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COMMUTING n-TUPLES OF CLOSED OPERATORS WHICH POSSESS SPECTRAL CAPACITY

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Abstract

This paper introduces the notions of commuting n-tuples of closed operetors which possess SVEP, SDP and spectral capacity respectively. A formula of analytic functional calculus for several commuting unbounded operators is found. With the help of the formula, it is proved that T has spectral capacity implies T has SDP and T has SDP implies T has SVEP. It is also proved that if T possesses spectral capacity $\mathscr{E}$ and f_j is analytic on a neighbourhood of $\sigma(T)$ for $j=1, 2,\cdots, k$, then $f(T)=(f_1(T), \cdots, f_k(T))$ is decomposable, and the spectral capacity $\mathscr{E}^*$ of $f(T)$ is uniquely determind by $\mathscr{E}^*(F)=\mathscr{E}(f^{-1}(F)\cap\sigma(T))$.

§ 1. Analytic Functional Calculus

In[1], X is a Banach space and $T_1, \cdots, T_n$ are closed operators on X. For any i, $1\leqslant i\leqslant n$, there exists $\xi_i\in\rho(T_i)\cap\mathbb{C}$. Let $a_i=(\xi_i-T_i)^{-1}$. If $a=(a_1, \cdots, a_n)$ is a commuting n-tuple of bounded operators, then $T=(T_1,\cdots T_n)$ is called a commuting n-tuple of close operators. Taylor spectrum, denoted by $\sigma(T)$, is the subset of $\hat{\mathbb{C}}^n$ $\left\{\left(\xi_1-\frac{1}{z_1}, \xi_2-\frac{1}{z_2}, \cdots, \xi_n-\frac{1}{z_n}\right)\middle|\ z=(z_1, \cdots, z_n)\in\sigma(a)\right\}$ where $\hat{\mathbb{C}}^n=\hat{\mathbb{C}}\times\hat{\mathbb{C}}\times\cdots\times\hat{\mathbb{C}}$ and $\hat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$. If $(i)=\{i_1, \cdots, i_p\}\subset\{1, \cdots, n\}$, $\tau=(t_1, \cdots, t_n)$ is an n-tuple of indeterminates, then $t_{(i)}$ denote $t_{i_1}\wedge\cdots\wedge t_{i_p}$. Let $D_{(i)}=\cap\{D_{T_{j_1}\cdots T_{j_k}}$ for any $\{j_1, \cdots, j_k\}\cap(i)=\varnothing\}$, where $D_{T_{j_1}\cdots T_{j_k}}$ is the domain of $T_{j_1\cdots j_k}$, and $D_{J_p}=\{\sum x_{(i)}t_{(i)}\,|\,x_{(i)}\in D_{(i)}, |(i)|=p\}$, $0\leqslant p\leqslant n$. J_p is a mapping from D_{J_p} into $D_{J_{p+1}}$: $\sum x_{(i)}t_{(i)}\to\sum\limits_{(i)}\sum\limits_{j}T_jx_{(i)}t_j\wedge t_{(i)}$. We shall use these notations in this paper.

Let U be an open set in $\hat{\mathbb{C}}^n$. We define

1. $\hat{A}_{(i)}(U, X)=\{f\,|\,f\in A(U, D_{(i)})$ and for any $\{j_1, \cdots, j_k\}\cap(i)=\varnothing$, $\prod\limits_{p=1}^{k}z_{j_p}f(z)\in A(U, D_{(i)})$ and $\prod\limits_{p=1}^{k}T_{j_p}\prod\limits_{p=1}^{k}z_{j_p}f(z)\in A(U, X)\}$;

$$A^p[\tau, \hat{A}(U, X)]=\{\sum f_{(i)}t_{(i)}\,|\,f_{(i)}\in\hat{A}_{(i)}(U, X)\},\ 0\leqslant p\leqslant n.$$

2. $C^\infty_{(i)(j)}(U, X)=\{f\,|\,f\in C^\infty(U, D_{(i)})$ and for any $\{i_1, \cdots, i_k\}\cap(i)=\varnothing$ and

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$\{j_1, \cdots, j_h\}\subset(j), \prod_{p=1}^{k} z_{i_p}\prod_{q=1}^{h} \bar z_{i_q}^2 f(z)\in C^\infty(U, D_{(i)})$ and $\prod_{p=1}^{k} T_{i_p}\prod_{p=1}^{k} z_{i_p}\prod_{q=1}^{h} \bar z_{j_q}^2 f(z)\in C^\infty(U, X)\}$;

$$\Lambda^p[\tau\cup d\bar z, \hat C^\infty(U, X)]=\{\sum f_{(i)(j)}\ |f_{(i)(j)}\in C^\infty_{(i)(j)}(U, X),$$
$$|(i)|+|(j)|=p\},\ 0\leqslant p\leqslant 2n.$$

For any $\psi=\sum f_{(i)}t_{(i)}\in\Lambda^p[\tau, \hat A(U, X)]$, J_p is a mapping from $\Lambda^p[\tau, \hat A(U, X)]$ into $\Lambda^{p+1}[\tau, \hat A(U, X)]$: $J_p\psi(z)=\sum_{(i)}\sum_j (z_j-T_j)\, f_{(i)}(z)t_j\wedge t_{(i)}$. $J_p\oplus\bar\partial$ is a mapping from $\Lambda^p[\tau\cup d\bar z, \hat C^\infty(U, X)]$ into $\Lambda^p[\tau\cup d\bar z, \hat C^\infty(U, X)]$:

$$(J_p\oplus\bar\partial)(\sum f_{(i)(j)}t_{(i)}\wedge d\bar z_{(j)})(z)$$
$$=\sum\sum_q(z_q-T_q)f_{(i)(j)}(z)t_q\wedge t_{(i)}\wedge d\bar z_{(j)}+\sum\sum_q\frac{\partial}{\partial\bar z_q}f_{(i)(j)}(z)d\bar z_q\wedge t_{(i)}\wedge d\bar z_{(j)}.$$

For conveniece, if $\xi=(\xi_1, \cdots, \xi_n)\in\mathbb{C}^n$, let $\mathbb{C}^n_\xi=\{z=(z_1, \cdots, z_n)\,|\,z\in\mathbb{C}^n,\ z_i\neq\xi_i,\ 1\leqslant i\leqslant n\}$ and $\frac{1}{\xi-U}=\left\{\left(\frac{1}{\xi_1-z_1}, \cdots, \frac{1}{\xi_n-z_n}\right)\Big|\ z\in U\right\}$.

Theorem 1.1. *Let $T=(T_1, \cdots, T_n)$ be a commuting n-tuple closed operators, $\xi_i\in\rho(T_i)\cap\mathbb{C}$, $a_i=(\xi_i-T_i)^{-1}$, $1\leqslant i\leqslant n$ and $a=(a_1, \cdots, a_n)$. Then the following diagram is commutative:*

$$\begin{array}{ccccccc}0\to\Lambda^0[\tau, \hat A(U, X)]&\xrightarrow{J}&\Lambda^1[\tau, \hat A(U, X)]&\to\cdots\to&\Lambda^n[\tau, \hat A(U, X)]&\to 0\\ \downarrow u_0&&\downarrow u_1&&\downarrow u_n&\\ 0\to\Lambda^0[\sigma, \hat A(V, X)]&\xrightarrow{\alpha}&\Lambda^1[\sigma, \hat A(V, X)]&\to\cdots\to&\Lambda^n[\sigma, \hat A(V, X)]&\to 0,\end{array}$$

where $V=\frac{1}{\xi-U}$ and u_p is an isomorphism from $\Lambda^p[\tau, \hat A(U, X)]$ onto $\Lambda^p[\sigma, \hat A(V, X)]$: for any $f_{(i)}t_{(i)}\in\Lambda^p[\tau, \hat A(U, X)]$, $u_p(f_{(i)}t_{(i)})(\lambda)=\prod_{j\in(i)}\left(\frac{T_j-\xi_j}{\lambda_j}\right)f_{(i)}\left(\xi-\frac{1}{\lambda}\right)s_{(i)}$. Hence, if let

$$H^p[\hat A(U, X), J]=\ker J_p/\operatorname{Im} J_{p-1},$$

then

$$H^p[\hat A(U, X), J]\cong H^p[\hat A(V, X), \alpha].$$

Proof For any $g_{(i)}s_{(i)}\in\Lambda^p[\sigma, \hat A(V, X)]$, we can define another mapping ν_p: $\nu_p(g_{(i)}s_{(i)})(z)=\prod_{j\in(i)}\left(\frac{-a_j}{\xi_j-z_j}\right)g_{(i)}\left(\frac{1}{\xi-z}\right)t_{(i)}$. It is clear that $v_p\circ u_p=1$ and $u_p\circ v_p=1$.

For any $z\in U$, $\lambda\in\frac{1}{\xi-z}$,

$$(\alpha_p\circ u_p)(f_{(i)}t_{(i)})(\lambda)=\alpha_p(\lambda)\left(\prod_{j\in(i)}\left(\frac{T_j-\xi_j}{\lambda_j}\right)f_{(i)}\left(\xi-\frac{1}{\lambda}\right)s_{(i)}\right.$$
$$=\sum_{k\in(i)}(\lambda_k-a_k)\left(\prod_{j\in(i)}\left(\frac{T_j-\xi_j}{\lambda_j}\right)f_{(i)}\left(\xi-\frac{1}{\lambda}\right)s_k\wedge s_{(i)}\right)$$
$$=\sum_{k\in(i)}\left(\xi_k-\frac{1}{\lambda_k}-T_k\right)\left(\prod_{j\in(i)\cup(k)}\left(\frac{T_j-\xi_j}{\lambda_j}\right)f_{(i)}\left(\xi-\frac{1}{\lambda}\right)s_k\wedge s_{(i)}\right)$$
$$=u_{p+1}\left(\sum_{k\in(i)}(z_k-T_k)f_{(i)}(z)t_k\wedge t_{(i)}\right)=(u_{p+1}\circ J_p)(f_{(i)}t_{(i)})(\lambda).$$

Thus we complete the proof.

Theorem 1.2. *Let T and a be as in Theorem 1.1. Then the following diagram* *is commutative:*

$$\begin{array}{ccccccc} 0\to\Lambda^0[\tau\cup d\bar z,\ \hat{O}^\infty(U,X)] & \to & \Lambda^1[\tau\cup d\bar z,\ \hat{O}^\infty(U,X)] & \to\cdots\to & \Lambda^{2n}[\tau\cup d\bar z,\ \hat{O}^\infty(U,X)] & \to 0 \\ \downarrow\omega_0 & & \downarrow\omega_1 & & \downarrow\omega_n & \\ 0\to\Lambda^0[\sigma\cup d\bar\lambda,\ \hat{O}^\infty(V,X)] & \to & \Lambda^1[\sigma\cup d\bar\lambda,\ \hat{O}^\infty(V,X)] & \to\cdots\to & \Lambda^{2n}[\sigma\cup d\bar\lambda,\ \hat{O}^\infty(V,X)] & \to 0, \end{array}$$

where $V=\dfrac{1}{\xi-U}$ *and* w_p *is an isomorphism from* $\Lambda^p[\tau\cup d\bar z,\ \hat{O}^\infty(U,X)]$ *onto* $\Lambda^p[\sigma\cup d\bar\lambda,\ \hat{O}^\infty(V,X)]$: *for any* $f_{(i)(j)}t_{(i)}\wedge d\bar z_{(j)}\in\Lambda^p[\tau\cup d\bar z,\ \hat{O}^\infty(U,X)]$,

$$w_p(f_{(i)(j)}t_{(i)}\Lambda d\bar z_{(j)})(\lambda)=(-1)^{|(j)|}\prod_{k\in(i)}\left(\frac{T_k-\xi_k}{\lambda_k}\right)\prod_{h\in(j)}\frac{1}{\bar\lambda_h^2}f_{(i)(j)}\left(\xi-\frac{1}{\lambda}\right)s_{(i)}\Lambda d\bar\lambda_{(j)}.$$

Hence, if let

$$H^p[\hat{O}^\infty(U,X),\ J\oplus\bar\partial]=\ker(J_p\oplus\bar\partial)/\mathrm{Im}(J_{p-1}\oplus\bar\partial),$$

then

$$H^p[\hat{O}(U,X),\ J\oplus\bar\partial]\cong H^p[\hat{O}(U,X),\ \alpha\oplus\bar\partial],\ 0\leqslant p\leqslant 2n.$$

Proof If $g_{(i)(j)}s_{(i)}\wedge d\bar\lambda_{(j)}\in\Lambda^p[\sigma\cup d\bar\lambda,\ \hat{O}^\infty(V,X)]$, then the mapping r_p:

$$r_p(g_{(i)(j)}s_{(i)}\wedge d\bar\lambda_{(j)})(z)=(-1)^{|(j)|}\prod_{k\in(i)}\left(\frac{-a_k}{\xi_k-z_k}\right)\prod_{h\in(j)}g_{(i)(j)}\left(\frac{1}{\xi-z}\right)t_{(i)}\wedge d\bar z_{(j)}$$

satisfies the condition $w_p\circ r_p=I$ and $r_p\circ w_p=I$.

For any $z\in U$, $\lambda\in\dfrac{1}{\xi-z}$

$$\begin{aligned}&(\alpha\oplus\bar\partial)\circ w_p(f_{(i)(j)}t_{(i)}\wedge d\bar z_{(j)})(\lambda)\\ &=\sum_{q\in(i)}(\lambda_q-a_q)\left[(-1)^{|(j)|}\prod_{k\in(i)}\left(\frac{T_k-\xi_k}{\lambda_k}\prod_{h\in(j)}\frac{1}{\bar\lambda_h^2}f_{(i)(j)}\left(\xi-\frac{1}{\lambda}\right)\right]s_q\wedge s_{(i)}\wedge d\bar\lambda_{(j)}\\ &\quad+\sum_{q\in(j)}\frac{\partial}{\partial\bar\lambda_p}\left[(-1)^{|(j)|}\prod_{k\in(i)}\left(\frac{T_k-\xi_k}{\lambda_k}\right)\prod_{h\in(j)}\frac{1}{\bar\lambda_h^2}f_{(i)(j)}\right]d\bar\lambda_q\wedge s_{(i)}\wedge d\bar\lambda_{(j)}\\ &=\sum_{q\in(i)}\left(\xi_q-\frac{1}{\lambda_q}-T_q\right)\left[(-1)^{|(j)|}\prod_{k\in(i)\cup\{q\}}\left(\frac{T_k-\xi_k}{\lambda_k}\right)\prod_{h\in(j)}\frac{1}{\bar\lambda_h^2}f_{(i)(j)}\right]s_q\wedge s_{(i)}\wedge d\bar\lambda_{(j)}\\ &\quad+\sum_{q\in(j)}(-1)^{|(j)|+1}\prod_{k\in(i)}\left(\frac{T_k-\xi_k}{\lambda_k}\right)\prod_{h\in(j)\cup\{q\}}\frac{1}{\bar\lambda_h^2}\frac{\partial}{\partial\bar z_q}f_{(i)(j)}d\bar\lambda_q\wedge s_{(i)}\wedge d\bar\lambda_{(j)}\\ &=(w_{p+1}\circ J_p)(f_{(i)(j)}t_{(i)}\wedge d\bar z_{(j)})(\lambda)+(w_{p+1}\circ\bar\partial)(f_{(i)(j)}t_{(i)}\wedge d\bar z_{(j)})(\lambda)\\ &=w_{p+1}\circ(J_p\oplus\bar\partial)(f_{(i)(j)}t_{(i)}\wedge d\bar z_{(j)})(\lambda).\end{aligned}$$

Suppose U is an open set in $\hat{\mathbb{C}}^n$. We shall denote by

$$\hat{O}^p(U,X)=\bigcup_{z\in U}\{(\psi)_z\,|\,\psi\in\Lambda^p[\tau,\ \hat{A}(W,X)]\text{ for some neighbourhood of }z\}$$

the sheaf of all germs of analytic forms of degree p, and denote by

$$\hat{B}^p(U,X)=\bigcup_{z\in U}\{(\varphi)_z\,|\,\varphi\in\Lambda^p[\tau\cup d\bar z,\ \hat{O}^\infty(W,X)]\text{ for some neighbourhood of }z\}$$

the sheaf of all germs of somooth forms of degree p. For any $z\in U$, we have operator J_p(or $J_p\oplus\bar\partial$) from $O^p(z)$ (or $B^p(z)$) into $O^{p+1}(z)$ (or $B^{p+1}(z)$) and define

$$H^p[\hat{A}(\{z\},X),\ J]=\ker J_p/\mathrm{Im}J_{p+1},$$

$$H^p[\hat{O}^\infty(\{z\},X),\ J\oplus\bar\partial]=\ker J_p\oplus\bar\partial/\mathrm{Im}J_{p-1}\oplus\bar\partial.$$

Obviously, $\hat{B}^p(U, X)$ is a fine sheaf.

By Theorems 1. 1 and 1. 2 we have the following corollary.

Corollary 1.3. *For any* $z\in\hat{\mathbb{C}}^n$ *and* $\lambda=\frac{1}{\xi-z}$,

$$H^p[\hat{A}(\{z\}, X), J]\cong H^p[\hat{A}(\{\lambda\},X), \alpha], 0\leqslant p\leqslant n;$$
$$H^p[\hat{C}^\infty(\{z\}, X), J\oplus\bar{\partial}]\cong H^p[\hat{C}^\infty(\{\lambda\}, X), \alpha\oplus\bar{\partial}], 0\leqslant p\leqslant 2n.$$

Corollary 1.4. *For any* $z\in\rho(T)$,

$$H^p[\hat{A}(\{z\}, X), J]=0, 0\leqslant p\leqslant n;$$
$$H^p[\hat{C}^\infty(\{z\}, X), J\oplus\bar{\partial}]=0, 0\leqslant p\leqslant 2n.$$

Proof If $z=(z_1, \cdots, z_n)\in\rho(T)$, we can choose $\xi_i\in\rho(T_i)$ but $\xi_i\neq z_i$, $1\leqslant i\leqslant n$. Then $\lambda=\frac{1}{\xi-z}\in\mathbb{C}^n\cap\rho(\alpha)$. Thus we have $H^p[\hat{A}(\{z\}, X), J]\cong H^p[\hat{A}(\{\lambda\}, X), \alpha]=H^p[A(\{\lambda\}, X), \alpha]=0$ by [2] Lemma 2.2. Similarly, $H^p[\hat{C}^\infty(\{z\}, X), J\oplus\bar{\partial}]\cong H^p[C^\infty(\{\lambda\}, X), \alpha\oplus\bar{\partial}]=0$.

Proposition 1.5. *For any open set* $U\subset\rho(T)$, *the sequence*

$$0\to\Lambda^0[\tau\cup d\bar{z}, \hat{C}^\infty(U, X)]\to\Lambda^1[\tau\subset d\bar{z}, \hat{C}^\infty(U, X)]\to\cdots\to\Lambda^{2n}[\tau\cup d\bar{z}, \hat{C}^\infty(U, X)]\to 0$$

is exact.

Proof For any $\psi\in\Lambda^p[\tau\cup d\bar{z}, \hat{C}^\infty(U, X)]$, $(\psi)_z$ is a section of the sheaf $\hat{B}^p$. Since $\hat{B}^p(U, X)$ is a fine sheaf and for $z\in U$, $H^p[\hat{C}^\infty(\{z\}, X), J\oplus\bar{\partial}]=0$, by [7] Propositions 6. 3. 2 and 6. 3. 6, the sequence is exact.

Corollary 1.6. *If f is analytic function on an neighbourhood U of $\sigma(T)$, then for any $x\in X$ there exists $\psi\in\Lambda^{n-1}[\tau\cup d\bar{z}, \hat{C}^\infty(V, X)]$, $V=U\cap\rho(T)$, such that* $tf(z)x=f(z)xt_1\wedge\cdots\wedge t_n=(J\oplus\bar{\partial})\psi$.

Proof By definition, $tf(z)x\in\Lambda^n[\tau\cup d\bar{z}, \hat{C}^\infty(V, X)]$ and $(J\oplus\bar{\partial})(tf(z)x)=0$, then the Corollary is an immediate consequence of Proposition 1. 5.

We shall define the single valued extention property. Before stating the definition, we shall have variant of Dilbeanlt-Grothendieck Lemma, like [4] Lemma 2.1.

For convenience, let

$$\Lambda^p_q[\tau\cup d\bar{z}, \hat{C}^\infty(U, X)]=\{\sum f_{(i)(j)}t_{(i)}\wedge d\bar{z}_{(j)}\,|\,|(i)|+|(j)|=p,\ |(i)|=q\}.$$

Lemma 1.7. *If G is an open polydisc in $\mathbb{C}^n$, then for any k, $0\leqslant k\leqslant n$, the sequence*

$$0\to\Lambda^k[\tau, \hat{A}(G, X)]\xrightarrow{i}\Lambda^k_k[\tau\cup d\bar{z}, \hat{C}^\infty(GX)]\xrightarrow{\bar{\partial}}\Lambda^{k+1}_k[\tau\cup d\bar{z}, \hat{C}^\infty(G, X)]$$
$$\to\cdots\xrightarrow{\bar{\partial}}\Lambda^{k+n}_k[\tau\cup d\bar{z}\hat{C}^\infty(G, X)]\to 0$$

is exact.

Proof First we show it is exact if $k=0$. Our proof is similar to that of [7] Theorems 5.8.1 and 5.8.2 but slightly complicated. We omit the details.

Now, suppose $k\geqslant 1$. If $\psi\in\Lambda^{k+p+1}_k[\tau\cup d\bar{z}, \hat{C}^\infty(G, X)]$ and $\psi=\sum\limits_{(i)}\sum\limits_{(j)}f_{(i)(j)}d\bar{z}_{(j)}\wedge t_{(i)}$, then $\bar\partial\psi=0$ is equivalent to $\bar\partial(\sum_{(j)} f_{(i)(j)}d\bar z_{(j)})=0$ for any (i). Similarly, we can construct $\sum g_{(i)(j)}d\bar z_{(j)}$ satisfying $g_{(i)(j)}\in C^\infty(G, D_{(i)})$ and $T_{j_1}\cdots T_{j_k}g_{(i)(j)}\in C^\infty(G, X)$ for any $\{j_1, \cdots, j_k\}\cap(i)=\varnothing$ and $\bar\partial(\sum_{(j)} g_{(i)(j)}d\bar z_{(j)})=\sum f_{(i)(j)}d\bar z_{(j)}$. Let $\varphi_{(i)}=\sum_{(j)} g_{(i)(j)}d\bar z_{(j)}$ and $\varphi=\sum_{(i)}\varphi_{(i)}\wedge t_{(i)}$. Then $\varphi\in\Lambda_k^{k+p}[\tau\cup d\bar z, \hat C^\infty(G, X)]$ and $\bar\partial\varphi=\sum\bar\partial\varphi_{(i)}=\psi$.

If $G=G_1\times\cdots\times G_n$ is an open polydisc in $\hat{\mathbb{C}}^n$, where $G_i=\{z\,|\,|z-b_i|<r_i\}$ or $G_i=\{z\,|\,|z|>r_i\}$, we denote by G^1 the polydisc in $\mathbb{C}^n$, $G^1=G_1^1\times\cdots\times G_n^1$, where $G_i^1=G_i$ if $G_i\subset\mathbb{C}$ and $G_i^1=\frac{1}{G_i}$ otherwise.

Lemma 1.8. *If G is an open polydisc in $\hat{\mathbb{C}}^n$, then the sequence in Lemma 1.7 is also exact.*

Proof Without loss of generality, we may assume $G_i=\{z\,|\,|z|>r_i\}$, $i>k$ and $G_i=\{z\,|\,|z-b_i|<r_i\}$, $i\leqslant k$. Define isomorphisms

$$\delta_p:\Lambda_k^{k+p}[\tau\cup d\bar z, \hat C^\infty(G, X)]\to\Lambda_k^{k+p}[\tau\cup d\bar\lambda, \hat C^\infty(G^1, X)],$$

$$\delta_p f_{(i)(j)}d\bar z_{(j)}\wedge t_{(i)}(A)=\prod_{q\in(i)\setminus\{1,\cdots,k\}}\frac{1}{\bar\lambda_q^2}f^*_{(i)(j)}d\bar\lambda_{(j)}\wedge t_{(i)},$$

where

$$f^*(\lambda)=f\Big(\lambda_1, \cdots, \lambda_k, \frac{1}{\lambda_{k+1}}, \cdots, \frac{1}{\lambda_n}\Big),\ 0\leqslant p\leqslant n,$$

and δ_*: $\Lambda^k[\tau, \hat A(G, X)]\to\Lambda^k[\tau, \hat A(G^1, X)]$, $\delta_* f_{(i)}t_{(i)}=f^*_{(i)}t_{(i)}$, It is easy to verify that the follow diagram is commutative.

$$\begin{array}{ccccccc}
0\to\Lambda^k[\tau, \hat A(G, X)] & \xrightarrow{i} & \Lambda_k^k[\tau\cup d\bar z, \hat C^\infty(G, X)] & \to\cdots\xrightarrow{\bar\partial} & \Lambda_k^{k+n}[\tau\cup d\bar z, \hat C^\infty(G, X)]\to 0\\
\downarrow\delta_* & & \downarrow\delta_0 & & \downarrow\delta_n\\
0\to\Lambda^k[\tau, \hat A(G^1, X)] & \xrightarrow{i} & \Lambda_k^k[\tau\cup d\bar\lambda, \hat C^\infty(G^1, X) & \to\cdots\xrightarrow{\bar\partial} & \Lambda_k^{k+n}[\tau\cup d\bar z, \hat C^\infty(G^1, X)]\to 0.
\end{array}$$

Then, it follows from Lemma 1.7 that the sequence is exact.

Definition 1.9. *If for any $z\in\hat{\mathbb{C}}^n$, $H^p[\hat A(\{z\}, X), J]=0$, $0\leqslant p\leqslant n-1$, then T is said to have single valued extension property* (abbrev. SVEP).

Theorem 1.10. *Suppose $T=(T_1, \cdots, T_n)$ is a commuting n-tuple of closed operators. Then the following coditions are equivalent:*

(1) $H^p[\hat A(\{z\}, X), J]=0$ *for all $z\in\hat{\mathbb{C}}^n$ and each $p=0, \cdots, n-1$;*

(2) $H^p[\hat C^\infty(\{z\}, X), J\oplus\bar\partial]=0$ *for all $z\in\hat{\mathbb{C}}^n$ and each $p=0, \cdots, n-1$;*

(3) $H^p[\hat C^\infty(U, X), J\oplus\bar\partial]=0$ *for each open set U in $\hat{\mathbb{C}}^n$ and $p=0, 1, \cdots, n-1$;*

(4) $H^p[\hat A(D, X), J]=0$ *for each open polydisc D in $\hat{\mathbb{C}}^n$ and $p=0, 1, \cdots, n-1$.*

Proof With the help of Lemmas 1.7 and 1.8, we can prove this Theorem using the same methods as in [2]. We omit the details.

Proposition 1.11. *If $T=(T_1, \cdots, T_n)$ is a commuting n-tuple of closed operators, $\xi_i\in\rho(T_i)\cap\mathbb{C}$, $a_i=(\xi_i-T_i)^{-1}$, $1\leqslant i\leqslant n$, and $a=(a_1, \cdots, a_n)$, then T has SVEP iff a has SVEP in the sense of* [2] *Definition* 1.1.

Proof For any $\lambda\in\hat{\mathbb{C}}^n\setminus\mathbb{C}^n$, if we regard $a=(a_1, \cdots, a_n)$ as a closed operator system, then $\lambda\in\rho(a)$. Hence $H^p[\hat{A}(\{\lambda\}, X), a]=0$ for each $p=0, 1, \cdots, n-1$. Then the proposition follows from Proposition 1.3.

Definition 1.12. *Suppose $T=(T_1, \cdots, T_n)$ have SVEP, the local spectrum $\sigma(T, x)$ relative to $x\in X$ is the complement of $\rho(T, x)=\bigcup\{U\subset\hat{\mathbb{C}}^n$ open $|$ there is $\psi\in\Lambda^{n-1}[\tau\cup d\bar{z}, \hat{C}^\infty(U, X)]$ such that $tx=(J\oplus\bar{\partial})\psi\}$.*

Proposition 1.13. *Let T and a be as in Proposition 1.11, for any $x\in X$,*

$$\sigma(T, x)=\xi-\frac{1}{\sigma(a, x)}$$

and $\pi_i(\sigma(T, x))=\sigma(T_i, x)$, $1\leqslant i\leqslant n$, where π_i is the projection of $\hat{\mathbb{C}}^n$ onto its i-th coordinate.

Proof Immediately from the definition and Theorem 1.2 and [2] Corollary 2.2.

In [1], Eschmeier defines the analytic functional calculus for n-tuples of closed operators. If f is an analytic functional on a neibourhood of $\sigma(T)$, then f_ξ is an analytic function on a neighbourhood of $\sigma(a)$, where $f_\xi(\lambda)=f\left(\xi-\frac{1}{\lambda}\right)$. $f(T)$ is defined to be $f_\xi(a)$. We try to define the analytic functional calculus directly.

Let $\hat{\mathbb{C}}_{\xi_i}=\hat{\mathbb{C}}\setminus\{\xi_i\}$, $\hat{\mathbb{C}}^n_\xi=\hat{\mathbb{C}}_{\xi_1}\times\cdots\times\hat{\mathbb{C}}_{\xi_n}$. Then $\hat{\mathbb{C}}_\xi$ is a local compact topology space. We notice that if we substitute compact set in $\mathbb{C}^n$ by compact set in $\hat{\mathbb{C}}^n_\xi$, then [5] Lemma 3.3 are true for the case of closed operator systems. Therefore, if f is analytic on a neighbourhood U of $\sigma(T)$, then the equality $tf(z)x-\chi=(J\oplus\bar{\partial})\psi$ has a solution $\chi\in\Lambda^n[\tau\cup d\bar{z}, \hat{C}^\infty_\xi(U, X)]$, where $\Lambda^n[\tau\cup d\bar{z}, \hat{C}^\infty_\xi(U, X)]$ is the family of forms with compact support in $\hat{\mathbb{C}}^n_\xi$. Let π be a chain homomorphism keeping the part of χ which contains $d\bar{z}_1, \cdots, d\bar{z}_n$, $T_\xi(z)=\prod_{i=1}^n\left(\frac{\xi_i-T_i}{\xi_i-z_i}\right)$ and $R_{Z-T}f(z)x=(-1)^n\pi\chi$. We can show $f(T)x=\left(\frac{1}{2\pi i}\right)^n\int_U T_\xi(z)R_{Z-T}f(z)x\,dz_1\wedge\cdots\wedge dz_n$ is generalized Lebesgue intergrable and $f\to f(T)$ is an algebric homomorphism satisfying the spectral mapping theorem. Because of [1] Theorems 2.2 and 2.4, it is sufficient to prove $f_\xi(a)=f(T)$.

Theorem 1.14. *Let $T=(T_1, \cdots, T_n)$ be a commuting n-tuple of closed operators, $\xi_i\in\rho(T)\cap\mathbb{C}$, $a_i=(\xi_i-T_i)^{-1}$, $1\leqslant i\leqslant n$, and $a=(a_1, \cdots, a_n)$. Suppose f is analytic on a neighbourhood U of $\sigma(T)$. Then $f_\xi(a)=f(T)$.*

Proof By applying Theorem 1.2, the equality $tf(z)x-\chi=(J\oplus\bar{\partial})\psi$ becomes $sf_\xi(\lambda)x-w_n\chi=(\alpha\oplus\bar{\partial})w_{n-1}\psi$. It is clear that supp.$w_n\chi$ is compact in $\mathbb{C}^n$ because supp.χ is compact in $\hat{\mathbb{C}}^n_\xi$. Moreover, since $U_\xi=\frac{1}{\xi-U}\subset\mathbb{C}^n$, we have $w_{n-1}\psi\in\Lambda^{n-1}[\sigma\cup d\bar{\lambda}, C^\infty(U_\xi, X)]$. By definition, $R_{\lambda-a}f_\xi(\lambda)x=(-1)^n\pi w_n\chi=(-1)^n w_n\pi\chi$. Let

$\chi=\chi_1+hd\bar{z}_1\wedge\cdots\wedge d\bar{z}_n$. Then $w_4\pi\chi=\prod_{i=1}^n\left(\frac{\xi_i-T_i}{\lambda_i}\right)\prod_{i=1}^n\frac{1}{\bar{\lambda}_i^2}h\left(\xi-\frac{1}{\lambda}\right)d\bar{\lambda}_1\wedge\cdots\wedge d\bar{\lambda}_n$.

Therefore

$$\begin{aligned}
&f_\xi(a)x\\
&=\left(\frac{1}{2\pi i}\right)^n\int_{U_\xi}(-1)^n\prod_{i=1}^n\left(\frac{\xi_i-T_i}{\lambda_i}\right)\prod_{i=1}^n\frac{1}{\bar{\lambda}_i}h\left(\xi-\frac{1}{\lambda}\right)d\bar{\lambda}_1\wedge\cdots\wedge d\bar{\lambda}_n\wedge d\lambda_1\wedge\cdots\wedge d\lambda_n\\
&=\left(\frac{1}{2\pi i}\right)^n\int_U(-1)^n\prod_{i=1}^n(\xi_i-T_i)(\xi_i-z_i)\prod_{i=1}^n(\bar{\xi}_i-\bar{z}_i)h(z)\\
&\quad\times\prod_{i=1}^n\frac{1}{|\xi_i-z_i|^2}d\bar{z}_1\wedge\cdots\wedge d\bar{z}_n\wedge dz_1\wedge\cdots\wedge dz_n\\
&=\left(\frac{1}{2\pi i}\right)^n\int_U(-1)^n\prod_{i=1}^n\left(\frac{\xi_i-T_i}{\xi_i-z_i}\right)h(z)d\bar{z}_1\wedge\cdots\wedge d\bar{z}_n\wedge dz_1\wedge\cdots\wedge dz_n\\
&=\left(\frac{1}{2\pi i}\right)^n\int_U T_\xi(z)R_{Z-T}f(z)xdz_1\wedge\cdots\wedge dz_n=f(T)x.
\end{aligned}$$

Remark. If $n=1$ and $\sigma(T)\subset U$, let $\sigma(T)\subset U_1\subset U_2\subset U$ and Γ be a close Jordan curve in $U\backslash\bar{U}_2$ enclosing $\sigma(T)$. Assume θ is a C^∞-scalar function, equal to 0 in U, and to 1 outside of U_2. Let $\psi(z)=(z-T)^{-1}f(z)x$. Then

$$\begin{aligned}
f(T)x&=\frac{1}{2\pi i}\int_U(-1)\frac{\xi-T}{\xi-z}\bar{\partial}(\theta\psi(z))dz=\frac{1}{2\pi i}\int_\Gamma(-1)\frac{\xi-T}{\xi-z}\theta\psi(z)dz\\
&=\frac{1}{2\pi i}\int_\Gamma\frac{\xi-T}{\xi-z}(z-T)^{-1}f(z)xdz\\
&=\frac{1}{2\pi i}\int_\Gamma\frac{f(z)}{z-\xi}dz+\frac{1}{2\pi i}\int_\Gamma(z-T)^{-1}f(z)xdz\\
&=f(\infty)x+\frac{1}{2\pi i}\int_\Gamma(z-T)^{-1}f(z)dz.
\end{aligned}$$

If $U=U_1\times\cdots\times U_n$, $\infty\in\sigma(T_i)\subset U_i$, we can show inductively that

$$\begin{aligned}
f(T)x=f(\infty,\cdots,\infty)x&+\sum_{i=1}^n\frac{1}{2\pi i}\int_{\Gamma_i}f(\infty,\cdots,z_i,\cdots,\infty)(z_i-T_i)^{-1}xdz_i\\
&+\sum_{i,j}\left(\frac{1}{2\pi i}\right)^n\int_{\Gamma_i}\int_{\Gamma_j}f(\infty,\cdots,z_i,\cdots,z_j,\cdots,\infty)(z_i-T_i)^{-1}(z_j-T_j)^{-1}xdz_idz_j\\
&+\cdots+\left(\frac{1}{2n_i}\right)^n\int_{\Gamma_1}\cdots\int_{\Gamma_n}f(z_1,\cdots,z_n)\prod_{i=1}^n(z_i-T_i)^{-1}xdz_1\cdots dz_n.
\end{aligned}$$

Let $\mathrm{Inv}(T)=\{Y\mid Y$ is a closed subspace of X and for any i, $T_i(Y\cap D_{T_i})\subset Y$, $\rho(T_i|y)\neq\varnothing\}$; $R(T)=\{Y\mid Y\in\mathrm{Inv}(T)$ and for any i, $\rho(T_i)\cap\rho(T_i|Y)\neq\varnothing\}$.

Corollary 1.15. *Suppose $Y\in R(T)$ and U is a neighbourhood of $\sigma(T)U\sigma(T|Y)$. If f is an analytic function on U, then $Y\in$ Inv $(f(T))$ and $f(T|Y)=f(T)|Y$.*

Proof For any $\xi_i\in\rho(T_i)\cap\rho(T_i|Y)$ and each $y\in Y$,

$$\begin{aligned}
f(T|Y)y&=\left(\frac{1}{2\pi i}\right)^n\int_U T_\xi(z)R_{z-T|y}f(z)ydz_1\wedge\cdots\wedge dz_n\\
&=\left(\frac{1}{2\pi i}\right)^n\int_U T_\xi(z)R_{z-T}f(z)ydz_1\wedge\cdots\wedge dz_n=f(T)y.
\end{aligned}$$

Hence $f(T)y\in Y$ and $f(T)y=f(T|Y)y$.

If T has SVEP, we can also define analytic functional calculus on the local spectrum. If f is analytic on a neighbourhood U of $\sigma(T, x)$, then there exists $\chi\in\Lambda^n[\tau\cup d\bar{z}, \hat{C}_\xi^\infty(U, X)]$ such that $tf(z)\ x-\chi=(J\oplus\bar{\partial})\psi$ holds on U. We define $f_T(x)=\left(\frac{1}{2\pi i}\right)^n\int T_\xi(z)(-1)^n\pi\chi\, dz_1\wedge\cdots\wedge dz_n$. The results of [2] can be extended to the case of closed operator n-tuples. In particular, if f is analytic on U, then $f(T)x=f_T(x)$ for any $x\in X$. Because of the similarity of the proof, we omit it in this paper. We have the following proposition.

Proposition 1.16. *Suppose $T=(T_1, \cdots, T_n)$ has SVEP and $\sigma(T, x)$ is compact for some $x\in X$. Then $x\in\bigcap_{i=1}^n D_{T_i}$ and $f(T)x=\left(\frac{1}{2\pi i}\right)^n\int R_{z-T}f(z)x\,dz_1\wedge\cdots\wedge dz_n$ for any analytic function f on the neighbourhood of $\sigma(T)$.*

Proof Since $g(z)=\prod_{i=1}^n(\xi_i-z_i)$ is analytic on a relative compact neighbourhood U of $\sigma(T)$, then tuere is χ with compact support in U such that $tf(z)x-\chi=(J\oplus\bar{\partial})\psi$. Hence $tg(z)f(z)x-g(z)\chi=(J\oplus\bar{\partial})g(z)\psi$. The result is

$$\left(\frac{1}{2\pi i}\right)^n\int R_{z-T}f(z)x\,dz_1\wedge\cdots\wedge dz_n=\left(\frac{1}{2\pi i}\right)^n\int\pi\chi\, dz_1\wedge\cdots\wedge dz_n$$

$$=\prod_{i=1}^n a_i\left[\left(\frac{1}{2\pi i}\right)\int T_\xi(z)\pi g(z)\chi\,dz_1\wedge\cdots\wedge dz_n\right]=\prod_{i=1}^n a_i(fg)_T(x)$$

$$=\prod_{i=1}^n a_i f_T(g_T(x))=\prod_{i=1}^n a_i\, f(T)\prod_{i=1}^n(\xi_i-T_i)x=f(T)x.$$

The reason why $g_T(x)=\prod_{i=1}^n(\xi_i-T_i)x$ is that $h(z)=\prod_{i=1}^n(\xi_i-z_i)^{-1}$ is analytic. Then $x=(hg)_T(x)=h_T(g_T(x))=\prod_{i=1}^n a_i[g_T(x)]$. Hence $x\in\bigcap_{i=1}^n D_{T_i}$ and $\prod_{i=1}^n(\xi_i-T_i)x=g_T(x)$.

§ 2. Spectral Decompositions

Lemma 2.1. *Suppose $T=(T_1, \cdots, T_n)$ is a commuting n-tuple of closed operators and $X=X_1+X_2$, $X_j\in(T)$, $j=1, 2$. For any $\xi_i\in\rho(T_i)\cap\rho(T_i|X)$, $\xi=(\xi_1, \cdots, \xi_n)$ and $U\subset\hat{\mathbb{C}}_\xi^n$, we have*

(1) *each $\varphi\in\Lambda^p[\tau, \hat{A}(U, x)]$ can be written as $\varphi=\varphi_1+\varphi_2$, where*

$$\varphi_j\in\Lambda^p[\tau, \hat{A}(U, X_j)],\ j=1, 2;$$

(2) *each $\psi\in\Lambda^p[\tau\cup d\bar{z}, \hat{C}^\infty(U, X)]$ can be written as $\psi=\psi_1+\psi_2$, where*

$$\psi_j\in\Lambda^p[\tau\cup d\bar{z}, \hat{C}^\infty(U, X_j)],\ j=1, 2.$$

Proof (1) It is clear that $\rho(T_i)\cap\rho(T_i|X_1)\subset\rho(T_i|X_2)$. Therefore $\xi_i\in\rho(T_i|X_2)$. If $\varphi\in\Lambda^p[\tau, \hat{A}(U, X)]$, then

$$u_\xi\varphi\in\Lambda^p\left[\alpha, \hat{A}\left(\frac{1}{\xi-U}, X\right)\right]=\Lambda^p\left(\alpha, A\left(\frac{1}{\xi-U}, X\right)\right]$$

by Theorem 1.1. Since a is a bounded n-tuple and $X_j \in \mathrm{Inv}(a)$ (Corollary 1.15), we have $u_p\varphi=\varphi_1^*+\varphi_2^*$, where $\varphi_j^* \in \Lambda^p\left[\alpha, A\left(\frac{1}{\xi-U} X_j,\right)\right]$, $j=1, 2$. Hence $\varphi=v_p\varphi_1^*+v_p\varphi_2^*$. Let $\varphi_j=v_p\varphi_j^* \in \Lambda^p[\tau, \hat{A}(U, X_j)]$, we are done.

(2) In the same way.

Lemma 2.2. *Suppose $T=(T_1, \cdots, T_n)$ is a commuting n-tuple of closed operators and $Y \in R(T)$. For any $\xi_i \in \rho(T_i) \cap \rho(T_i|Y)$, $\xi=(\xi_1, \cdots, \xi_n)$* and $U \subset \hat{\mathbb{C}}^n_\xi$, *we have*

(1) *each $\tilde{f} \in \Lambda^p[\tau, \hat{A}(U, X/Y)]$ can be written as $\tilde{f}=f/Y$, where*

$$f \in \Lambda^p[\tau, \hat{A}(U,X)];$$

(2) *each $\tilde{g} \in \Lambda^p[\tau \cup d\bar{z}, \hat{C}^\infty(U, X/Y)$ can be written as $\tilde{g}=g/Y$, where*

$$g \in \Lambda^p[\tau \cup d\bar{z}, \hat{C}^\infty(U, X)].$$

Proof It follows from [8] Proposition 3.1 that T_i^Y is a closed operator and $\sigma(T_i^Y) \subset \sigma(T_i) \bigcup \sigma(T_i|Y)$. Hence $\xi_i \in \rho(T_i) \cap \rho(T_i|Y) \subset \rho(T_i^Y)$ and $(\xi_i-T_i)^{-1}=a_i^y$. Therefore, using the same methods as in Lemma 2.1, we obtain (1), (2).

Lemma 2.3. *If $T=(T_1, \cdots, T_n)$ is a commuting n-tuple of closed operators, then*

(1) *If $X=X_1+X_2$, $X_j \in R(T)$, $j=1, 2$, for any $(i) \subset \{1, \cdots, n\}$ and each $x \in D_{(i)}$, $x=x_1+x_2$, where $x_j \in X_j \cap D_{(i)}$, $j=1, 2$.*

(2) *If $Y \in R(T)$ and $\tilde{D}_{(i)}$ denote $\cap \{D_{T^Y_{j_1}\cdots T^Y_{j_k}} | \{j_1, \cdots, j_k\} \cap (i)=\varnothing\}$, then a given $\tilde{x} \in \tilde{D}_{(i)}$ can be written as $\tilde{x}=x/Y$, where $x \in D_{(i)}$.*

Proof (1) Choose $\xi_i \in \rho(T_i) \cap \rho(T_i|X_1) \cap \rho(T_i|X_2)$. If $x \in D_{(i)}$, then

$$\prod(\xi_i-T_i)x=x_1^*+x_2^*,\ x_j^* \in X_j,\ j=1, 2.$$

Let $x_1=\prod_{j\bar{\in}(i)} a_j x_1^*$, $x_2=\prod_{k\bar{\in}(i)} a_j x_2^*$. Then $x_j \in X_j \cap D_{(i)}$ and $x=x_1+x_2$.

(2) Choose $\xi_i \in \rho(T_i|Y) \cap \rho(T_i)$. If $\tilde{x} \in \tilde{D}_{(i)}$, then $\prod_{j\bar{\in}(i)}(\xi_j-T_j^Y)\tilde{x}=x^*/Y$ for some x^*. Let $x=\prod_{j\bar{\in}(i)} a_j x^*$. Then $x \in D_{(i)}$ and $x/Y=\tilde{x}$.

Definition 2.4. *Let $T=(T_1, \cdots, T_n)$ be a commuting n-tuple of closed operators. If there is a mapping $\mathscr{E}$ from the family of closed subsets $\mathscr{F}(\hat{\mathbb{C}}^n)$ of $\hat{\mathbb{C}}^n$ into* Inv (T) *satisfying:*

(1) $\mathscr{E}(\phi)=\{0\}$, $\mathscr{E}(\hat{\mathbb{C}}^n)=X$;

(2) $\mathscr{E}\left(\bigcap_{n=1}^\infty F_n\right)=\bigcap_{n=1}^\infty \mathscr{E}(F_n)$ *for any sequence* $\{F_n\} \subset \mathscr{F}(\hat{\mathbb{C}}^n)$;

(3) $X=\sum_{j=1}^m \mathscr{E}(\bar{G}_j)$ *for any finite open cover* $\{G_j\}_{j=1}^m$ *of* $\hat{\mathbb{C}}^n$;

(4) $\sigma(T|\mathscr{E}(F)) \subset F$ *for each* $F \in \mathscr{F}(\hat{\mathbb{C}}^n)$; *then T is said to possess a spectral capacity.*

Proposition 2.5. *If $T=(T_1,\cdots, T_n)$ possesses a spectral capacity $\mathscr{E}$, then for each* $F \in \mathscr{F}(\hat{\mathbb{C}}^n)$, (1) $\sigma(T|\mathscr{E}(F)) \subset \sigma(T)$, (2) $\sigma(T_i|\mathscr{E}(F)) \subset \sigma(T_i)$, $1 \leqslant i \leqslant n$, (3) $\mathscr{E}(F) \in$

$R(T)$.

Proof For any $z\overline{\in}\sigma(T)$, there are open sets D and D_1 such that $z\in D$, $D\cup D_1=\hat{\mathbb{C}}^n$ and $\bar{D}\cap\sigma(T)=\varnothing$. The spectral capacity $\mathscr{E}$ provides the decomposition of X: $X=\mathscr{E}(\bar{D})+\mathscr{E}(\bar{D}_1)$. For any $x\in\mathscr{E}(\bar{D})$, there is $\psi\in\Lambda^{n-1}[\tau\cup d\bar{z},\hat{C}^\infty(U,\ \mathscr{E}(\bar{D}))]$ such that $tx=(J\oplus\bar{\partial})\psi$, where $U=\hat{\mathbb{C}}^n\backslash\bar{D}\supset\sigma(T)$. By definition $R_{z-T}\,x=0$, hence

$$x=\left(\frac{1}{2\pi i}\right)^n\int T_\xi(z)R_{z-T}x\,dz_1\wedge\cdots\wedge dz_n=0.$$

Therefore $\mathscr{E}(\bar{D})=\{0\}$ and $X=\mathscr{E}(\bar{D}_1)$. Then. $\sigma(T|\mathscr{E}(F))=\sigma(T|\mathscr{E}(F)\cap\mathscr{E}(\bar{D}_1))=\sigma(T|\mathscr{E}(F\cap\bar{D}_1))\subset\bar{D}_1$. Thus $z\overline{\in}\sigma(T|\mathscr{E}(F))$ and $\sigma(T|\mathscr{E}(F))\subset\sigma(T)$. Furthermore

$$\sigma(T_i)=\pi_i\sigma(T)\supset\pi_i\sigma(T|\mathscr{E}(F))=\sigma(T_i|\mathscr{E}(F)).$$

Therefore $\rho(T_i)\cap\rho(T_i|\mathscr{E}(F))=\rho(T_i)\neq\varnothing$ and (1), (2), (3) all hold.

Proposition 2.6. *Suppose $T=(T_1,\ \cdots,\ T_n)$ possesses spectral capacity $\mathscr{E}$. Then supp. $\mathscr{E}=\sigma(T)$.*

Proof If $F\in\mathscr{F}(\hat{\mathbb{C}}^n)$ and $\mathscr{E}(F)=X$, then $\sigma(T)=\sigma(T|\mathscr{E}(F))\subset F$. Hence $\sigma(T)\subset$ supp. $\mathscr{E}$. On the other hand, by the proof of Proposition 2.5, we know that if $z\overline{\in}\sigma(T)$, then there exists D such that $z\in D$ and $\mathscr{E}(\bar{D})=\{0\}$. Hence $z\overline{\in}$ supp. $\mathscr{E}$.

Theorem 2.7. *Let $T=(T_1,\ \cdots,\ T_n)$ possess spectral capacity $\mathscr{E}$. Suppose f_j is analytic on a neighbourhood of $\sigma(T)$, $1\leqslant j\leqslant m$. Then $f(T)=(f_1(T),\ \cdots,\ f_n(T))$ is a decomposable n-tuple in the sense of* [4] *Definition* 3.1. *The spectral capacitp $\mathscr{E}^*$ of $f(T)$ is uniquely determind by $\mathscr{E}^*(F)=\mathscr{E}(f^{-1}(F)\cap\sigma(T))$.*

Proof For any $F\in\mathscr{F}(\mathbb{C}^m)$ let $\mathscr{E}^*(F)=\mathscr{E}(f^{-1}(F)\cap\sigma(T))$. We have

(1) $\mathscr{E}^*(\phi)=\{0\}$, $\mathscr{E}^*(\mathbb{C}^m)=\mathscr{E}(f^{-1}(\mathbb{C}^m)\cap\sigma(T))=\mathscr{E}(\sigma(T))=X$;

(2) $\mathscr{E}^*(\cap F_n)=\mathscr{E}(f^{-1}(\cap F_n)\cap\sigma(T))=\cap\mathscr{E}(f^{-1}(F_n)\cap\sigma(T))=\cap\mathscr{E}^*(F_n)$;

(3) If $\{G_j\}_{j=1}^k$ is an open cover of $\mathbb{C}^m$, then $\sigma(T)\subset\bigcup\limits_{j=1}^k(f^{-1}(\bar{G}_j)\cap\sigma(T))$, and $\sum\limits_{j=1}^k\mathscr{E}^*(\bar{G}_j)=\sum\limits_{j=1}^k\mathscr{E}(f^{-1}(\bar{G}_j)\cap\sigma(T))=X$;

(4) By Corollary 1.15, $\mathscr{E}^*(F)\in$ Inv $(f(T))$ and $f(T|\mathscr{E}^*(F))=f(T)|\mathscr{E}^*(F)$. Therefore

$$\begin{aligned}\sigma(f(T)|\mathscr{E}^*(F))&=\sigma(f(T|\mathscr{E}^*(F)))=f(\sigma(T|\mathscr{E}^*(F)))\\&=f(\sigma(T|\mathscr{E}(f^{-1}(F)\cap\sigma(T))))\subset f(f^{-1}(F)\cap\sigma(F))\subset F.\end{aligned}$$

Thus $f(T)$ is decomposable and $\mathscr{E}^*$ is the unique spectral capacity.

Corollary 2.8. *If $T=(T_1,\ \cdots,\ T_n)$ possesses spectral capacity, then T has SVEP and $\mathscr{E}(F)=\{x|\sigma(T,x)\subset F\}$ for each $F\in\mathscr{F}(\hat{\mathbb{C}}^n)$.*

Proof In view of Corollary 2.7, $a=(a_1,\ \cdots,\ a_n)$ is decomposable. Hence a has SVEP. It follows from 1.11 that T has SVEP. Furthermore,

$$\mathscr{E}(F)=\mathscr{E}_a\left(\frac{1}{\xi-F}\cap\mathbb{C}^n\right)=\left\{x|\sigma(a,\ x)\subset\frac{1}{\xi-F}\right\}=\{x|\sigma(T,\ x)\subset F\}.$$

Definition 2.9. *Let $T=(T_1,\ \cdots,\ T_n)$ be a commuting n-tuple of closed operators* *and* $Y\in \operatorname{Inv}(T)$. *If for any* $Z\in\operatorname{Inv}(T)$, $\sigma(T|Z)\subset\sigma(T|Y)$ *implies* $Z\subset Y$, *then* Y *is called a spectral maximal space of* T. *The family of all spectral maximal space of* T *is denoted by* $SM(T)$.

Proposition 2.10. *Let* $T=(T_1, \cdots, T_n)$ *possess spectral capacity* $\mathscr{E}$. *Then* $Y\in SM(T)$ *iff* $Y=\mathscr{E}(\sigma(T|Y))$.

Proof If $F\in\mathscr{F}(\hat{\mathbb{C}}^n)$, then $\mathscr{E}(F)=X_T(F)=\{x|\sigma(T, x)\subset F\}$. Suppose $Z\in\operatorname{Inv}(T)$ satisfy $\sigma(T|Z)\subset\sigma(T|\mathscr{E}(F))$. Then for any $x\in Z$,

$$x\in X_{T|Z}(\sigma(T|Z))\subset X_T(\sigma(T|Z))=\mathscr{E}(\sigma(T|Z))\subset\mathscr{E}(\sigma(T|\mathscr{E}(F))\subset\mathscr{E}(F).$$

Conversely, if $Y\in SM(T)$, then $\mathscr{E}(\sigma(T|Y))\subset Y$ since $\sigma(T|\mathscr{E}(\sigma(T|Y)))\subset\sigma(T|Y)$. For any $y\in Y$, $y\in X_T(\sigma(T|Y))=\mathscr{E}(\sigma(T|Y))$, thus $Y\subset\mathscr{E}(\sigma(T|Y))$ and $Y=\mathscr{E}(\sigma(T|Y))$ is obtained.

Proposition 2.11. *Let* $T=(T_1, \cdots, T_n)$ *possess spectral capacity* $\mathscr{E}$. *Then for any* $F\in\mathscr{F}(\hat{\mathbb{C}}^n)$, $\sigma(T^{\mathscr{E}(F)})\subset\hat{\mathbb{C}}^n\setminus\mathring{F}$.

Proof For any $z\in\mathring{F}$, we have to show $z\in\rho(T^{\mathscr{E}(F)})$. Suppose $\tilde{\psi}=\sum\tilde{x}_{(i)}t_{(i)}$ and $J_p(z)\tilde{\psi}=0$. By Lemma 2. 3, we may assume $\tilde{x}_{(i)}=x_{(i)}/\mathscr{E}(F)$. Hence

$$J_p(z)\tilde{\psi}=\sum_{(i)}\sum_{j}(z_j-T_j)x_{(i)}/\mathscr{E}(F)t_j\wedge t_{(i)}=0.$$

If G is an open set satisfying $Z\in G\subset\overline{G}\subset\mathring{F}$, then $\hat{\mathbb{C}}^n\setminus\overline{G}$ and $\mathring{F}$ is an open cover of $\hat{\mathbb{C}}^n$. Thus $X=X_1+X_2$, where $X_1=\mathscr{E}(\hat{\mathbb{C}}^n\setminus G)$ and $X_2=\mathscr{E}(F)$. Since X_1, $X_2\in R(T)$, we have $x_{(i)}=z_{(i)}+y_{(i)}$, $z_i\in D_{(i)}\cap X_1$ and $y_{(i)}\in D_{(i)}\cap X_2$ (Lemma 2.3). Set $\psi_1=\sum z_{(i)}t_{(i)}$, $\psi_2=\sum y_{(i)}t_{(i)}$ and $\psi=\psi_1+\psi_2$. Then $J_p(z)\psi_1/\mathscr{E}(F)=J_p(z)\psi/\mathscr{E}(F)=0$. Consequently, $\prod_{j\in(i)}(z_j-T_j)z(i)\in\mathscr{E}(F)\cap\mathscr{E}(\hat{\mathbb{C}}^n\setminus F)=X_1\cap X_2$ or $J_p(z)\psi_1/(X_1\cap X_2)=0$. Let $\tilde{T}_j=(T_j|X_1)^{X_1\cap X_2}$, $\tilde{T}=(\tilde{T}_1, \cdots,\tilde{T}_n)$. Then $\tilde{T}$ is a commuting n-tuple of closed operators. If S is a mapping from X/X_2 into $X_1/(X_1\cap X_2)$: $x/X_2=(x_1+x_2)/X_2\to x_1/(X_1\cap X_2)$, then $T_j^{X_2}=S^{-1}T_jS$. By [1] Theorem 2.1, $\sigma(T^{X_2})=\sigma(\tilde{T})\subset\sigma(T|X_1)\cup\sigma(T|X_1\cap X_2)\subset\hat{\mathbb{C}}^n\setminus G$. Since $z\in G$, we have $z\overline{\in}\sigma(T)$. Hence there exists $\tilde{\varphi}$ such that $J_p(z)\tilde{\varphi}=\sum z_{(i)}/(X_1\cap X_2)t_{(i)}=\psi_1/(X_1\cap X_2$. Let $\tilde{\varphi}=\varphi/(X_1\cap X_2)$. Then $J_p(z)\varphi-\psi_1\in\Lambda^p[\tau(z), X_1\cap X_2]$ and $J_p(z)\varphi/\mathscr{E}(F)=\psi_1/\mathscr{E}(F)=\psi/\mathscr{E}(F)=\tilde{\psi}$. Thus $z\in\rho(T^{\mathscr{E}(F)})$ is obtained. Since z is an arbitrary point of $\mathring{F}$, we have $\sigma(T^{\mathscr{E}(F)})\subset\hat{\mathbb{C}}^n\setminus\mathring{F}$.

Proposition 2.12. *Let* $T=(T_1, \cdots, T_n)$ *possess spectral capacity* $\mathscr{E}$. *Suppose* $F\in\mathscr{F}(\hat{\mathbb{C}}^n)$ *and* $\{G_j\}_{j=1}^m$ *is an open cover of* F. *Then* $\mathscr{E}(F)\subset\sum_{j=1}^m\mathscr{E}(\overline{G}_j)$.

Proof Choose $\xi_i\in\rho(T_i)$. Set $a_i=(\xi_i-T_i)^{-1}$ and $a=(a_1, \cdots, a_n)$. Then a is decomposable and $\mathscr{E}_a(F)=\mathscr{E}\left(\left(\xi-\frac{1}{F}\right)\cap\mathbb{C}^n\right)$ for each $F\in\mathscr{F}(\hat{\mathbb{C}}^n)$. With the help of [3] § 2 Theorem, we have

$$\mathscr{E}(F)=\mathscr{E}_a\left(\left(\xi-\frac{1}{F}\right)\cap\mathbb{C}^n\right)\subset\sum_{j=1}^m\mathscr{E}_a\left(\left(\xi-\frac{1}{G_j}\cap\mathbb{C}^n\right)\right)=\sum_{j=1}^m\mathscr{E}(\overline{G}_j).$$

Proposition 2.13. *Let* $T=(T_1, \cdots, T_n)$ *possess spectral capacity* $\mathscr{E}$. *Then*

(1) T_j *possesses spectral capacity* $\mathscr{E}_j$: $F\to\mathscr{E}(\hat{\mathbb{C}}\times\cdots\times F\times\cdots\times\hat{\mathbb{C}})$ *for any* j;

(2) $\mathscr{E}(F)=\cap\{\sum_{j=1}^{m}[\mathscr{E}_1(\bar{D}_{1j})\cap\mathscr{E}_2(\bar{D}_{2j})\cap\cdots\cap\mathscr{E}_n(\bar{D}_{nj})]\,|\,F\subset\bigcup_{j=1}^{m}(D_{1j}\times\cdots\times D_{nj}),\}D_{1j}\times\cdots\times D_{nj}$ *is a polydisc in* $\hat{\mathbb{C}}^n$ *for any* $F\in\mathscr{F}(\hat{\mathbb{C}}^n)$.

Proof (1) Obvious.

(2) In view of Proposition 2. 12, $\mathscr{E}(F)$ is contained in the right side of the equality. If x belongs to the right side, then $\sigma(T, x)=\cap\left\{\bigcap_{j=1}^{m}D_{1j}\times\cdots\times D_{nj}\,\middle|\text{ for any }\bigcup_{j=1}^{m}D_{1j}\times\cdots\times D_{nj}\supset F\right\}=F$. Therefore $x\in\mathscr{E}(F)$ and the equality is obtained.

Definition 2.14. *Let* $T=(T_1,\cdots,T_n)$ *be a commuting n-tuple of closed operators. Suppose for any open cover* $\{G_j\}_{j=1}^{m}$ *of* $\hat{\mathbb{C}}^n$, *there are* $X_j\in\mathrm{Inv}(T)$, $1\leqslant j\leqslant m$, *such that* $\sigma(T|X_j)\subset G_j$ *for each* j *and* $X=\sum_{j=1}^{m}X_j$. *Then* T *is called to have the spectral decomposition property* (abbrev. SDP).

Theorem 2.15. *Suppose* $T=(T_1,\cdots,T_n)$ *has SDP. Then* T *has SVEP.*

Proof We have to show $H^p[\hat{A}(\{z\}, X), J]=0$ for each $z\in\hat{\mathbb{C}}^n$, $p=0,\cdots,n-1$. Suppose $z\in U_1\times\cdots\times U_n$, $\psi\in A^p[\tau, \hat{A}(U, X)]$ and $J_p\psi=0$. For any fixed $\xi_i\in\rho(T_i)\cap\mathbb{C}$, $1\leqslant i\leqslant n$, if there is i such that $z_i=\xi_i$, then $z=(z_1,\cdots,z_n)\in\rho(T)$ and $H^p[\hat{A}(\{z\}, X), J]=0$ by Proposition 1.4. If for any i, $z_i\neq\xi_i$, then there exist open sets D_i, D_i^1 in $\hat{\mathbb{C}}$ such that $\xi_i\in D_i^1\subset\bar{D}_i^1\subset D_i\subset\bar{D}_i\subset\rho(T_i)\setminus\bar{U}_j$. Set $G_i=\hat{\mathbb{C}}\times\cdots\times D_i\times\cdots\times\hat{\mathbb{C}}$, $G_i^1=\hat{\mathbb{C}}\times\cdots\times D_i^1\times\cdots\times\hat{\mathbb{C}}$, $G=\bigcup_{i=1}^{n}G_i$ and $G^1=\bigcup_{i=1}^{n}G_i$. We choose another open set V_i such that $z_i\in V_i\subset\bar{V}\subset U_i$ and $V=V_1\times\cdots\times V_n$. Then $U\setminus\bar{G}^1$, $\hat{\mathbb{C}}^n\setminus(V\cup\bar{G}^1)$ and $\{G\}_{j=1}^{n}$ is an open cover of $\hat{\mathbb{C}}^n$. Hence there are X_1, X_2, Y_j $(1\leqslant j\leqslant n)\in\mathrm{Inv}(T)$ such that $\sigma(T|X_1)\subset U\setminus\bar{G}^1$, $\sigma(T|X_2)\subset\hat{\mathbb{C}}^n\setminus(V\cup\bar{G}^1)$ and $\sigma(T|Y_j)\subset G_j$, $1\leqslant j\leqslant n$. By the prejection property, we have $\sigma(T_j|Y_j)\subset D_j$. Since $D_j\cap\sigma(T_j)=\phi$, Y_j must be $\{0\}$. Obviously $\xi_j\bar{\in}\pi_j(U\setminus\bar{G}^1)$. Then $\xi_j\in\rho(T_j|Y_1)$ since

$$\sigma(T_j|X_1)=\pi_j(\sigma(T|X_1))\subset\pi_j(U\setminus\bar{G}^1).$$

In the same way, we have $\xi_j\in\rho(T_j|X_2)$. Thus $X=X_1+X_2$, $X_j\in R(T)$. It is easy to prove $\xi_j\in\rho(T_j|X_1\cap X_2)$. Then $X_1\cap X_2\in R(T|X_1)$. By the proof of Proposition 2.4, $\sigma(T^{X_2})\subset\sigma(T|X_1)U\sigma(T|X_1\cap X_2)$. It is not difficult to prove $\sigma(T_j|X_1\cap X_2)\subset U_j$. The result is $\sigma(T^{X_2})\subset\sigma(T|X_1)\cup\sigma(T|X_1\cap X_2)\subset U$. Hence $\sigma(a_j^{X_2})\subset\frac{1}{\xi_j-U_j}$. By Theorem 2.1 and Remark 2.1 of [4], we have

$$H^p[\hat{A}(U, X/X_2), J]\cong H^p\left[A\left(\frac{1}{\xi-U}, X/X_2,\right)\alpha\right]=0.$$

Since $J_p\psi/X_2=0$, there exists $\tilde{\varphi}\in A^p[\tau, \hat{A}(U, X/X_2)]$ such that $J_{p-1}\tilde{\varphi}=\psi/X_2$. Let $\varphi/X_2=\tilde{\varphi}$. Then $\psi^*=\psi-J_{p-1}\varphi\in A^p[\tau, \hat{A}(U, X)]$. Since $\sigma(T|X_2)\cap V=\phi$, we have $\rho\in A^{p-1}[\tau, \hat{A}(V, X_2)]$ such that $\psi^*=J_{p-1}\rho$. Thus $\psi=J_{p-1}(\rho+\varphi)$ and $H^p[\hat{A}(\{z\},$

$X), J] = 0$ is obtained.

Theorem 2.16. *Suppose $T=(T_1, \cdots, T_n)$ has SDP. Then for any $F\in\mathscr{F}(\hat{\mathbb{C}}^n)$, $X_T(F)\in R(T)$ and for any compact set F in $\mathbb{C}^n$, $T_i|X_T(F)\,(j=1, 2, \cdots, n)$ are bounded.*

Proof Choose $\xi_i\in\rho(T_i)$, $1\leqslant i\leqslant n$. For any $z\bar{\in}F$, there exist polydiscs D, D^1 such that $z\in D^1\subset\bar{D}^1\subset D\subset\hat{\mathbb{C}}^n\backslash F$ and $\rho(T_i)\backslash D_i\neq\phi$. In a way similar to the proof of Theorem 2. 15, we have $X_j\in R(T)$, $j=1,2$, such that $\xi_j\in\rho(T_i|X_j)$, $1\leqslant i\leqslant n$, $j=1$, 2, and $\sigma(T|X_1)\subset\hat{\mathbb{C}}^n\backslash\{z\}$, $\sigma(T|X_2)\subset D^1$. For any $x\in X_T(F)$, $x=x+x_2$. Let $U=D\cap(\hat{\mathbb{C}}^n\backslash\bar{D}^1)$. Since $x\in X_T(F)$ and $F\cap U=\varnothing$, there exists $\psi^*\in\Lambda^{n-1}\,[\tau\bigcup d\bar{z},\ \hat{C}^\infty(U, X)]$ such that $tx=(J\oplus\bar{\partial})\psi^*$. Because $U\cap\sigma(T|X_2)=\varnothing$, there is another form $\psi_2\in\Lambda^{n-1}[\tau\cup d\bar{z},\ \hat{C}^\infty(U,\ X)]$ such that $tx_2=(J\oplus\bar{\partial})\psi^*$. Set $\psi_1^*=\psi-\psi_2$. Then $tx_1=(J\oplus\bar{\partial})\,\psi_1^*$. Since $\sigma(T^{X_1})\subset\sigma(T|X_1)\cup\sigma(T|X_1\cap X_2)\subset D'$, there is $\tilde{\varphi}\in\Lambda^{n-1}\,[\tau\cup d\bar{z},\ \hat{\mathbb{C}}^\infty(U,\ X/X_1)]$ such that $\psi_1^*/X_1=(J\oplus\bar{\partial})\tilde{\varphi}$. If $\tilde{\varphi}=\varphi/X_1$, then $\psi_1=\psi^*-(J\oplus\bar{\partial})\varphi\in\Lambda^{n-1}[\tau\cup d\bar{z},\ \hat{C}^\infty(U,\ X)]$ and $(J\oplus\bar{\partial})\psi_1=tx_1$. Multiplying suitable C^∞–scalar functions θ_1 and θ_2, we have $x_j=tx_j-(J\oplus\bar{\partial})\theta_j\psi_j\in\Lambda^n[\tau\cup d\bar{z},\ \hat{C}_\xi^\infty(D,\ X_j)]$, $j=1$, 2. Thus $\left(\frac{1}{2\pi i}\right)^n\int_D T_\xi(z)\,(-1)^n\pi\chi\, dz_1\wedge\cdots\wedge dz_n=\left(\frac{1}{2\pi i}\right)^n\int_D T_\xi(z)\,(-1)^n\pi\chi_1\, dz_1\wedge\cdots\wedge dz_n+\chi_2$, where $\chi=x_1+x_2$. Since $x\in X_T(F)$and $D\cap F=\varnothing$, we have

$$\left(\frac{1}{2\pi i}\right)^n\int_D T_\xi(z)\,(-1)^n\pi\chi\, dz_1\wedge\cdots\wedge dz_n=0.$$

Therefore $x_2\in X_1$ and $x=x_1+x_2\in X_1$. Since x is arbitary in $X_T(F)$, we have $X_T(F)\subset X_1$. Let X_z denote X_1. Then $X_T(F)\subset\bigcap_{z\bar{\in}F}X_z$. It is obvious that $\bigcap X_z\in X_T(F)$. Therefore $X_T(F)=\bigcap X_z$ is a closed invariant space of T. Because for any $z\bar{\in}F$, $\xi_i\in\rho(T_i|X_z)$, ξ_i must be contained in $\rho(T_i|X_T(F))$. Thus $X_T(F)\in R(T)$ is obtained.

If F is compact in $\mathbb{C}^n$, then $X_T(F)\subset\bigcap_{i=1}^n D_{T_i}$ by Proposition 1. 14. The restriction $T_i|X_T(F)$ is closed and defined on Banach space $X_T(F)$, it must be bounded by colsed graph theorem.

Theorem 2.17. *Let $T=(T_1, \cdots, T_n)$ have SDP. Suppose f_j is an analytic function on a neighbourhood of $\sigma(T)$, $1\leqslant j\leqslant m$. Then $f(T)=(f_1(T), \cdots, f_m(T))$ has SDP.*

Proof Suppose $\{G_j\}_{j=1}^k$ is an open cover of $\mathbb{C}^m$. Then $\{f^{-1}(G_j)\}_{j=1}^k$ is an open cover of $\sigma(T)$. Using the same method as in the proof of Theorem 2.16, we can find $X_j\in R(T)$, $1\leqslant j\leqslant k$, such that $X=\sum_{j=1}^k X_j$ and $\sigma(T|X_j)\subset f^{-1}(G_j)$. Consequently $X_j\in\operatorname{Inv}(f(T))$ and $f(T)|X_j=f(T|X_j)$. Therefore

$$\sigma(f(T)|X_j)=\sigma(f(T|X_j))=f(\sigma(T|X_j))\subset f(f^{-1}(G_j))=G_j,\ 1\leqslant j\leqslant k.$$

By definition $f(T)$ has SDP.

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