}{\partial x^2}\,dx\,d\tau\right.$$

$$-\int_0^t\int_R \frac{\partial a}{\partial x}\frac{\partial^2 u_1}{\partial x^2}\frac{\partial^3 u_1}{\partial x^3}\,dx\,d\tau-\int_0^t\int_R \frac{\partial^2 a}{\partial x^2}\frac{\partial u_1}{\partial x}\frac{\partial^3 u_1}{\partial x^3}\,dx\,d\tau$$

$$+\int_0^t\int_R \frac{\partial^2 b}{\partial x^2}\frac{\partial^3 u_1}{\partial x^3}\frac{\partial v}{\partial x}\,dx\,d\tau+\int_0^t\int_R \frac{\partial b}{\partial x}\frac{\partial^3 u_1}{\partial x^3}\frac{\partial^2 v}{\partial x^2}\,dx\,d\tau$$

$$+\mu^2\int_0^t\tau\int_R (Df_1Du_2+Df_2Dv)\,dx\,d\tau+\mu^3\left(\int_0^t\tau\int_R \frac{\partial b}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2 u_1}{\partial x^2}\,dx\,d\tau\right.$$

$$\left.-\int_0^t\tau\int_R \frac{\partial a}{\partial x}\frac{\partial u_1}{\partial x}\frac{\partial^2 u_1}{\partial x^2}\,dx\,d\tau\right)+\mu^4\left(\int_0^t\tau^2\int_R (D^2f_1D^2u_2\right.$$

$$\left.+D^2f_2D^2v)\,dx\,d\tau\right)+\mu^5\left(\int_0^t\tau^2\int_R \frac{\partial a}{\partial t}(D^3u_1)^2\,dx\,d\tau\right.$$

$$-\int_0^t\tau^2\int_R \frac{\partial a}{\partial x}D^3u_1D^3u_2\,dx\,d\tau+\frac{1}{2}\int_0^t\tau^2\int_R \frac{\partial c}{\partial t}(D^3v)^2\,dx\,d\tau$$

$$\left.+\int_0^t\tau^2\int_R \frac{\partial b}{\partial x}D^3vD^3u_2\,dx\,d\tau+\int_0^t\tau^2\int_R (F_1D^3u_2+F_2D^3v)\,dx\,d\tau\right)$$

$$+\mu^6\left(\int_0^t\tau^2\int_R \frac{\partial^2 b}{\partial x^2}\frac{\partial^3 u_1}{\partial x^3}\frac{\partial v}{\partial x}\,dx\,d\tau+\int_0^t\tau^2\int_R \frac{\partial b}{\partial x}\frac{\partial^2 v}{\partial x^2}\frac{\partial^3 u_1}{\partial x^3}\,dx\,d\tau\right.$$

$$\left.-\int_0^t\tau^2\int_R \frac{\partial a}{\partial x}\frac{\partial^2 u_1}{\partial x^2}\frac{\partial^3 u_1}{\partial x^3}\,dx\,d\tau-\int_0^t\tau^2\int_R \frac{\partial^2 a}{\partial x^2}\frac{\partial u_1}{\partial x}\frac{\partial^3 u_1}{\partial x^3}\,dx\,d\tau\right), \tag{2.33}$$

$$\widetilde{R}_3(u_{10}, u_{20}, v_0)=\frac{1}{2}\left(\frac{a_0}{c_0}\|u_{10}\|^2_{H^2}+\frac{1}{c_0}\|u_{20}\|^2_{H^2}+\|v_0\|^2_{H^2}\right)$$
$$+\frac{1}{2}\int_R a(u_{10}, v_0)(D^3u_{10})^2\,dx+\|D^3u_{20}\|^2+\int_R c(u_{10}, v_0)(D^3v_0)^2\,dx$$
$$-\mu\left(\int_R u_{20}\frac{\partial u_{10}}{\partial x}\,dx+\int_R \frac{\partial u_{20}}{\partial x}\frac{\partial^2 u_{10}}{\partial x^2}\,dx+\int_R \frac{\partial^2 u_{20}}{\partial x^2}\frac{\partial^3 u_{10}}{\partial x^3}\,dx\right). \tag{2.34}$$

It follows from the assumptions (i), (ii), the inequality $ab\leqslant\frac{\varepsilon a^2}{2}+\frac{b^2}{2\varepsilon}$, $E\leqslant E_0$ that when $\mu$ is appropriately small, that is, there exists a constant $\mu_0>0$ such that when $\mu\leqslant\mu_0$, we have

$$C_4N^2(0, t)\leqslant I_1(t)+I_2(t), \quad \forall t\in[0, T], \tag{2.35}$$

where $C_4>0$ is a positive constant independent of $T$ and the solution $(u_1, u_2, v)$.

On the other hand, it is easy to see that

$$|\widetilde{R}_3(u_{10}, u_{20}, v_0)|\leqslant C_5\|(u_{10}, u_{20}, v_0)\|^2_{H^3}. \tag{2.36}$$

For our punpose it remains to estimare $H(t)$.

It follows from

$$u^2(x, t)=2\int_{-\infty}^{x} uu_x\,dx \tag{2.37}$$

that for $0\leqslant\tau\leqslant t\leqslant T$,

$$(1+\tau)^{\frac{1}{2}}|(u_1, u_2, v)(\tau)|^2_{L^\infty}\leqslant\|(u_1, u_2, v)(\tau)\|^2$$
$$+(1+\tau)\|D(u_1, u_2, v)(\tau)\|^2\leqslant N^2(0, t). \tag{2.38}$$

Similarly, we have

$$(1+\tau)^{\frac{3}{2}}|D(u_1, u_2, v)(\tau)|^2_{L^\infty}\leqslant(1+\tau)\|D(u_1, u_2, v)(\tau)\|^2$$
$$+(1+\tau)^2\|D^2(u_1, u_2, v)(\tau)\|^2\leqslant 2N^2(0, t), \tag{2.39}$$
$$(1+\tau)^2|D^2(u_1, u_2, v)(\tau)|^2_{L^\infty}\leqslant(1+\tau)^2\|D^2(u_1, u_2, v)(\tau)\|^2$$
$$+(1+\tau)^2\|D^3(u_1, u_2, v)(\tau)\|^2\leqslant 2N^2(0, t). \tag{2.40}$$

Therefore, by the assumption of theorem $N^2(0, t)\leqslant E^2\leqslant E_0^2$,

$$\left|\int_0^t\int_R \frac{\partial a}{\partial x}D^3u_1D^3u_2\,dx\,d\tau\right|\leqslant CN(0, t)\left(\int_0^t\|D^3u_1\|^2d\tau+\int_0^t\|D^3u_2\|^2d\tau\right)$$
$$\leqslant CEN^2(0, t). \tag{2.41}$$

Similarly, by the expressions (2.7) of $f_1$, $f_2$,

$$\left|\int_0^t\int_R f_1u_2\,dx\,d\tau\right.\leqslant CEN^2(0, t), \tag{2.42}$$

$$\left|\int_0^t\int_R f_2v\,dx\,d\tau\right|\leqslant CE\int_0^t(\|u_2\|^2+\|Du_1\|^2+\|Dv\|^2)d\tau\leqslant CEN^2(0, t). \tag{2.43}$$

From the expressions (2.16), we obtain

$$\|F_1\|\leqslant C(|Du_1|_{L^\infty}\|D^3a\|+|D^2a|_{L^\infty}\|D^2u_1\|+|Da|_{L^\infty}\|D^3u_1\|+|Db|_{L^\infty}\|D^3v\|$$
$$+|D^2b|_{L^\infty}\|D^2v\|+|Dv|_{L^\infty}\|D^3b\|), \tag{2.44}$$

$$\left|\int_R F_2D^3v\,dx\right|\leqslant C(|D(u_1, v)|_{L^\infty}\|D^3v\|\|D^4v\|+|D^2(u_1, v)|_{L^\infty}\|D^3v\|\|D^4v\|$$
$$+|(Du_1, Dv, D^2u_1, D^2v)|_{L^\infty}(\|D^3u_1\|+\|D^3v\|)\|D^3v\|). \tag{2.45}$$

Therefore

$$\left|\int_0^t\int_R F_1D^3u_2+F_2D^3v\,dx\,d\tau\right|\leqslant CEN^2(0,t). \tag{2.46}$$

Furthermore, by (2.38)—(2.40) we obtain

$$\left|\int_0^t\tau^2\int_R(F_1D^3u_2+F_2D^3v)\,dx\,d\tau\right|$$
$$\leqslant C\left(\sup_{0\leqslant\tau\leqslant t}|D(u_1, v)|_{L^\infty}\int_0^t\tau^2(\|D^3u_1\|^2+\|D^3u_2\|^2+\|D^3v\|^2+\|D^4v\|^2)d\tau\right.$$
$$+\sup_{0\leqslant\tau\leqslant t}\tau^{\frac{1}{2}}|D^2(u_1, v)|_{L^\infty}\left(\int_0^t\tau\|D^2(u_1, v)\|^2d\tau\right.$$
$$\left.\left.+\int_0^t\tau^2\|D^3(u_1, v)\|^2d\tau\right)\right)\leqslant CEN^2(0, t), \tag{2.47}$$

$$\left|\int_0^t\tau\int_R\frac{\partial a}{\partial x}\frac{\partial u_1}{\partial x}\frac{\partial^2u_1}{\partial x^2}\,dx\,d\tau\right|\leqslant C\sup_{0<\tau<t}\tau^{\frac{1}{2}}|D(u_1, v)|_{L^\infty}$$
$$\times\left(\int_0^t\|Du_1\|^2d\tau+\int_0^t\tau\|D^2u_1\|^2d\tau\right)\leqslant CEN^2(0, t), \tag{2.48}$$

$$\left|\int_0^t \tau^2 \int_R \frac{\partial^2 a}{\partial x^2} \frac{\partial u_1}{\partial x} \frac{\partial^3 u_1}{\partial x^3}\, dx\, d\tau\right| \leqslant C\,(\sup_{0\leqslant\tau\leqslant t} \tau^{\frac{1}{2}} |Du_1(\tau)|_{L^\infty}$$
$$\times\left(\int_0^t \tau \|D^2(u_1, v)\|^2 d\tau + \int_0^t \tau^2 \|D^3 u_1\|^2 d\tau\right) + (\sup_{0\leqslant\tau\leqslant t} \tau^{\frac{1}{2}} |D(u_1, v)|_{L^\infty})^2$$
$$\times\left(\int_0^t \|Du_1\|^2 d\tau + \int_0^t \tau^2 \|D^3 u_1\|^2 d\tau\right)\right) \leqslant CEN^2(0, t). \tag{2.49}$$

The other terms of $H(t)$ in (2.33) can be estimated in the same way as above. Thus we arrive at

$$H(t) \leqslant C_6 EN^2(0, t). \tag{2.50}$$

Let

$$\varepsilon_2 = \min\left(\varepsilon_1, \frac{C_4}{2C_6}\right). \tag{2.51}$$

Thus when $E \leqslant \varepsilon_2$, it follows from (2.35), (2.36), (2.50) that

$$N^2(0, t) \leqslant \frac{2C_5}{C_4} \|(u_{10}, u_{20}, v_0)\|^2_{H^3}. \tag{2.52}$$

Let $C_2 = \frac{2C_5}{\sqrt{C_4}}$, thus the proof is completed.

## § 3. The Proof of Main Theorem

Based on the local existerce theorem and the a priori estimates obtained in the previous sections, we are going to prove our Main Theorem.

Let

$$\varepsilon_0 = \min\left(\varepsilon_1, \varepsilon_2, \frac{\varepsilon_2}{C_1}, \frac{\varepsilon_1}{C_2}, \frac{\varepsilon_2}{C_2\sqrt{1+C_1^2}}\right). \tag{3.1}$$

Thus when $\|(u_{10}, u_{20}, v_0)\|_{H^3} \leqslant \varepsilon_0$, by Theorem 1 we have a unique local solution

$$(u_1, u_2, v) \in X(0, \delta; C_1\|(u_{10}, u_{20}, v_0)\|_{H^3}). \tag{3.2}$$

Since $C_1\|(u_{10}, u_{20}, v_0)\|_{H^3} \leqslant C_1\varepsilon_0 \leqslant \varepsilon_2$, Theorem 2 gives

$$(u_1, u_2, v) \in X(0, \delta; C_2\|(u_{10}, u_{20}, v_0)\|_{H^3}). \tag{3.3}$$

Thus by $C_2\|(u_{10}, u_{20}, v_0)\|_{H^3} \leqslant \varepsilon_1$ and Theorem 1 with $t_1 = \delta$, we again have

$$(u_1, u_2, v) \in X(\delta, 2\delta; C_1C_2\|(u_{10}, u_{20}, v_0)\|_{H^3}). \tag{3.4}$$

It follows from the definition of $N^2(0, t)$ that

$$N^2(0, 2\delta) \leqslant N^2(0, \delta) + N^2(\delta, 2\delta). \tag{3.5}$$

Thus (5.3)—(5.5) lead to

$$(u_1, u_2, v) \in X(0, 2\delta; C_2\sqrt{1+C_1^2}\|(u_{10}, u_{20}, v_0)\|_{H^3}). \tag{3.6}$$

Now by the definition (3.1) of $\varepsilon_0$, we have

$$C_2\sqrt{1+C_1^2}\|(u_{10}, u_{20}, v_0)\| \leqslant \varepsilon_2. \tag{3.7}$$

Thus Theorem 2 again gives

$$(u_1, u_2, v) \in X(0, 2\delta; C_2\|(u_{10}, u_{20}, v_0)\|_{H^3}). \tag{3.8}$$

Using the same arguments on $[n\delta, (n+1)\delta]$ and $[0, (n+1)\delta]$

successively ($n=2, 3, \cdots$), we obtain the global existence and uniqueress of solutions. Thus for $0\leqslant t\leqslant +\infty$, we have

$$N^2(0, t)\leqslant C_2^2\|(u_{10}, u_{20}, v_0)\|^2_{H^3}. \tag{3.9}$$

It follows from the definition of $N$ and (3.9) that

$$\|D(u_1, u_2, v)(t)\|\leqslant C(1+t)^{-\frac{1}{2}}, \quad \forall t\geqslant 0, \tag{3.10}$$

$$\|D^2(u_1, u_2, v)(t)\|_{H^1}\leqslant C(1+t)^{-1}, \quad \forall t\geqslant 0. \tag{3.11}$$

Combining (3.10), (3.11) with (2.38)—(2.40), we obtain the decay rates of solutions. Thus the proof is completed.

**Remark.** Consider the Cauchy problem of the radiation hydrodynamic system with damping term $\alpha v$:

$$\begin{cases} \dfrac{\partial u}{\partial t}-\dfrac{\partial v}{\partial x}=0, \\ \dfrac{\partial v}{\partial t}-\dfrac{R\theta}{u^2}\dfrac{\partial u}{\partial x}+\left(\dfrac{R}{u}+\dfrac{16\sigma}{3C}\theta^3\right)\dfrac{\partial \theta}{\partial x}+\alpha v=0, \\ \left(\dfrac{R}{r-1}+\dfrac{16\sigma u}{C}\theta^3\right)\dfrac{\partial \theta}{\partial t}-\dfrac{\partial}{\partial x}\left(\dfrac{16\sigma A}{3}\dfrac{\theta^{3+\beta}}{u}\dfrac{\partial \theta}{\partial x}\right)+\left(\dfrac{R\theta}{u}+\dfrac{16\sigma}{3C}\theta^4\right)\dfrac{\partial v}{\partial x}=0, \\ t=0:\ u=u_0(x),\ v=v_0(x),\ \theta=\theta_0(x), \end{cases}$$

where $R$, $\sigma$, $\alpha$, $\gamma>1$, $A$, $\beta$, $C$ are positive constants (see [17]).

Using similar arguments we can get the global existence of solutions near equilibrium and the same decay rates.

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