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ON THE CONSTRUCTION OF SIMPLE GROUPS OVER FORMALLY REAL FIELDS

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Abstract

The problem of classifying the semi-simple algebraic groups defined over $\mathbb{R}$ is reduced to the problem of finding all admissible ($\mathbb{R}$)-indices (i.e. the Satake diagrams). In this paper, by a similar way as in the construction of Chevalley groups and the twisted groups, for all admissible ($\mathbb{R}$)-indices (except two indices given in table II′), a uniform construction of simple groups of adjoint type defined over formally real fields is given. Moreover, the quasi-Bruhat decomposition of such groups is given also in this paper. Thus a uniform proof of the existence of simple algebraic groups of adjoint type defined over $\mathbb{R}$ is given.

The proof of the existence of semi-simple algebraic groups defined over an algebraically closed field $\mathbb{K}$ for all possible types of simple root system was given by C. Chevalley[1].

$SL(l+1, \mathbb{K})$ are simply connected simple algebraic groups defined over $\mathbb{K}$ of type A_l and $PSL(l+1, \mathbb{K})$ are adjoint groups of type A_l. Groups of type B_l, C_l. D_l can be obtained in similar ways. A group of type G_2 can be constructed explicitly, as the automorphism group of 8-dimensional Cayley algebra. Groups of types F_4, E_6, E_7, E_8 can be treated in similar ways.

C, Chevalley[2] gave a uniform construction of the groups of adjoint type over $\mathbb{K}$ for all irreducible simple root systems (This of course presupposes the existence theorem for simple Lie algebras over $\mathbb{C}$). Thus C. Chevalley[2] gave a uniform proof of the existence of simple algebraic groups defined over $\mathbb{K}$ (adjoint type) for all possible types $A-G$.

In C. Chevalley [1], expose 23, there is a construction of a simply connected group of any given type, provided some group of that type already exists.

For a given root system, we denote by L the lattice with $L_0 \subset L \subset L_1$ where L_0 and L_1 are the roots and the weight lattices respectively. R. Steinberg[3] gave a uniform proof of the existence of semi-simple algebraic groups defined over $\mathbb{K}$, of which global roots and characters are L_0 and L respectively.

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Using Weil's theorem, an existence proof for a reductive group with given root datum is given by M. Demasure and A. Grothendieck[4].

The classification of semi-simple algebraic groups defined over perfect field k is considered by J. Tits[5] and I. Satake[6].

The problem of classifying the semi-simple algebraic groups defined over k is reduced to the problem of finding the admissible (k)-indices of semi-simple algebraic groups defined over k and all possible semi-simple anisotropic kernels for a given admissible (k)-index. J. Tits[5] enumerated all possible admissible (k)-indices when k is $\mathbb{R}$(real numbers field), $\mathfrak{p}$ ($\mathfrak{p}$-adic fields), and $\mathfrak{n}$ (number fields).

For each admissible $(\mathbb{R})$-index of types A_l, B_l, C_l, D_l J. Tits[5] gave a classical group which is identical with the simple algebraic group defined over $\mathbb{R}$ with such index. For each admissible $(\mathbb{R})$-index of types G_2, F_4, E_6, E_7, E_8 (except certain indices, for example, the index E VI, i.e., $E^9_{7,4}$ in [5]), J. Tits[5] constructed the simple algebraic group defined over $\mathbb{R}$ with such index by means of Cayley algebra, Jordan algebra, associative division algebra, division Cayley algebra.

In this paper, by a similar way as in [2, 8, 9], for all possible admissible $(\mathbb{R})$-indices (except two indices given in the toble II′), a uniform construction of simple groups of adjoint type over formally real fields is given. Thus for all possible admissible $(\mathbb{R})$-indices, a uniform proof of the existence of simple algebraic groups of adjoint type defined over $\mathbb{R}$ is given.

We shall use the contents and notations in [7, 10] in our paper.

Obviously, all admissible $(\mathbb{R})$-indices are the Satake diagrams (B, θ), where B (denoted by Π in our paper) is a simple root system of a simple Lie algebra $\mathbb{L}$ over $\mathbb{C}$ and θ is a Cartan involution[10,11]. We denote by Φ the root system with the fundamental system Π. Clearly, we have $\theta(\Phi)=\Phi$. For each $r\in\Phi$, we define $\bar{r}=-\theta(r)$ and $r'=\frac{1}{2}(r+\bar{r})$ (The notation r' in our paper is the notation $\bar{r}$ in [10]). We can define an ordering on Φ such that $r'\geqslant 0$ for each positive root r $(r>0)$.

We define $\Pi^0=\{\alpha\in\Pi\,|\,\alpha=-\bar{\alpha}\}$ and $\Phi^0=\{\alpha\in\Phi\,|\,\alpha=-\bar{\alpha}\}$. Clearly, Π^0 is a subset of Π consisting of all simple roots which correspond to the black nodes in the Satake diagram (Π, θ) and Φ^0 is a root system with the fundamental system Π^0. Furthermore, we define $\Pi^*=\Pi\backslash\Pi^0$ and $\Phi^*=\Phi\backslash\Phi^0$, and we define

$$\Phi_1=\{r\in\Phi^*\,|\,r=\bar{r}\},\ \Phi_{11}=\Phi^*\backslash\Phi_1,\ \Phi_{11b}=\{r\in\Phi_{11}\,|\,r+\bar{r}\in\Phi\},$$

$$\Phi_{11a}=\Phi_{11}\backslash\Phi_{11b},\quad \Phi_{1b}=\{r+\bar{r}\,|\,r\in\Phi_{11b}\},\quad \Phi_{1a}=\Phi_1\backslash\Phi_{1b}.$$

Proposition 1. *For each Satake diagram (Π, θ) there exist a Chevally basis $C_b=\{h_r,\ r\in\Pi;\ e_r,\ r\in\Phi\}$ of $\mathbb{L}$ and an involutive automorphism ρ_θ of $\mathbb{L}$ such that $\rho_\theta(e_r)=K_r e_{\bar{r}}$, $r\in\Phi$, e_r, $e_{\bar{r}}\in C_b$, $k_r=\pm1$ which satisfies conditions (A): $k_r=1$ if*

$r\in\Phi_{11}\cup\Phi_{1a}$, $k_r=-1$ *if* $r\in\Phi_{1b}$.

Proof It follows from the properties of Satake diagram that there exist a Chevalley basis $\tilde{C}_b=\{h_r,\ r\in\Pi;\ \tilde{e}_r,\ r\in\Phi\}$ of $\mathbb{L}$ and an involutive automorphism ρ_θ of $\mathbb{L}$ such that $\rho_\theta(\tilde{e}_r)=\tilde{k}_r\tilde{e}_{\bar{r}}$, $r\in\Phi$, $\tilde{e}_r$, $\tilde{e}_{\bar{r}}\in\tilde{C}_b$, $\tilde{k}_r=\pm1$. Let $e_r=\eta_r\tilde{e}_r$, $r\in\Phi$, $\tilde{e}_r\in\tilde{C}_b$, $\eta_r=\pm1$. By a similar argument as in 13.6.2 of [7] (or Lemma 2.1 of [9]), we can choose the appropriate η_r such that $\rho_\theta(e_r)=k_re_{\bar{r}}$, $r\in\Phi$, $k_r=\pm1$ which satisfies the condition (A). It follows from p. 58 of [7] that $C_b=[h_r,\ r\in\Pi;\ e_r,\ r\in\Phi\}$ is a Chevalley basis of $\mathbb{L}$.

We assume that K is a field of characteristic $p\neq2$. Let f be a non-trivial involutive automorphism of K. We write $\bar{t}=f(t)$ for each $t\in K$ and we define a map σ for all $r\in\Phi$ and $t\in K$ by $\sigma(x_r(t))=x_{\bar{r}}(k_r\bar{t})$, where $x_r(t)=\exp(t\,ad\ e_r)$ and $x_{\bar{r}}(k_r\bar{t})=\exp(k_r\bar{t}\,ad e_{\bar{r}})$. It is easily verified that the map σ can be extended to an involutive automorphism σ of the Chevalley group

$$G=\mathbb{L}(K)=\langle x_r(t),\ r\in\Phi,\ t\in K\rangle.$$

We define $U^1=\{u\in U\,|\,\sigma u=u\}$, $V^1=\{v\in V\,|\,\sigma v=v\}$ and $G^1=\langle U^1,\ V^1\rangle$. Clearly, G^1 is determined by the Satake diagram $(\Pi,\ \theta)$ and K, f, so we denote by $\mathbb{L}(\Pi,\ \theta;\ K,f)$ the group G^1.

We shall define some notations and terminology which will be used.

I. If $r\in\Phi_1$; $X^1_r(t)=x_r(t)$, $t\in K_r=\{t\in K\,|\,t=k_r\bar{t}\}$, $W^1_r=W_r$,

$N^1_r(u)=n_r(u)$, $u\in K^*_r=K_r\backslash\{0\}$, $h^1_r(v)=h_r(v)$, $v\in K^{*\prime}_r=\{v\in K\,|\,v=\bar{v}\}\backslash\{0\}$,

IIa. If $r\in\Phi_{11a}$: $X^1_r(t)=x_r(t)x_{\bar{r}}(\bar{t})$, $t\in K_r=K$, $W^1_r=w_rw_{\bar{r}}$,

$N^1_r(u)=n_r(u)n_{\bar{r}}(\bar{u})$, $u\in K^*_r=K_r\backslash\{0\}$, $h^1_r(v)=h_r(v)h_{\bar{r}}(\bar{v})$, $v\in K^{*\prime}_r=K^*=K\backslash\{0\}$.

IIb. If $r\in\Phi_{11b}$: $X^1_r(t)=x_r(t)x_{\bar{r}}(\bar{t})x_{r+\bar{r}}\left(-\dfrac{1}{2}N_{r,\bar{r}}t\bar{t}\right)$, $t\in K_r=K$, $W^1_r=w_{r+\bar{r}}$,

$N^1_r(u)=n_{r+\bar{r}}(u)$, $u\in K^*_{r+\bar{r}}$, $h^1_r(v)=h_r(v)h_{\bar{r}}(\bar{v})$, $v\in K^{*\prime}_r=K\backslash\{0\}$.

Furthermore, we define $W^1=\langle W^1_r,\ r\in\Phi^*\rangle$, $N^1=\langle N^1_r(u),\ h^1_r(v),\ r\in\Phi,\ u\in K^*_r,\ v\in K^{*\prime}_r\rangle$ and $H^1=N^1\cap H$. We denote by Φ^+ the positive root system containing Π and denote by Φ^- the negative root system $\{-r\,|\,r\in\Phi^+\}$.

The following proposition is easily verified.

Proposition 2. (a) *Let* $r\in\Phi^{*+}=\Phi^*\cap\Phi^+$ *and* $t\in K_r$. *Then* $X^1_r(t)\in U^1$. (b) *Each element* u *of* U^1 *has an unique expression in the form*

$$u=\prod_{i=1}^{m}X_{r_i}(t_i),\ r_i\in\Phi^{*+}_1=\{r\in\Phi^{*+}\,|\,r\leqslant\bar{r}\},\ t_i\in K^*_{r_i},\ i=1,2,\cdots,m,\ r_1\leqslant r_2\leqslant\cdots\leqslant r_m.$$

We define $J_1(u)=r_1$ and $J(u)=\{r_1,\ r_2,\ \cdots,\ r_m\}$.

It is clear that similar results hold for V^1 in terms of negative roots.

Proposition 3. (a) $H^1\subset N^1\subset G^1$ *and* $N^1/H^1\cong W^1=\langle W^1_r,\ r\in\Pi^*;\ W^1_0\rangle$, *where* $W^1_0=\langle w_\alpha,\ \alpha\in\Pi^0\rangle\cap W^1$, (b) *Let* $r\in\Phi^*$ *and* $v\in K^{*\prime}_r$. *Then* $h^1_r(v)\in H^1$.

Proof For each $r\in\Phi^*\backslash\Phi_{11b}$, we have $N^1_r(u)=X^1_r(u)X^1_{\bar{r}}(-u^{-1})X^1_r(u)$, $u\in K^*_r$.

Since $r\pm\bar{r}\notin\Phi$, we have $N_r^1(u)\in G^1$, and $h_r^1(v)\in G^1$, $v\in K_r^{*\prime}$.

Clearly, the image of $N_r^1(u)$ under the natural homomorphism from N onto W is just W_r^1 for $r\in\Phi^*$, $u\in K_r^*$, so we have $N^1/H^1\cong W^1$. It follows from the properties of Satake diagram that $W^1=\langle W_r^1,\ r\in\Pi^*,\ W_0^1\rangle$.

For all $r\in\Phi^*/\Phi_{11b}$ the statement b is obvious by the statement 1. By a similar argument as in 13.7.2 of [7] (cf. the case in which J has type A_2), we have $h_r^1(v)\in H^1$ for all $r\in\Phi_{11b}$, $v\in K_r^{*\prime}$ since $J=\{r,\ \bar{r},\ r+\bar{r}\}$ has type A_2. It follows that $H^1\subset N^1\subset G^1$.

Let N_r^1 denote the element $N_r^1(\xi_r)$, ξ_r be a fixed element of K_r^*.

We define $G_0=\langle x_\alpha(t),\ \alpha\in\Phi^0,\ t\in K;\ H\rangle$, $Y^1=G^1\cap G_0$ and $B^1=Y^1U^1$.

Proposition 4. *Let $y\in Y^1$, $u\in U^1$, $v\in V^1$ and $n\in N^1$. Then $yuy^{-1}\in U^1$, $yvy^{-1}\in V^1$ and $nyn^{-1}\in Y^1$.*

Proof Suppose $r\in\Phi^{*+}$ and $\alpha\in\Phi^0$. If $ir+j\alpha\in\Phi$, i, j being a pair of positive integers, then we have $ir+j\alpha\in\Phi^{*+}$ since $\alpha'=\frac{1}{2}(\alpha+\bar{\alpha})=0$. Thus it follows from Chevalley's commutator formula that $yuy^{-1}\in U^1$. Similarly, we have $yvy^{-1}\in V^1$. Suppose $w\in W^1$ and $\alpha\in\Phi^0$. Then we have $(w(\alpha))'=0$. Thus it follows from Proposition 3(a) that $nyn^{-1}\in Y^1$.

By Proposition 4, we have the following proposition immediately.

Proposition 5. *$Y^1U^1=U^1\,Y^1$ and B^1 is a subgroup of G^1.*

For each $r\in\Phi^*$, we define

$$I(r)=\left\{s\in\Phi^*\,|\,s'=kr',\ k=1,\ 2,\ \text{or}\ \frac{1}{2}\right\}$$

and $I_1(r)=\{s\in I(r)\,|\,s\leqslant\bar{s}\}$. If $I_1(r)=\{r_1,\ r_2,\ \cdots,\ r_n\}$, we define $X_r(T)=\prod_{i=1}^{n}X_{r_i}^1(t_i)$, $t_i\in K_{r_i}$, $T=(t_1,\ t_2,\cdots,\ t_n)$ and $D(T)=t_1\bar{t}_1+t_2\bar{t}_2+\cdots+t_n\bar{t}_n$. The following lemma is easily verified.

Lemma 1. *Suppose $r\in\Pi^*$. Then $I(r)=\{s\in\Phi^{*+}\,|\,W_r^1(s)\in\Phi^-\}$.*

For each $r\in\Pi^*$ there exists a Satake diagram $(\Pi_r,\ \theta)$ whose $\mathbb{R}$-rank is equivalent to one such that $I(r)=\Phi_r^{*+}=\Phi_r^*\cap\Phi_r^+$ where $\Phi_r^*=\{s\in\Phi_r\,|\,s\neq-\bar{s}\}$, Φ_r being the root system with fundamental system Π_r, and Φ_r^+ is the positive root system of Φ_r containing Π_r.

All Satake diagrams whose $\mathbb{R}$-rank is equivalent to one are given in table I.

Lemma 2. *Suppose $r\in\Pi^*$, $I_1(r)=\{r_1,\ r_2,\ \cdots,\ r_n\}$ and $D(T)\neq0$, $T\in K_{r_1}\times K_{r_2}\times\cdots\times K_{r_n}$. Then $X_{-r}(T)=X_r(T^*)N_r^1X_r(T^{*\prime})y$, $y\in Y^1$, T^*, $T^{*\prime}\in K_{r_1}\times K_{r_2}\times\cdots\times K_{r_n}$.*

Proof We shall consider the cases 1–6 given in table 1 separately.

Case 1. We have $I(r)=I_1(r)=\{r\}$ $(r=\bar{r})$ and $X_{-r}(T)=X_{-r}^1(t_1)=x_{-r}(t_1,)$ $t_1=t\in K_r$. Since $D(T)\neq0$ we have $t_1=t\neq0$, so we have

$$X_{-r}(T)=x_r(t^{-1})h_r(-t^{-1})n_rx_r(t^{-1})=X_r(T^*)N_r^1X_r(T^{*\prime})y,$$

by 6, 4.4 of [7], where $y=h_r(-t)\in Y^1$, $n_r=N_r^1\in N^1$, $T^*=(t^{-1})$, $T^{*\prime}=(t)\in K_r$, $(\xi_r=1)$.

Case 2. We have $I(r)=\{r, \bar{r}\}$, $I_1(r)=\{r\}(r\neq\bar{r})$ and $X_{-r}(T)=X^1_{-r}(t_1)=x_{-r}(t_1)x_{-\bar{r}}(\bar{t}_1)$, $T=(t_1)$, $t_1\in K_r$. Since $D(T)\neq 0$, we have $t=t_1\neq 0$, so by 6.4.4 of [7] we have

$$X_{-r}(T)=x_r(t^{-1})x_{\bar{r}}(\bar{t}^{-1})h_r(-t^{-1})h_{\bar{r}}(\bar{t}^{-1})n_rn_{\bar{r}}x_r(t^{-1})x_{\bar{r}}(\bar{t}^{-1}),=X_r(T^*)N_r^1X_r(T^{*\prime})y,$$

where

$$y=h_r(-t)h_{\bar{r}}(-\bar{t})\in Y^1,\ n_rn_{\bar{r}}=N_r^1\in N^1,\ T^*=(t^{-1}),\ T^{*\prime}=(t)\in K_r, (\xi_r=1).$$

Case 3. We have $I(r)=\{r=r_1=\varphi_2-\varphi_3,\ r_2=\varphi_2-\varphi_4,\ \bar{r}_1=\varphi_1-\varphi_4,\ \bar{r}_2=\varphi_1-\varphi_3\}$, $I_1(r)=\{r_1, r_2\}$. We write $\alpha_1=\varphi_1-\varphi_2$, $\alpha_2=\varphi_3-\varphi_4$. By 4.1.2. of [7] we have

$$N_{r_1,\ \alpha_1}=N_{-\bar{r}_2,r_1}=N_{\alpha_1,-\bar{r}_2}=c_1,\ N_{r_1,-r_2}=N_{-r_2,\alpha_2}=N_{\alpha_2,r_1}=c_2,$$

$$N_{-\alpha_1,-r_2}=N_{-r_2,\bar{r}_1}=N_{\bar{r}_1,-\alpha_1}=c_3,\ N_{\bar{r}_1,-\bar{r}_2}=N_{-\bar{r}_2,-\alpha_2}=N_{-\alpha_2,\bar{r}_1}=c_4.$$

Since ρ_θ is an involutive automorphism of $\mathbb{L}$, we have $-c_2c_4=-c_1c_3=1$. We have

$$X_{-r}(T)=X^1_{-r_1}(t_1)X^1_{-r_2}(t_2)=x_{-r_1}(t_1)x_{-\bar{r}_1}(\bar{t}_1)x_{-r_2}(t_2)x_{-\bar{r}_2}(\bar{t}_2),$$

$$T=(t_1, t_2),\ t_i\in K_{r_i}\ i=1, 2.$$

Since $D(T)\neq 0$, we have $T\neq(0, 0)$. If either t_1 or t_2 is equal to zero, the assertion is verified from case 2. Assuming $t_1\neq 0$ and $t_2\neq 0$, by 5.2.2 and 6.4.4 of [7] we have

$$\begin{aligned}X_{-r}(T)&=x_{\bar{r}_1}(\bar{t}_1^{-1})h_{\bar{r}_1}(-\bar{t}_1^{-1})n_{\bar{r}_1}x_{\bar{r}_1}(\bar{t}_1^{-1})x_{-r_1}(t_1)x_{-r_2}(t_2)x_{-\bar{r}_2}(\bar{t}_2)\\&=x_{\bar{r}_1}(\bar{t}_1^{-1})h_{\bar{r}_1}(-\bar{t}_1^{-1})n_{\bar{r}_1}x_{-r_1}(t_1)x_{-r_2}(t_2)x_{\alpha_1}(-c_3\bar{t}_1^{-1}t_2)x_{-\bar{r}_2}(\bar{t}_2)x_{\alpha_2}(c_4\bar{t}_1^{-1}\bar{t}_2)x_{\bar{r}_1}(\bar{t}_1^{-1})\\&=x_{\bar{r}_1}(\bar{t}_1^{-1})h_{\bar{r}_1}(-\bar{t}_1^{-1})n_{\bar{r}_1}x_{-r_1}(d)x_{-r_2}(t_2)x_{-\bar{r}_2}(\bar{t}_2)x_{\bar{r}_1}(\bar{t}_1^{-1})y_1,\end{aligned}$$

where $d=t_1-c_1c_3\bar{t}_1^{-1}t_2\bar{t}_2$ and $y_1=x_{\alpha_1}(-c_3\bar{t}_1^{-1}t_2)x_{\alpha_2}(c_4\bar{t}_1^{-1}\bar{t}_2)$. Since $D(T)\neq 0$, we have $d=\bar{t}_1^{-1}D(T)\neq 0$. Thus we have

$$\begin{aligned}X_{-r}(T)=&x_{\bar{r}_1}(\bar{t}_1^{-1})h_{\bar{r}_1}(-\bar{t}_1^{-1})x_{r_1}(d^{-1})h_{r_1}(-d^{-1})n_{\bar{r}}n_rx_{-r_2}(t_2)x_{-\bar{r}_2}(\bar{t}_2)\\&\times x_{-\bar{r}_1}(-d^{-1}t_2\bar{t}_2)x_{r_1}(d^{-1})y_2x_{\bar{r}_1}(\bar{t}_1^{-1}(y_1),\end{aligned}$$

where $y_2=x_{-\alpha_1}(-c_1d^{-1}\bar{t}_2)x_{-\alpha_2}(c_2d^{-1}t_2)$. We write $T^*=(t_1^*, t_2^*)$ and $T^{*\prime}=(t_1^{*\prime}, t_2^{*\prime})$, where $t_1^*=t_1D(T)^{-1}$, $t_2^*=\bar{t}_2D(T)^{-1}$, and $t_1^{*\prime}=t_1$, $t_2^{*\prime}=t_2$. Finally we have

$$\begin{aligned}X_{-r}(T)&=x_{r_1}(t_1^*)x_{\bar{r}_1}(\bar{t}_1^*)x_{r_2}(t_2^*)x_{\bar{r}_2}(\bar{t}_2^*)n_{\bar{r}}n_rx_{r_1}(t_1^{*\prime})x_{r_1}(\bar{t}_1^{*\prime})x_{r_2}(t_2^{*\prime})x_{r_1}(\bar{t}_2^{*\prime})y\\&=X^1_{r_1}(t_1^*)X^1_{r_2}(t_2^*)N_r^1X^1_{r_1}(t_1^{*\prime})X^1_{r_2}(t_2^{*\prime})y=X_r(T^*)N_r^1X_r(T^{*\prime})y,\end{aligned}$$

where $y=h_{\bar{r}}(-t_1)h_{r_1}(-d)y_1y_2$ and $n_{\bar{r}}n_r=N_r^1\in N^1(\xi_r-1)$. Since $y\in G_0$ and $X_{-r}(T)$, $X_r(T^*)$, $X_r(T^{*\prime})$, $N_r^1\in N^1$, we have $y\in Y^1$.

By a similar argument, this lemma can be verified for other cases.

Henceforth we assume that $K=K_0(\sqrt{-1})$, where K_0 is a formally real field (or ordered field) and f is the conjugation of K. Thus if $T\neq(0, 0,\cdots, 0)\in K\times K\cdots\times K$, then $D(T)\neq 0$.

Lemma 3. *Suppose $r\in\Pi^*$. Then the subset $B^1\cup B^1N_r^1B^1$ is a subgroup of G^1.*

Proof Clearly, we have $(B^1)^{-1}=B^1$ and $(B^1N_r^1B^1)^{-1}=B^1N_r^1B^1$. In order to prove that the subset is closed under multiplication, it is sufficient to show

$N^1_rB^1N^1_r \subset B^1 \cup B^1N^1_rB^1$. By Lemma 1, if $I_1(r) = \{r_1, r_2, \cdots, r_n\}$, each element b of B^1 has an expression in the form $b = X_r(T')u_ry_1$, where $y_1 \in Y^1$, $T' \in K_{r_1} \times K_{r_2} \times \cdots \times K_{r_n}$ and $u_r \in U^1$ satisfies $N^1_ru_rN^1_r \in U^1$. Suppose first $T' \neq (0, 0, \cdots, 0)$. Then $N^1_rX_r(T')N^1_r = X_{-r}(T)$, $D(T) \neq 0$. It follows from Lemma 2 that

$$N^1_rbN^1_r = N^1_rX_r(T')u_ry_1N^1_r = X_{-r}(T)u'_ry'_1 = X_r(T^*)N^1_rX_r(T^{*\prime})yu'_ry'_1,$$

where $u'_r = N^1_ru_rN^1_r \in U^1$, $y'_1 = N^1_ry_1N^1_r \in Y^1$, and T^*, $T^{*\prime}$ and y are given as in Lemma 2. By Proposition 4 we have $N^1_rB^1N^1_r \subset B^1N^1_rB^1$.

Suppose $T' = (0, 0, \cdots, 0)$. Then we have $N^1_rB^1N^1_r \subset B^1$. The proof is complete.

Using Lemmas, 1, 2, 3, we have the following lemma by a similar argument as uesd in 8, 1.5 of [7]

Lemma 4. *Suppose $r \in \Pi^*$ and $n \in N^1$. Then $B^1nB^1N^1_rB^1 \subseteq B^1nN^1_rB^1 \cup B^1nB^1$.*

Theorem 1. *Let $K = K_0(\sqrt{-1})$, K_0 being a formally real field. Then the subset $B^1N^1B^1$ is a subgroup of G^1 and $G^1 = B^1N^1B^1$.*

Proof Clearly, we have $(B^1N^1B^1)^{-1} = B^1N^1B^1$. In order to prove that the subset is closed under multiplication, it is sufficient to show $B^1n_1B^1n_2B^1 \subseteq B^1N^1B^1$ for each pair $n_1, n_2 \in N^1$. By Proposition 3 we have $n_2 = N^1_{r_1}N^1_{r_2}\cdots N^1_{r_q}\tilde{n}_0$, where $r_i \in \Pi^*$ $i = 1, 2, \cdots, q$ and $\tilde{n}_0$ is an element of N^1 whose image under the natural homomorphism from N onto W belongs to W^1_0. Thus we have $\tilde{n}_0 \in Y^1$, so we have

$$B^1n_1B^1n_2B^1 = B^1n_1B^1N^1_{r_1}N^1_{r_2}\cdots N^1_{r_q}B^1, \quad r_i \in \Pi^*, \ i = 1, 2, \cdots, q.$$

By Lemma 4 we have

$$B^1n_1B^1n_2B^1 \subseteq (B^1n_1B^1 \cup B^1n_1N^1_{r_1}B^1)B^1N^1_{r_2}B^1 \cdots B^1N^1_{r_q}B^1 \subseteq \cdots \subseteq BNB.$$

Thus the subset $B^1N^1B^1$ is a subgroup of G^1. Since B^1, $N^1 \subset B^1N^1B^1$ we have $G^1 = B^1N^1B^1$. The proof is complete.

By Theorem 1 we have

$$G^1 = \bigcup_{w \in W^1} B^1n_wB^1 = \bigcup_{w \in W^1} U^1Y^1n_wU^{1-}_w, \quad U^{1-}_w = U^-_w \cap U^1, \ (n_w \in N^1).$$

This decomposition of G^1 into the double coset of B^1 is called the quasi-Bruhat decomposition of G^1. Clearly, the quasi-Bruhat decomposition of G^1 is the generalized Bruhat decomposition of twisted groups and Chevalley groups.

Lemma 5. *Let $r_1, r_2 \in \Phi^*$ such that $r_1 \neq r_2$ and $r_1 \neq \bar{r}_2$. Then there exists $h(\chi) \in H^1$ such that $\chi(r_1) = 1$, $\chi(r_2) \neq \chi(r_1)$ or $\chi(r_2) = 1$, $\chi(r_1) \neq \chi(r_2)$.*

Proof We define $s' = \frac{1}{2}(s + \bar{s})$ for each $s \in \Phi^*$ and denote by F the set $\{r_1, r_2\}$. We shall consider separately the different possibilities:

I. Suppose $I(r_1) \neq I(r_2)$:

a. Suppose $(r'_1, r'_2) = 0$. Then we put $h(\chi) = h^1_{r_2}(2)$.

b. Suppose $(r'_1, r'_2) \neq 0$. (1) If $F \cap \Phi_{11b} = \emptyset$, then we put $h(\chi) = h^1_r(-1)$, where $r \in F$ satisfies $(r', r') \geqslant (s', s')$, $s \in F \backslash \{r\}$. (2), If there is a root r of F such that $r \notin \Phi_{11b}$ and $F \backslash \{r\} \in \Phi_{11b}$, then we put $h(\chi) = h^1_r(-1)$. (3), If $F \subset \Phi_{11b}$, then we put

$h(\chi)=h^1_{r_2}(-1)$.

II. Suppose $I(r_1)=I(r_2)$:

a. Suppose $F\not\subset\Phi_{11b}$. Then we put $h(\chi)=h^1_{r_2}(5^{-1}(3+4\sqrt{-I}))$.

b. Suppose $F\subset\Phi_{11b}$. Then we put

$$h(\chi)=h^1_{r_1}(d)h^1_{r_2}(d^{-3})h^1_r(2),\ d=I+\sqrt{-1},\ r=r_1+\bar{r}_1.$$

c. Suppose $r_1\in\Phi_{11b}$ and $r_2\in\Phi_{1b}$. Then we put $h(\chi)=h'_{r_2}(-1)$.

It is easilly verified that $h(\chi)$ put above are the elements of H^1 as required.

Corollaray 1. *Let $n_w\in N^1$ such that $w\notin W^1_0$. Then there exists $h(\chi)\in H^1$ such that*

$$n_w h(\chi)n_w^{-1}=h(\chi')\neq h(\chi)$$

and $\chi(\alpha)=1$ for all $\alpha\in\Phi^0$.

Lemma 6. *Suppose $y\in Y^1$ and $y\neq I$. Then there exists $u\in U^1$ such that $yuy^{-1}u^{-1}\neq 1$ or there exists $v\in V^1$ such that $yvy^{-1}v^{-1}\neq I$.*

Proof We assume that $y\in Z^1$ where Z^1 is the centre of G^1. Clearly, y has a unique expression in the form $y=u_1h(\chi_0)n_wu$, where $u_1\in U\cap G_0$, $u\in U^-_w\cap G_0$ $h(\chi_0)\in H$ and $w\in W_0=\langle w_\alpha,\ \alpha\in\Phi^0\rangle$. Obviously, for each $\alpha\in\Phi^0$ there is $h(\chi)\in H^1$ such that $\chi(\alpha)\neq 1$. Thus we have $u_1=u=I$. It is clear that if $w\neq 1$, then there is $r\in\Phi^*$ such that $w(r)\neq r$, so we have $y=h(\chi_0)$. If $h(\chi_0)\neq 1$, then there exists $r\in\Phi^*$ such that $\chi_0(r)\neq 1$. We have a contradiction if $y\neq I$. The proof is complete.

Henceforth we assume that R^1 is an arbitrary normal subgroup of G^1 such that $R^1\neq\{1\}$. For each $r\in\Phi^*$ we define $X^1_r=\{X^1_r(t)\mid t\in K_r\}$.

Lemma 7. *There is a root $r^*\in\Phi^{*+}$ such that $R^1\cap X^1_{r^*}\neq\{1\}$, where*

$$X^1_{r^*}=\{X^1_{r^*}(t)\mid t\in K_{r^*}\}.$$

Proof It follows from Theorem 1 That there is $x\in R^1$ such that $x\neq 1$ and x has an expression $x=bn_w=\tilde{u}yn_w$, $b\in B^1$, $w\in W^1$, $\tilde{u}\in U^1$, $y\in Y^1$. We shall consider separately the different possibilities.

I. Suppose $w\in W^1_0$. Then $x=\tilde{u}y_1$, $y_1=yn_w\in Y^1$.

a. Suppose $\tilde{u}\neq I$. Then $x_1=h^1_r(2)\ xh^1_r(2)^{-1}x^{-1}=u\in R^1\cap U^1$, $x_1=u\neq I$, where $r=J_1(\tilde{u})$. b. Suppose $\tilde{u}=I$. Then there is $u\in R^1\cap U^1$, $u\neq I$ or $v\in R^1\cap V^1$, $v\neq I$ by Lemma 6.

II. Suppose $w\notin W^1_0$. Then $x_1=h(\chi)xh(\chi)^{-1}x^{-1}=h(\chi'_1)u_1\in R^1$ where $h(\chi)$ is the element of H^1 given in Corollary 1 and $u_1\in U^1$. moreover, we have

$$h(\chi'_1)=h(\chi)h(\chi')^{-1}\neq I.$$

a. Suppose $u_1\neq I$. Then we have $u\in R^1\cap U^1$, $u\neq I$ by a similar argument as in case 1, a. mentioned above. b. Suppose $u_1=I$. Then there exists $r\in\Phi^{*+}$ such that $\chi'_1(r)\neq 1$. Then $u=h(\chi'_1)X^1_r(t)h(\chi'_1)^{-1}X^1_r(t)^{-1}\in R^1$, $t\in K^*_r$. Clearly, we have $u\in R^1\cap U^1$, $u\neq I$.

We may summarize the results mentioned above in a statement as follows: there is $u\in R^1\cap U^1$, $u\neq I$ or there is $v\in R^1\cap V^1$, $v\neq I$.

We assume that there exists $u \in R^1 \cap U^1$ such that $u \neq I$. Then

$$J(u) = \{r_1, r_2, \cdots, r_n\}, \ n \geqslant I.$$

If $n=1$ lemma is established immediately in this case. Now we assume that $n>1$. Let $u_1^* = h(\chi)uh(\chi)^{-1}u^{-1}$ where $h(\chi)$ is the element of H^1 given in Lemma 5. Then $u_1^* \in R^1 \cap U^1$ and $J(u_1^*) = \{r_1^*, r_2^*, \cdots, r_m^*\}$ such that either $r_1 \prec r_1^*$ or $r_1 = r_1^*$, $r_2 \prec r_2^*$. Finally, after a finfte number of the step used above we have

$$u^* = X_{r^*}^1(t) \in R^1 \cap X_{r^*}^1, \ u^* \neq I.$$

We assume that there exists $v \in R^1 \cap V^1$ such that $v \neq I$. Then by a similar argument used above we have $X_{-r^*}^1(t) = v^* \in R^1 \cap V^1$, $t \in K_{r^*}^*$. Thus we have $u^* = N_{r^*}^1 v^* N_{r^*}^1 \in R^1 \cap X_{r^*}^1$, $u^* \neq I$. This finishes the proof.

Theorem 2. *Let $K = K_0(\sqrt{-1})$, K_0 being a formally real field. Then G^1 is simple.*

Proof By Lemma 7 there is $r^* \in \Phi^{*+}$ such that $X_{r^*}^1(t) \in R^1$, $t \in K_{r^*}^*$. If $r^* \notin \Phi_{11b}$ we put $h(\chi) = h_{r^*}^1(2)$, if $r^* \in \Phi_{11b}$ we put $h(\chi) = h_{r_1}^1(2)$, $r_1 = r^* + \bar{r}^*$. Clearly, we have $h(\chi) \in R^1$. It follows from the properties of the Satake diagram that for each $r \in \Phi^*$ there exists $w \in W^1$ such that $\chi(w(r)) \neq 1$. For an arbitrary $c \in K_s$ we have $h(\chi)\ X_s^1(t)h(\chi)^{-1}X_s^1(t)^{-1} = X_s^1(c)$, $s = w(r)$, $t = (I - \chi(s))^{-1}c$. It is clear that $X_s^1(c) \in R^1$ and $X_s^1 \subset R^1$. We have $X_r^1 \subset R^1$ since $n_w^{-1} X_s^1 n_w = X_r^1$, $n_w \in N^1$. It follows that $U^1 \subset R^1$ and $V^1 \subset R^1$, so G^1 is simple

We denote by N_0^1 the subgroup generated by all elements of N which map to the elements of W_0^1 under the natural homomorphism of N onto W. We define $\Pi_1^* = \{r \in \Pi^* |_{r \prec \bar{r}}\}$. Clearly, Π_1^* can be expressed in the form $\Pi_1^* = \{r_i,\ i \in I\}$ where I is a finite set. We define $W^{1'} = \{w_i,\ i \in I\}$, $w_i = W_{r_i}^1$, $i \in I$.

Theorem 3. *Let $K = K_0(\sqrt{-1})$, K_0 being a formally real field. Then G^1 has a (B^1, N^1) pair.*

Proof We shall verify that group G^1 satisfies the axioms BN. 1——BN. 5.

BN. 1: By Theorem 1 we have $G^1 = \langle B^1, N^1 \rangle$ immediately.

BN. 2: It is easily verified that $B^1 \cap N^1 = N_0^1$ and N_0^1 is a normal subgroup of N^1.

BN. 3: Obviously, we have $N^1/B^1 \cap N^1 = N^1/N_0^1 \cong W^1/W_0^1 \cong W^{1'}$.
Such a $W^{1'}$ is generated by a set of elements w_i, $i \in I$, $w_i^2 = I$.

BN. 4: If n_i is an element of N^1 which maps to w_i, $i \in I$ under the natural homomorphism from N onto W, then there is $r_i \in \Pi_1^*$ such that $n_i = N_{r_i}^1 h$, $h \in H^1$. Thus we have $B^1 n_i B^1 n B^1 \subseteq B^1 n_i n B^1 \cup B^1 n B^1$ for any element n of N^1 by Lemma 4.

BN. 5: Since $N_{r_i}^1 X_{r_i}^1 N_{r_i}^1 = X_{-r_i}^1$, $i \in I$, we have $n_i B^1 n_i \neq B^1$

Theorem 4. *Group $G^1 = \mathbb{L}(\Pi, \theta; \mathbb{C}, f)$ is a simple algebraic group defined over $\mathbb{R}$ and the admissible $(\mathbb{R})$-index of G^1 is just (Π, θ).*

Proof Clearly, $G = \mathbb{L}(\mathbb{C})$ is a simple algebraic group defined over $\mathbb{C}$. The conjugation f of fiele $\mathbb{C}$ can be extended a field automorphism f of G. For each $x\in G^1$ we have $f(x)\in G^1$. Thus it follows from proposition I.I.I of [6] that G^1 is defined over $\mathbb{R}$. Then by Theorem 6 of [3], G^1 is an algebraic group defined over $\mathbb{R}$. It follows from Theorem 2 that G^1 is a simple algebraic group defined over $\mathbb{R}$. It is easily verified that the admissible $(\mathbb{R})$-index of G^1 is just (Π, θ).

Let $K=K_0(\sqrt{-1})$, K_0 being a formally real field and $\emptyset$ be the empty set. The following statements are easily verified:

I. Suppose $\Pi^0=\emptyset$ (The all nodes of the Satake diagram (Π, θ) are white). Then $G^1=\mathbb{L}(\Pi, \theta; K, f)$ is a twisted group ${}^2\mathbb{L}(K)$ or Chevalley group $\mathbb{L}(K_0)$. It is clear that for the index (Π, θ) the semisimple anisotropic kernel is equal to $\{I\}$. Thus we have $Y^1=H^1$ and $B^1=H^1U^1$. It follows that B^1 is a Borel subgroup of G^1.

II. Suppose $\Pi^0\neq\emptyset$ (There are the black nodes in the Satake diagram (Π, θ)). Then $G^1=\mathbb{L}(\Pi, \theta; K, f)$ may be interpreted as a generalized twisted group and generalized Chevalley group. It is clear that for the index (Π, θ) the semisimple anisotropic kernels which are not equal to $\{I\}$ are contained in Y^1. Thus we have $Y^1\neq H^1$, $H^1\subset Y^1$ and $B^1\neq H^1U^1$. It follows that B^1 is not a Borel subgroup of G^1.

Let p be a p-adic field. In a similar ways as in this paper, for each admissible (p)-index we can construct a simple algebraic group of adjoint type defined over p which has such an admissible (p)-index.

In table II we give all Satake diagrams and all Satake diagrams which correspond to the groups of twisted type or Chevalley type.

Table I

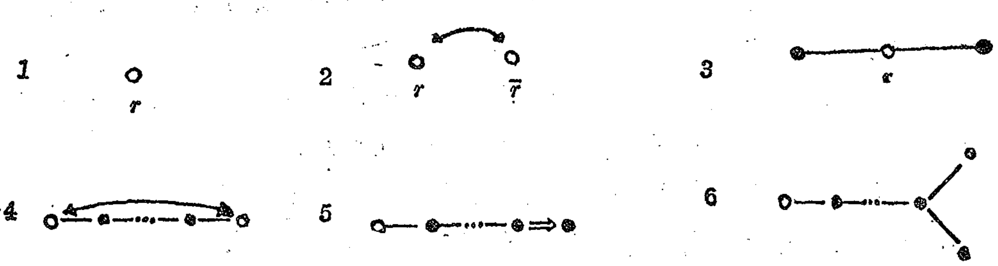


Table II (Satake diagrams)

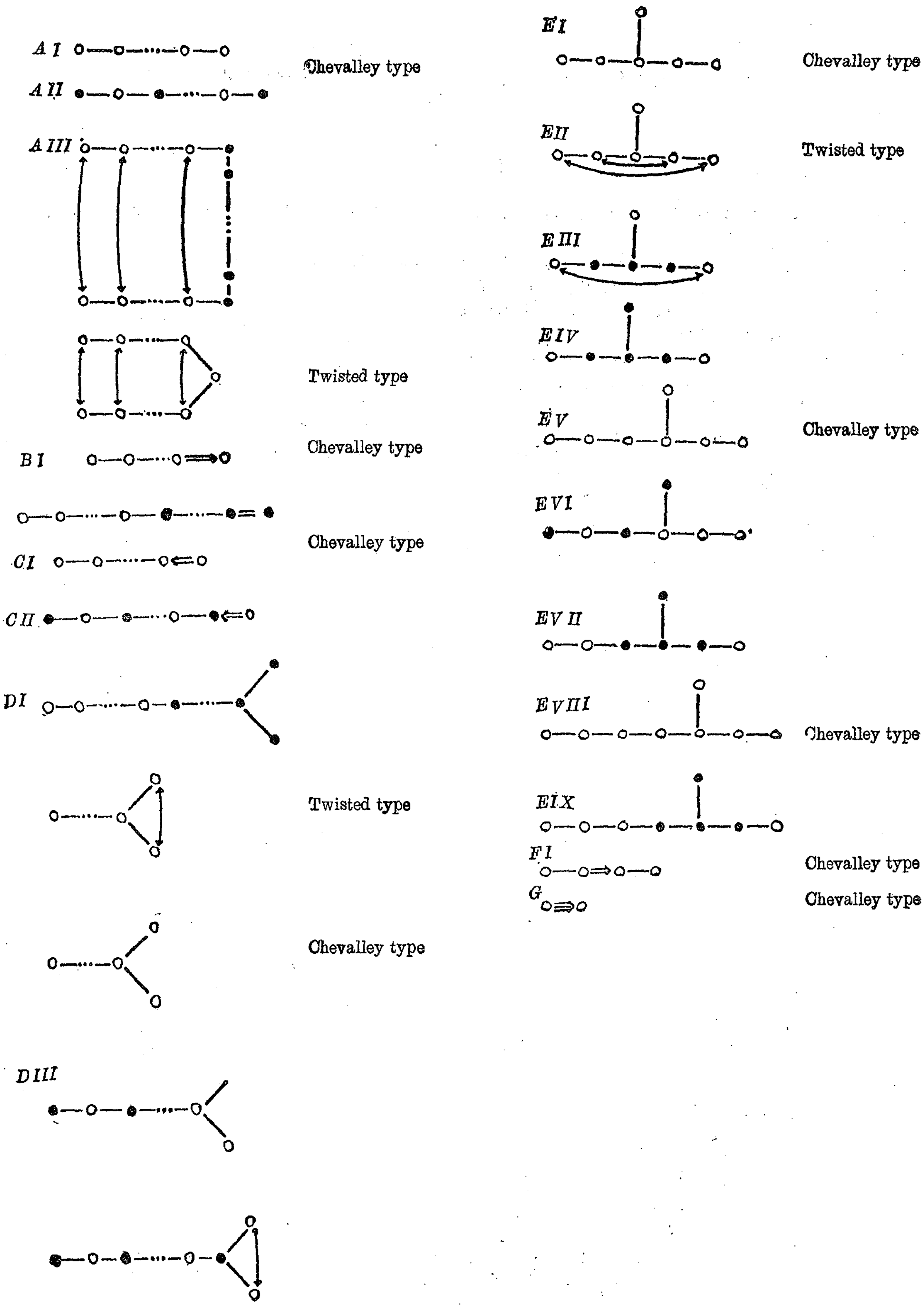
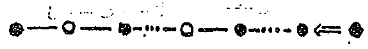
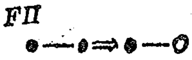


Table II′

CEII



FII



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