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ON THE HOMOLOGY OF THE VIRASORO ALGEBRA

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Abstract

This paper gives the structure of the homology of the Witt algebra and the Virasoro algebra with coeffieionts in a Verma module. Let $s_k=(3k^2+k)/2$, $t_k=(3k^2-k)/2$, $k\in\mathbb{Z}_+$, and $P=\{-s_k, -t_k | k\in\mathbb{Z}_+\}$. Then the author obtains the homology of the Witt algebra with coefficients in an irreducible module $L(\lambda)$ with highest weight $\lambda\notin P$, and the homology of the Virasoro algebra with coefficients in some irreducible modules.

§ 1. Introduction

In [2] the author obtained the homology of Kao-Moody Lie algebras with coefficients in a Verma module. In this paper, we determine the structure of the Witt algebra and the Virasoro algebra with coefficients in a Verma module. Let $s_k=(3k^2+k)/2$, $t_k=(3k^2-k)/2$, $k\in\mathbb{Z}_+$ (the set of the non-negative integers), the non-negative integers s_k, $t_k (k\in\mathbb{Z}_+)$ are called Euler's pentagonal numbers. Let $P=\{-s_k, -t_k | k\in\mathbb{Z}_+\}$. Then we also obtain the homology of the Witt algebra with coefficients in an irreducible module $L(\lambda)$ with highest weight $\lambda\notin P$.

Let $\boldsymbol{g}$ be the complex Lie algebra with basis $\{e_i\}_{i\in\mathbb{Z}}$, where

$$[e_i, e_j]=(j-i)e_{i+j},\ i, j\in\mathbb{Z}.$$

$\boldsymbol{g}$ is also known as the Witt algebra. Let $\boldsymbol{h}=\boldsymbol{C}e_0$ (it is a Cartan subalgebra of $\boldsymbol{g}$), $\boldsymbol{n}=\bigoplus_{i\in N}\boldsymbol{C}e_i$, $\boldsymbol{n}^-=\bigoplus_{i\in N}Ce_{-i}$, and identify $\boldsymbol{h}^*$ with $\boldsymbol{C}$. In [4], Goncharova computed the cohomology $H^*(\boldsymbol{n}, \boldsymbol{C})$, or equivalently, the homology $H_*(\boldsymbol{n}^-, \boldsymbol{C})$. She proved that

$$\begin{cases} H_k(\boldsymbol{n}^-, \boldsymbol{C})_\nu=0, \text{ unless } \nu=-s_k \text{ or } \nu=-t_k, \\ H_k(\boldsymbol{n}^-, \boldsymbol{C})_\nu=\boldsymbol{C}(\nu), \text{ if } \nu=-s_k \text{ or } \nu=-t_k, \end{cases} \tag{1.1}$$

where $\boldsymbol{C}$ is regarded as 1-dimensional trivial $\boldsymbol{n}^-$-module, $H_k(\boldsymbol{n}^-, C_\nu)$ is the ν-weightspace of $H_k(\boldsymbol{n}^-, \boldsymbol{C})$ relative to the action of $\boldsymbol{h}$ and $\boldsymbol{C}(\nu)$ is the 1-dimensional $\boldsymbol{h}$-module, where $\boldsymbol{h}$ acts as ν (cf. [8, Theorem 6.6]).

Similarly, we can also obtain the homology of $\boldsymbol{n}$ with the trivial coefficient:

$$H_k(\boldsymbol{n}, \boldsymbol{C})=\boldsymbol{C}(s_k)\oplus\boldsymbol{C}(t_k),\ k\geqslant 0. \tag{1.2}$$

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In § 2, we shall use (1.2) and the methods in [2] to compute the homology of the Witt algebra with coefficients in a Verma module (see Theorem 2.8). In [10], Rocha-Caridi and Wallach obtained character formulas for the irreducible highest weight modules over the Witt algebra. Using [10, Theorem A], we obtain the homology of the Witt algebra with coefficients in an irreducible module $L(\lambda)$ with highest weight $\lambda \notin P$ (see Corollary 2.10).

Now we consider the central extension $\tilde{\boldsymbol{g}}$ of $\boldsymbol{g}$ defined as the complex Lie algebra with basis $\{E_0', E_i\}_{i\in\mathbb{Z}}$ and bracket relations

$$[E_0', X]=0, \text{ for all } X\in\tilde{\boldsymbol{g}},$$

$$[E_i, E_j]=(j-i)E_{i+j}+\delta_{i,-j}(i^3-i)E_0'/12, \ i, j\in\mathbb{Z}.$$

$\tilde{\boldsymbol{g}}$ is known as the Virasoro algebra. In § 3, using the computation of the spectral sequences, we obtain the homology of the Virasoro algebra with coefficients in a Verma module (see § 3, Theorem 3.8) . Using the relation between Verma modules and irreducible highest weight modules, we also obtain the homology of the Virasoro algebra with coefficients in some irreducible modules (see § 3).

In this paper, we denote the universal enveloping algebra of a complex Lie algebra $\boldsymbol{a}$ by $U(\boldsymbol{a})$. Unless the contrary is stated explicitly, $\boldsymbol{a}$-module will always be assumed to be left $\boldsymbol{a}$-module.

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§ 2. The Homology of the W.tt Algebra

For the Witt algebra $\boldsymbol{g}$, we let $\boldsymbol{b}=\boldsymbol{h}\oplus\boldsymbol{n}$, where $\boldsymbol{h}=\boldsymbol{C}e_0$ and $\boldsymbol{n}=\bigoplus_{i\in N}\boldsymbol{C}e_i$. If $\lambda\in\boldsymbol{h}^*$, we let $\boldsymbol{C}(\lambda)$ be the one-dimensional $\boldsymbol{b}$-module, where $\boldsymbol{n}$ acts trivially and $\boldsymbol{h}$ acts via λ. Let $M(\lambda)=U(\boldsymbol{g})\otimes_{U(\boldsymbol{b})}\boldsymbol{C}(\lambda)$. We call $M(\lambda)$ the Verma module with highest weight λ. Let $L(\lambda)$ denote the unique irreducible quotient of $M(\lambda)$. First we shall compute the homology of $\boldsymbol{g}$ in $M(\lambda)$.

Applying [3, Proposition 5.5.4] to the case of the Witt algebra, we have the following lemma.

Lemma 2.1. *For $\lambda\in\boldsymbol{h}^*$, we have*

$$(M(\lambda))^*\cong\mathrm{Hom}_{U(\boldsymbol{b})}(U(\boldsymbol{g}), \boldsymbol{C}(-\lambda)).$$

By [2, Proposition 3.4], we have the duality theorem between the homology groups and the cohomology groups of Lie algebras (possibly infinite-dimensional). Then

$$H_j(\boldsymbol{g}, M(\lambda)^t)^*\cong H^j(\boldsymbol{g}, (M(\lambda)^*)), \text{ for } j\geqslant 0, \tag{2.2}$$

where the right $\boldsymbol{g}$-module corresponding to the left $\boldsymbol{g}$-module $M(\lambda)$ is denoted by

$M(\lambda)^t$ and $(M(\lambda))^*$ is the dual $\boldsymbol{g}$-module of $M(\lambda)$.

By Lemma 2.1 and (2.2), we have

$$H_j(\boldsymbol{g}, M(\lambda)^t)^* \cong H^j(\boldsymbol{g}, \mathrm{Hom}_{U(b)}(U(\boldsymbol{g}), \boldsymbol{C}(-\lambda)), \quad \text{for } j\geqslant 0. \tag{2.3}$$

By (2.3) and [1, Proposition 4.2], we have

$$H^j(\boldsymbol{g}, \mathrm{Hom}_{U(b)}(U(\boldsymbol{g}), \boldsymbol{C}(-\lambda)) \cong H^j(\boldsymbol{b}, \boldsymbol{C}(-\lambda)), \quad \text{for } j\geqslant 0.$$

Then using (2.3), we have the following lemma.

Lemma 2.4. *For* $\lambda\in \boldsymbol{h}^*$, *we have*

$$H_j(\boldsymbol{g}, M(\lambda)^t)^* \cong H^j(\boldsymbol{b}, \boldsymbol{C}(-\lambda)), \text{ for } j\geqslant 0.$$

Since $\boldsymbol{n}$ is an ideal of $\boldsymbol{b}$ and $\boldsymbol{b}/\boldsymbol{n}\cong\boldsymbol{h}$, for the left $\boldsymbol{b}$-module $\boldsymbol{C}(\lambda)$, there is the Hochschild-Serre spectral sequence $\{E_r^{p,q}\}$, whose E_∞ term is associated with $H^*(\boldsymbol{b}, \boldsymbol{C}(-\lambda))$, that is

$$E_2^{p,q}=H^p((\boldsymbol{h}, H^q(\boldsymbol{n}, \boldsymbol{C}(-\lambda))\underset{p}{\Rightarrow} H^n(\boldsymbol{b}, \boldsymbol{C}(-\lambda)), \tag{2.5}$$

for $n=p+q$ (see [5, § 3, Corollary 4]).

If $\boldsymbol{C}(\lambda)$ is regarded as the trivial $\boldsymbol{n}$-module, then $H_*(\boldsymbol{n}, \boldsymbol{C}(\lambda))=H_*(\boldsymbol{n}, \boldsymbol{C})$. If $H_*(\boldsymbol{n}, \boldsymbol{C}(\lambda))$ and $H_*(\boldsymbol{n}, \boldsymbol{C})$ are regarded as $\boldsymbol{h}$-modules, then $H_q(\boldsymbol{n}, \boldsymbol{C}(\lambda))\cong H_q(\boldsymbol{n}, \boldsymbol{C})\otimes\boldsymbol{C}(\lambda)$ (the standard action of $\boldsymbol{h}$ on $H_*(\boldsymbol{n}, \boldsymbol{C}(\lambda))$ and $H_*(\boldsymbol{n}, \boldsymbol{C})$ are quite similar to that of [2, § 3, Remark]).

By the above formula, (1.2) and [2, Proposition 3.4], we have

$$H^q(\boldsymbol{n}, \boldsymbol{C}(-\lambda)) \cong H_q(\boldsymbol{n}, \boldsymbol{C}(\lambda))^* \cong \boldsymbol{C}(-s_q-\lambda)\oplus\boldsymbol{C}(-t_q-\lambda). \tag{2.6}$$

Let $\boldsymbol{a}$ be an abelian Lie algebra. It is easy to check that for $p\geqslant 0$,

$$H^p(\boldsymbol{a}, V^u)=\begin{cases}\mathrm{Hom}_{\boldsymbol{C}}(\wedge^p(a), V^u), & \text{if } u=0,\\ 0, & \text{if } u\neq 0,\end{cases} \tag{2.7}$$

where V^u is an $\boldsymbol{a}$-module such that $a\cdot v=u(a)v$, for $a\in\boldsymbol{a}$, $v\in V^u$ and $u\in\boldsymbol{a}^*$ and $\wedge(\boldsymbol{a})$ is the exterior algebra of $\boldsymbol{a}$.

Since $\boldsymbol{h}$ is abelian, by (2.6) and (2.7), we have

$$E_2^{p,q}=H^p(\boldsymbol{h}, H^q(\boldsymbol{n}, \boldsymbol{C}(-\lambda))=H^p(\boldsymbol{h}, \boldsymbol{C}(-s_q-\lambda)\oplus\boldsymbol{C}(-t_q-\lambda))$$
$$=\begin{cases}0, & \text{if } \lambda\neq -s_q \text{ or } -t_q,\\ (\wedge^p(\boldsymbol{h}))^*, & \text{if } \lambda=-s_q \text{ or } -t_q \text{ and } q\neq 0,\\ \wedge^p(\boldsymbol{h}))^*\oplus(\wedge^p(\boldsymbol{h}))^*, & \text{if } \lambda=0 \text{ and } q=0,\end{cases}$$

where $(\wedge^p(\boldsymbol{h}))^*=\mathrm{Hom}_{\boldsymbol{C}}(\wedge^p(\boldsymbol{h}), \boldsymbol{C})$. Then we obtain the three cases:

1) If $\lambda\neq -s_q$ or $-t_q$, for all $q\in\mathbb{Z}_+$, then $E_2^{p,q}=0$, for $p\geqslant 0$ and $q>0$. Thus the spectral sequence $\{E_r\}$ collapses. By (2.5), we have

$$H^n(\boldsymbol{b}, \boldsymbol{C}(-\lambda))=E_2^{n,0}=0, \text{ for } n\geqslant 0.$$

2) If $\lambda=-s_{q_0}$ or $-t_{q_0}$, for some $q_0\in N$, then

$$E_2^{p,q}=\begin{cases}0, & \text{for } p\geqslant 0 \text{ and } q\neq q_0,\\ (\wedge^p(\boldsymbol{h}))^*, & \text{for } p\geqslant 0 \text{ and } q=q_0.\end{cases}$$

Then the spectral sequence $\{E_r\}$ collapses and

$$H^n(\boldsymbol{b}, \boldsymbol{C}(-\lambda)) = E_2^{n-q_0, q_0} = (\wedge^{n-q_0}(\boldsymbol{h}))^*, \text{ for } n \geqslant 0.$$

3) If $\lambda = 0$, then the same argument just used shows that

$$H^n(\boldsymbol{b}, \boldsymbol{C}(-\lambda)) = (\wedge^n(\boldsymbol{h}))^* \oplus (\wedge^n(\boldsymbol{h}))^*, \text{ for } n \geqslant 0.$$

Using the above remark and Lemma 2.4 we obtain the following theorem.

Theorem 2.8. *Let $\boldsymbol{g}$ be the Witt algebra, $\boldsymbol{h}$ the Cartan subalgebra of $\boldsymbol{g}$ and $M(\lambda)$ the Verma module with highest $\lambda \in \boldsymbol{h}^*$. Then the homology of $\boldsymbol{g}$ with coefficients in $M(\lambda)$ is given as follows:*

1) *If $\lambda \neq -s_q$ or $-t_q$, for all $q \in \mathbb{Z}_+$, then*

$$H_j(\boldsymbol{g}, M(\lambda)^t) = 0, \text{ for all } j \geqslant 0.$$

2) *If $\lambda = -s_{q_0}$ or $-t_{q_0}$, for some $q_0 \in \boldsymbol{N}$, then*

$$H_j(\boldsymbol{g}, M(\lambda)^t) = \begin{cases} \boldsymbol{C}, & \text{for } j = q_0, q_0+1, \\ 0, & \text{otherwise.} \end{cases}$$

3) *If $\lambda = 0$, then*

$$H_j(\boldsymbol{g}, M(\lambda)^t) = \begin{cases} \boldsymbol{C} \oplus \boldsymbol{C}, & \text{for } j = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

Next we consider the homology groups $H_*(\boldsymbol{g}, L(\lambda)^t)$ of the Witt algebra $\boldsymbol{g}$ with coefficients in an irreducible highest weight module $L(\lambda)$ $(\lambda \in \boldsymbol{C})$.

In [6], Kac gave the irreducibility criterion for Verma modules: $M(\lambda)$ is irreducible, i.e., $M(\lambda) \cong L(\lambda)$ iff $\lambda \neq -\frac{1}{24}(m^2-1)$, for all $m \in \boldsymbol{N}$. Then this condition implies that

$$H_*(\boldsymbol{g}, L(\lambda)^t) = H_*(\boldsymbol{g}, M(\lambda)^t).$$

If $\lambda = -\frac{1}{24}(m^2-1)$, $m \in \boldsymbol{N}$, Rocha-Coridi and Wallach proved the following theorem.

Theorem 2.9 ([10, Theorem A]). *Let $k \in \mathbb{Z}_+$. Then these are resolutions:*

1) $$\cdots \xrightarrow{d_{k+i+1}} M(-s_{k+i}) \oplus M(-t_{k+i}) \xrightarrow{d_{k+i}} \cdots \xrightarrow{d_{k+2}}$$

$$M(-s_{k+1}) \oplus M(-t_{k+1}) \xrightarrow{d_{k+1}} M(\nu_k) \xrightarrow{\varepsilon_k} L(\nu_k) \to 0$$

for $\nu_k \in \{-s_k, -t_k\}$,

$$-s_k = -\frac{1}{24}[(6k+1)^2-1] = -\frac{1}{2}(3k^2+k)$$

and

$$-t_k = -\frac{1}{24}[(6k-1)^2-1] = -\frac{1}{2}(3k^2-k).$$

2) $$0 \to M(\nu_{k+2}) \xrightarrow{j_{k+2}} M(\nu_k) \xrightarrow{\varepsilon_k} L(\nu_k) \to 0$$

for $$\nu_k = -\frac{1}{24}[(6k)^2-1], \ k \geqslant 1.$$

3) $$0 \to M(\nu_{k+1}) \xrightarrow{j_{k+1}} M(\nu_k) \xrightarrow{\varepsilon_k} L(\nu_k) \to 0$$

for $$\nu_k=-\frac{1}{24}[(6k+3)^2-1].$$

4) $0\to M(\delta_{k+2})\xrightarrow{j_{k+1}}M(\gamma_k)\xrightarrow{e_k}L(\gamma_k)\to 0$

for $$\gamma_j=-\frac{1}{24}[(6j+2)^2-1],\quad \delta_j=-\frac{1}{24}[(6j+4)^2-1].$$

5) $0\to M(\gamma_{k+1})\xrightarrow{j_{k+1}}M(\delta_k)\xrightarrow{e_k}L(\delta_k)\to 0$

for γ_j,δ_j *as in* 4). *Here* j_k *is the unique (up to scalar) embedding,* e_k *is the canonical projection, and* d_k *is given in* [10, *Theorem A*].

By Theorem 2.8, Theorem 2.9 and the long exact sequence therem, we can easily obtain the following corollay.

Corollary 2.10. *If* $\lambda\notin P=\{-s_k,\ -t_k\,|\,k\in\mathbb{Z}_+\}$, *then*

$$H_j(\mathfrak{g},\ L(\lambda)^t)=0,\ \textit{for}\ j\geqslant 0.$$

§ 3. The Homoloy of the Virasoro Algebras

For the Virasora algebra $\tilde{\mathfrak{g}}$, we let $\tilde{\mathfrak{h}}=CE_0\oplus CE'_0, \tilde{\mathfrak{n}}=\bigoplus_{i\in N}CE_i$ and $\tilde{\mathfrak{b}}=\tilde{\mathfrak{h}}\oplus\tilde{\mathfrak{n}}$. We identify a pair $(\lambda,\ c)\in C^2$ with the linear functional on $\tilde{\mathfrak{n}}$ taking value λ on E_0 and c on E'_0 and denote by $M(\lambda,\ c)$ and $L(\lambda,\ c)$ the Verma module with highest weight $(\lambda,\ c)$ and its unique irreducible quotient, respectively.

Since $[CE'_0,\ \tilde{\mathfrak{g}}]=0$ and $\tilde{\mathfrak{g}}/CE'_0\cong\mathfrak{g}$ (the Witt algebra), by [1. Chap. 16, § 7, (6a)], we have

$$E^2_{i,j}=H_i(\mathfrak{g},\ H_j(CE'_0,\ M(\lambda,\ c)^t))\underset{i}{\Rightarrow}H_n(\tilde{\mathfrak{g}},\ M(\lambda,\ c)^t), \tag{3.1}$$

for $n=i+j$. Note that if $\mathfrak{a}$ is a complex Lie algebra and N^t is a right $\mathfrak{a}$-module, then

$$H_*(\mathfrak{a},\ N^t)=\mathrm{Tor}_*^{U(\mathfrak{a})}(N^t,\ C).$$

Then we can obtain (3.1) from [11, Theorem 11.39]. Hence we can apply the computation of the spectral sequences in [11, Chap. 11].

Now we compute $H_j(CE'_0,\ M(\lambda,\ c)^t)$ directly by the definition of homology of Lie algebra (see [2, § 3]) and note the action of $\mathfrak{g}$ on it. By [2, (3.1)], $H_*(CE',\ M(\lambda,\ c)^t)$ is the homology groups of the complex

$$\cdots\to M(\lambda,\ c)^t\otimes_C\wedge^2(CE'_0)\xrightarrow{\partial_2}M(\lambda,\ c)^t\otimes_C\wedge^1(CE'_0)\xrightarrow{\partial_1}M(\lambda,\ c)^t\otimes_C\wedge^0(CE'_0)\to 0, \tag{3.2}$$

where $\wedge^j(CE'_0)=0$ for $j\geqslant 2$, and $\partial_1(v\otimes E'_0)=v\cdot E'_0$ for all $v\in M(\lambda,\ c)^t$. Then we have

$$H_0(CE'_0,\ M(\lambda,\ c)^t)=M(\lambda,\ c)^t/\mathrm{im}\,\partial_1=\begin{cases}0, \text{ for } & c\neq 0,\\ M(\lambda,\ 0)^t, & \text{for } c=0;\end{cases}$$

$$H_1(\boldsymbol{C}E'_0,\ M(\lambda,\ c)^t)=\ker\partial_1/\operatorname{im}\partial_2=\begin{cases}0, & \text{for } c\neq 0,\\ M(\lambda,\ 0)^t, & \text{for } c=0;\end{cases}$$

$H_j(\boldsymbol{C}E'_0,\ M(\lambda,\ c)^t)=0$, for $j\geqslant 2$ and all $c\in\boldsymbol{C}$.

If $c\neq 0$, since $H_*(\boldsymbol{C}E'_0,\ M(\lambda,\ c)^t)=0$, by (3.1), we have

$$H_*(\tilde{\boldsymbol{g}},\ M(\lambda,\ c)^t)=0,\ \text{for al 1 } \lambda\in\boldsymbol{C} \text{ and } c\neq 0. \tag{3.3}$$

Now we need to consider only the case of $c=0$. Since $[\tilde{\boldsymbol{g}},\ \boldsymbol{C}E'_0]=0$, the trivial adjoint action of $\tilde{\boldsymbol{g}}$ on $\boldsymbol{C}E'_0$ induces the trivial action on $\wedge^j(\boldsymbol{C}E'_0)$ and the tensor product action on $M(\lambda,\ 0)^t\otimes\wedge^j(\boldsymbol{C}E'_0)$. Since the action of $\tilde{\boldsymbol{g}}$ commutes with the maps ∂_j, the action of $\tilde{\boldsymbol{g}}$ on the complex (3.2) induces the action of $\tilde{\boldsymbol{g}}$ on the homology spaces $H_j(\boldsymbol{C}E'_0,\ M(\lambda,\ 0)^t)\ (j\geqslant 0)$. If $j=0,\ 1$, then the action of $\tilde{\boldsymbol{g}}$ on $H_j(\boldsymbol{C}E'_0 M(\lambda,\ 0)^t)=M(\lambda,\ 0)^t$ is the original one. Since $\boldsymbol{C}E'_0$ acts trivially on $M(\lambda,\ 0)^t$, we can induce the action of $\boldsymbol{g}\cong\tilde{\boldsymbol{g}}/\boldsymbol{C}E'_0$ on it. Then $M(\lambda,\ 0)$ can be regarded as a $\boldsymbol{g}$-module and is isomorphic to the Verma module $M(\lambda)$ of the Witt algebra $\boldsymbol{g}$.

If $\lambda=0$, then by Theorem 2.8, 3) and (3.1), we have

$$\begin{aligned}E_{i,j}&=H_i(\boldsymbol{g},\ H_j(\boldsymbol{C}E'_0,\ M(\lambda,\ 0)^t))\\ &=\begin{cases}\boldsymbol{C}\oplus\boldsymbol{C}, \text{ if } (i,\ j)=(0,\ 0),\ (0,\ 1),\ (1,\ 0),\ (1,\ 1),\\ 0, \text{ otherwise.}\end{cases}\end{aligned} \tag{3.4}$$

By the definition of the spectral sequence (see [11, Theorem 11.10 and p. 313, Definition]), we have

$$E^{r+1}_{i,j}=\ker d^r_{i,j}/\operatorname{im} d^r_{i+r,j-r+1},\quad \text{for } r\geqslant 1,$$

where $d^r_{i,j}$ is a map from $E^r_{i,j}$ into $E^r_{i-r,j+r-1}$ whose bidegree is $(-r,\ r-1)$. Since $E^3_{i,j}$ is a subquotient of $E^2_{i,j}$ by (3.4), we have

$$E^3_{i,j}=E^2_{i,j}=0,\ \text{for } (i,\ j)\neq(0,\ 0),\ (0,\ 1),\ (1,\ 0),\ (1,\ 1).$$

Similarly, $E^r_{i,j}=0$, for $r\geqslant 2$. Hence

$$E^\infty_{i,j}=0,\ \text{for } (i,\ j)\neq(0,\ 0),\ (0,\ 1),\ (1,\ 0),\ (1,\ 1).$$

Now we consider $E^3_{0,n}$, $n\geqslant 0$. Since $E^2_{2,n-1}=0$, $\operatorname{im} d^2_{2,n-1}=0$. Since $E^2_{-2,n+1}=0$, the map $d^2_{0,n}$ is 0. Hence $\ker d^2_{0,n}=E^2_{0,n}$. Therefore we have $E^3_{0,n}\cong E^2_{0,n}$. Similarly, $E^{r+1}_{0,n}\cong E^r_{0,n}$, $r\geqslant 2$. Thus $E^\infty_{0,n}\cong E^2_{0,n}$, for $n\geqslant 0$. By the same manner we have $E^\infty_{1,j}\cong E^2_{1,j}$, for $j=0,\ 1$. Then if $\lambda=0$, then

$$E^\infty_{i,j}\cong E^2_{i,j}.\quad \text{for } i,\ j\geqslant 0. \tag{3.5}$$

For convenience, we denote $H_n(\tilde{\boldsymbol{g}},\ M(\lambda,\ 0)^t)$ by H_n, for $n\geqslant 0$. By the definition of convergence of the spectral sequence (see [11, p317, Definition]), there exists a bounded filtration $\{\Phi^i H\}$ such that

$$E^\infty_{i,j}\cong\Phi^i H_n/\Phi^{i-1}H_n, \tag{3.6}$$

for all $i,\ j\ (n=i+j)$.

Since $E^2_{i,j}$ is the spectral sequence in the first quadrant, the filtration $\{\Phi^i H\}$ defined by [11, p.324, Definition] is bounded, and for each n, we have

$$0=\Phi^{-1}H_n\subset\Phi^0H_n\subset\cdots\subset\Phi_nH_n=H_n. \tag{3.7}$$

For $n=0$, by (3.6) and (3.7), we have

$$E^\infty_{0,0}\cong\Phi^0H_0/\Phi^{-1}H_0\cong H_0.$$

For $n=1$, by (3.6) and (3.7), we have

$$E^\infty_{1,0}\cong H_1/\Phi^0H_1,$$
$$E^\infty_{0,1}\cong\Phi^0H_1/\Phi^{-1}H_1=\Phi^0H_1.$$

Hence $E^\infty_{1,0}\cong H_1/E^\infty_{0,1}$. Then we have

$$\dim H_1=\dim E^\infty_{1,0}+\dim E^\infty_{0,1}.$$

For $n=2$, $E^\infty_{2,0}\cong H_2/\Phi^1H_2$, $E^\infty_{1,1}\cong\Phi^1H_2/\Phi^0H_2$, $E^\infty_{0,2}\cong\Phi^0H_2$. Hence we have

$$H_2\cong E^\infty_{1,1}.$$

It is obvious that for $n\geqslant3$, we have $H_n=0$. Now we have proved that for $\lambda=0$,

$$\dim H_n(\tilde{\mathfrak{g}},M(0,0)^t)=\begin{cases}2, & \text{if } n=0,\\ 4, & \text{if } n=1,\\ 2, & \text{if } n=2,\\ 0, & \text{if } n\geqslant3.\end{cases}$$

For $\lambda=-s_{q_0}$ or $-t_{q_0}$, for some $q_0\in N$ and $\lambda\notin P$, the computation is similar to that for $\lambda=0$. Combining these results we obtain the following theorem.

Theorem 3.8. *Let $\tilde{\mathfrak{g}}$ be the Virasoro algebra, $\tilde{\mathfrak{h}}$ the Cartan subalgebra of $\tilde{\mathfrak{g}}$, and $M(\lambda, c)$ the Verma module with highest weight $(\lambda, c)\in\tilde{\mathfrak{h}}^*$. Then the homology of $\tilde{\mathfrak{g}}$ with coefficients in $M(\lambda, c)$ is given as follows:*

1) *If $\lambda=c=0$, then*

$$\dim H_n(\tilde{\mathfrak{g}},M(0,0)^t)=\begin{cases}2, & \text{if } n=0,\\ 4, & \text{if } n=1,\\ 2, & \text{if } n=2,\\ 0, & \text{if } n\geqslant3;\end{cases}$$

2) *If $\lambda=-s_{q_0}$ or $-t_{q_0}$, for some $q_0\in N$, and $c=0$, then*

$$\dim H_n(\tilde{\mathfrak{g}},M(\lambda,0)^t)=\begin{cases}0, & \text{if } 0\leqslant n<q,\\ 1, & \text{if } n=q_0,\\ 2, & \text{if } n=q_0+1,\\ 1, & \text{if } n=q_0+2,\\ 0, & \text{if } n\geqslant q_0+3;\end{cases}$$

3) *If (λ, c) $(\in\tilde{\mathfrak{h}}^*)$ is not as in case 1) or 2), then*

$$H_*(\tilde{\mathfrak{g}},M(\lambda,c)^t)=0.$$

Remark. 1) By Corollary 2.10, we have

$$H_*(\tilde{\mathfrak{g}},L(\lambda,0)^t)=0,\quad \text{for } \lambda\notin P.$$

2) By [10, Theorem (0.15) and Corollary (0.17)], we have

i) $M(\lambda,1)\cong L(\lambda,1)$, if $\lambda\neq-m^2/4$ for all $m\in\mathbb{Z}_+$;

ii) there is a resolution

$$0 \to M(-1/4(m+2)^2, 1) \to M(-m^2/4, 1) \to L(-m^2/4, 1) \to 0,$$

if $\lambda = -m^2/4$ for some $m \in \mathbb{Z}_+$.

Hence we have

$$H_*(\tilde{\mathfrak{g}}, L(\lambda, 1)^t) = 0, \quad \text{for all } \lambda \in \boldsymbol{C}.$$

3) By [10, Theorem B(i)] we have

$$H_*(\tilde{\mathfrak{g}}, L(\lambda, 25)^t) = 0, \quad \text{for all } \lambda \in \boldsymbol{C}.$$

4) By [10, Theorem B(ii)] we have

$$H_*(\tilde{\mathfrak{g}}, L(\lambda, 26)^t) = 0, \text{ if } \lambda \neq s_k - 1 \text{ or } t_k - 1 \quad \text{for all } k \geqslant 1.$$

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