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REPRESENTATION OF LINEAR FUNCTIONALS ON LIPSCHITZ SPACES Λ_φ OF FUNCTIONS HOLOMORPHIC IN THE UNIT BALL OF $\mathbb{C}^n$

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Abstract

In this paper three spaces $\Lambda_\varphi(B)$, $\lambda_\varphi(B)$ and $\Gamma_\varphi(B)$ of functions holomorphic in the unit ball of $\mathbb{C}^n$ are defined and the representations of linear functionals on $\lambda_\varphi(B)$ and $\Gamma_\varphi(B)$ are obtained.

§ 1. Introduction

Let B denote the unit ball of $\mathbb{C}^n$, S denote its boundary, σ denote the positive rotation-invariant measure on S with $\sigma(S)=1$ and $A(B)$ denote the class of functions holomorphic in B and continuous on $\bar{B}$.

Let Φ denote the class of functions φ: $[0, 1]\to[0, \infty)$ satisfying the following conditions:

(i) φ is continuous, increasing and $\varphi(0)=0$, $\varphi(t)\neq 0$ if $t\neq 0$,

(ii) $t/\varphi(t)$ is increasing and $t/\varphi(t)\to 0$ as $t\to 0$,

(iii) $\int_0^\delta \varphi(t)/t\,dt=O(\varphi(\delta))$ and $\int_\delta^1 \varphi(t)/t^2\,dt=O(\varphi(\delta)/\delta)$ for $\delta>0$.

Definition 1. *Let $\varphi\in\Phi$. A function $f\in A(B)$ is said to be of class $\Lambda_\varphi(B)$ if its boundary function satisfies the Lipschitz condition*

$$|f(\exp(i(\theta+h))\zeta)-f(\exp(i\theta)\zeta)|\leqslant K\varphi(|h|)$$

for $\zeta\in S$ and θ, $h\in\mathbb{R}$, where the constant K is independent of θ, h and ζ.

In the case $\varphi(t)=t^\alpha$, $0<\alpha\leqslant 1$, we get the usual Lipschitz spaces.

Recently G.D. Lyevshina[3] studied the representation of linear functionals on Lipschitz spaces Λ_φ in the unit disc. In the present paper we will study the same problem in the unit ball of $\mathbb{C}^n$.

§ 2. The Spaces $\Lambda_\varphi(B)$ and $\lambda_\varphi(B)$

If f is holomorphic in B with expansion in terms of homogeneous polynomials

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given by $f(z)=\sum_{k=0}^{\infty}F_k(z)$, as usual, for $\alpha>0$, we define the fractional derivative $f^{[\alpha]}$ and the fractional integral $f_{[\alpha]}$ of order α of f by the following formulas respectively

$$f^{[\alpha]}(z)=\sum_{k=0}^{\infty}\frac{\Gamma(k+\alpha+1)}{k!}F_k(z),$$

$$f_{[\alpha]}(z)=\sum_{k=0}^{\infty}\frac{k!}{\Gamma(k+\alpha+1)}F_k(z),$$

$f^{[\alpha]}$, $f_{[\alpha]}$ are holomorphic in B[4, p.389].

Theorem 1. *Suppose f is holomorphic in B and $\varphi\in\Phi$. Then $f\in\Lambda_\varphi(B)$ if and only if*

$$|f^{[1]}(r\zeta)|\leqslant K\frac{\varphi(1-r)}{1-r} \tag{1}$$

for $\zeta\in S$ and $r\in[0,1)$, where the constant K is independent of ζ and r.

Proof In the case $n=1$, the proof of theorem may be found in [5, Lemma 2 and Lemma 3]. We now assume $n>1$ and let $f\in\Lambda_\varphi(B)$. Since $f\in A(B)$, if let $f_\zeta(\lambda)=f(\lambda\zeta)$, $\zeta\in S$, $\lambda\in\mathbb{C}$, $|\lambda|<1$, then $f_\zeta\in A(U)$, where U is the unit disc and $A(U)$ the disc algebra of U. Application of the necessity of the case $n=1$ gives

$$|f_\zeta'(\lambda)|\leqslant K\frac{\varphi(1-|\lambda|)}{1-|\lambda|},$$

where the constant K is independent of ζ. Thus (1) follows from the equality

$$f^{[1]}(r\zeta)=rf_\zeta'(r)+f(r\zeta). \tag{2}$$

Conversely, if f satisfies (1), since

$$|r'f_\zeta(r')-r''f_\zeta(r'')|=\left|\int_{r'}^{r''}[tf_\zeta(t)]'\,dt\right|=\left|\int_{r'}^{r''}f^{[1]}(t\zeta)dt\right|$$

$$\leqslant K\int_{r'}^{r''}\frac{\varphi(1-t)}{1-t}dt\leqslant K[\varphi(1-r')+\varphi(1-r'')]\to 0$$

as $r'\to1$, $r''\to1$ by the properties (i) and (iii) of φ, we see that $f(r\zeta)$ converges, uniformly on S as $r\to1$, to a limit that we call $f(\zeta)$; this extends f to S. The continuity of f in B and $f(\zeta)$ on S shows that f is continuous on $\bar{B}$, in other words, $f\in A(B)$. Combining the equality (2) and inequality (1), we have

$$|f_\zeta'(\lambda)|\leqslant K\frac{\varphi(1-|\lambda|)}{1-|\lambda|}.$$

By the sufficiency of the case $n=1$, f_ζ is of class $\Lambda_\varphi(U)$. Thus

$$|f(\exp(i(\theta+h))\zeta)-f(\exp(i\theta)\zeta)|\leqslant K\varphi(|h|)$$

for $\zeta\in S$ and θ, $h\in\mathbb{R}$. This completes the proof.

On the basis of Theorem 1, we have

Definition 2. *Let $\varphi\in\Phi$ and $f\in\Lambda_\varphi(B)$, define the norm of f as*

$$\|f\|_{\Lambda_\varphi}=\sup_{z\in B}\frac{1-|z|}{\varphi(1-|z|)}|f^{[1]}(z)|. \tag{3}$$

Definition 3. *Let $\varphi\in\Phi$ and $f\in\Lambda_\varphi(B)$, we say that f is to be of class $\lambda_\varphi(B)$ if*

$$\frac{1-|z|}{\varphi(1-|z|)}|f^{[1]}(z)|\to 0, \text{ as } |z|\to 1.$$

Theorem 2. (i) *$\Lambda_\varphi(B)$ is a Banach space with the norm* (3).

(ii) *$\lambda_\varphi(B)$ is a closed subspace of $\Lambda_\varphi(B)$.*

(iii) *If $f\in\Lambda_\varphi(B)$, then $\|f_r\|_{\Lambda_\varphi}\leqslant\|f\|_{\Lambda_\varphi}$ for $r\in(0, 1)$,* where

$$f_r(z)=f(rz).$$

(iv) *$f\in\lambda_\varphi(B)$ if and only if $f\in\Lambda_\varphi(B)$ and $\|f_r-f\|_{\Lambda_\varphi}\to 0$ as $r\to 1$.*

Proof (i) Let $\{f_k\}$ be a Cauchy sequence in $\Lambda_\varphi(B)$. By the definition of the norm of $\Lambda_\varphi(B)$,

$$|f_k^{[1]}(z)-f_l^{[1]}(z)|\leqslant\frac{\varphi(1-|z|)}{1-|z|}\|f_k-f_l\|_{\Lambda_\varphi}\leqslant\frac{\varphi(1)}{1-\rho}\|f_k-f_l\|_{\Lambda_\varphi}$$

for $z\in\rho B$, where ρB is the ball of radius ρ centered at the origin. Therefore $\{f_k^{[1]}\}$ converges uniformly on any compact subset of B, so it converges to a function g holomorphic in B. Let $f=g_{[1]}$, then $f^{[1]}=g$. We now prove that $f\in\Lambda_\varphi(B)$. Since $\{f_k\}$ is a Cauchy sequence in $\Lambda_\varphi(B)$, there is a constant M with $\|f_k\|_{\Lambda_\varphi}\leqslant M$, $k=1, 2, \cdots$. For given $\rho\in(0,1)$, there exists a positive integer k_0 such that

$$|f_k^{[1]}(z)-f^{[1]}(z)|<\varphi(1-\rho)$$

for $z\in\rho B$ and $k>k_0$, and

$$\frac{1-|z|}{\varphi(1-|z|)}|f^{[1]}(z)|\leqslant\frac{1-|z|}{\varphi(1-|z|)}|f^{[1]}(z)-f_k^{[1]}(z)|+\|f_k\|_{\Lambda_\varphi}\leqslant 1+M \tag{4}$$

for $z\in\rho B$. Letting $\rho\to 1$ in (4) gives $f\in\Lambda_\varphi(B)$. To prove $\{f_k\}$ converges to f in $\Lambda_\varphi(B)$, we note that

$$\frac{1-|z|}{\varphi(1-|z|)}|f_k^{[1]}(z)-f_l^{[1]}(z)|\leqslant\|f_k-f_l\|_{\Lambda_\varphi}<\varepsilon$$

for $z\in B$ and k, l sufficiently large. Fixing k and letting $l\to\infty$ yield

$$\|f_k-f\|_{\Lambda_\varphi}\leqslant\varepsilon$$

for k sufficiently large. This completes the proof.

(ii) Let $\{f_k\}\subset\lambda_\varphi(B)$ and $\{f_k\}$ converges to f in $\Lambda_\varphi(B)$. For given $\varepsilon>0$, there exists a positive integer k_0 such that $\|f_k-f\|_{\Lambda_\varphi}<\varepsilon$ for $k>k_0$. Fix $k>k_0$, there is an $r_0\in(0, 1)$ such that

$$\frac{1-|z|}{\varphi(1-|z|)}|f_k^{[1]}(z)|<\varepsilon$$

for $r_0<|z|<1$ since $f_k\in\lambda_\varphi(B)$. Thus

$$\frac{1-|z|}{\varphi(1-|z|)}|f^{[1]}(z)|\leqslant\|f_k-f\|_{\Lambda_\varphi}+\frac{1-|z|}{\varphi(1-|z|)}|f_k^{[1]}(z)|<2\varepsilon$$

for $r_0<|z|<1$. Hence $f\in\lambda_\varphi(B)$.

(iii) It follows from the property (ii) of φ.

(iv) Let $f\in\lambda_\varphi(B)$. Since

$$\sup_{|z|=\rho}\frac{1-|z}{\varphi(1-|z|)}|f_r^{[1]}(z)-f^{[1]}(z)|\leqslant 2\sup_{|z|=\rho}\frac{1-|z|}{\varphi(1-|z|)}|f^{[1]}(z)|\to 0$$

as $\rho\to 1$, there is a $\rho_0\in(0, 1)$ such that

$$\sup_{|z|=\rho}\frac{1-|z|}{\varphi(1-|z|)}|f_r^{[1]}(z)-f^{[1]}(z)|<\varepsilon \tag{5}$$

for given $\varepsilon>0$, $\rho\in(\rho_0, 1)$ and $r\in(0, 1)$. On the other hand, there exists an $r_0\in(0, 1)$ such that

$$|f^{[1]}(rz)-f^{[1]}(z)|<\varepsilon\varphi(1-\rho_0)$$

for $z\in\rho_0\bar{B}$ and $r\in(r_0, 1)$. Thus

$$\frac{1-|z|}{\varphi(1-|z|)}|f^{[1]}(rz)-f^{[1]}(z)|<\varepsilon \tag{6}$$

for $z\in\rho_0\bar{B}$ and $r\in(r_0, 1)$. Combining (5) and (6) shows that (6) is true in B. This gives $\|f_r-f\|_{\Lambda_\varphi}\to 0$ as $r\to 1$.

The converse follows from the facts that $t/\varphi(t)\to 0$ as $t\to 0$ and $\lambda_\varphi(B)$ is a closed subspace of $\Lambda_\varphi(B)$.

§ 3. The Space $\Gamma_\varphi(B)$

Definition 4. *Let $\varphi\in\Phi$. A function g holomorphic in B is said to be of class $\Gamma_\varphi(B)$, if*

$$\int_0^1\int_S\varphi(1-r)|g^{[1]}(r\zeta)|d\sigma(\zeta)dr<\infty.$$

The norm of $g\in\Gamma_\varphi(B)$ is defined as

$$\|g\|_{\Gamma_\varphi}=\int_0^1\int_S\varphi(1-r)|g^{[1]}(r\zeta)|d\sigma(\zeta)dr. \tag{7}$$

Theorem 3. (i) *$\Gamma_\varphi(B)$ is a Banach space with the norm* (7),

(ii) *$\|g_r-g\|_{\Gamma_\varphi}\to 0$ as $r\to 1$ for every $g\in\Gamma_\varphi(B)$,*

(iii) *If there is a constant M with $\|g_r\|_{\Gamma_\varphi}\leqslant M$ for every $r\in(0, 1)$, then $g\in\Gamma_\varphi(B)$* and $\|g\|_{\Gamma_\varphi}\leqslant M$.

To prove this theorem we need the following Lemma.

Lemma 1. *If $g\in\Gamma_\varphi(B)$, then for $r\in(0, 1)$*

$$\sup_{|z|\leqslant r}|g(z)|\leqslant 2^{n+2}[r\varphi(1-r)(1-r)^n]^{-1}\|g\|_{\Gamma_\varphi}.$$

Proof For $t\in(0, 1)$, $g^{[1]}_{(1+t)/2}$ is holomorphic in $\bar{B}$, the Cauchy integral formula gives

$$|g^{[1]}(t\zeta)|\leqslant\int_S\left|g^{[1]}\left(\frac{1+t}{2}\eta\right)\right|\left|1-\frac{2t}{1+t}\langle\zeta, \eta\rangle\right|^{-n}d\sigma(\eta)$$
$$\leqslant\frac{2^n}{(1-t)^n}\int_S\left|g^{[1]}\left(\frac{1+t}{2}\eta\right)\right|d\sigma(\eta),$$

where $\zeta\in S$. Thus

$$
\begin{aligned}
|g(r\zeta)| &\leqslant \frac{1}{r}\int_0^r |g^{[1]}(t\zeta)|\,dt \\
&\leqslant \frac{2^n}{r\varphi(1-r)(1-r)^{n-1}}\int_0^r \frac{\varphi(1-t)}{1-t}\int_S \left|g^{[1]}\left(\frac{1+t}{2}\eta\right)\right| d\sigma(\eta)\,dt \\
&\leqslant \frac{2^{n+2}}{r\varphi(1-r)(1-r)^n}\int_0^1 \varphi(1-\rho)\int_S |g^{[1]}(\rho\eta)|\,d\sigma(\eta)\,d\rho \\
&= 2^{n+2}[r(1-r)^n\varphi(1-r)]^{-1}\|g\|_{\Gamma_\varphi}
\end{aligned}
$$

as claimed.

Proof of Theorem 3. (i) Let $\{g_k\}$ be a Cauchy sequence in $\Gamma_\varphi(B)$, it converges uniformly on any compact subset of B by Lemma 1, so it converges to a function g holomorphic in B, and

$$\int_S |g^{[1]}(r\zeta)|\,d\sigma(\zeta) = \lim_{k\to\infty}\int_S |g_k^{[1]}(r\zeta)|\,d\sigma(\zeta)$$

for any $r\in(0, 1)$. By Fatou's theorem

$$
\begin{aligned}
\int_0^1 \varphi(1-r)\int_S |g^{[1]}(r\zeta)|\,d\sigma(\zeta)\,dr &\leqslant \varliminf_{k\to\infty}\int_0^1 \varphi(1-r)\int_S |g_k^{[1]}(r\zeta)|\,d\sigma(\zeta)\,dr \\
&= \varliminf_{k\to\infty}\|g_k\|_{\Gamma_\varphi} < \infty.
\end{aligned}
$$

Therefore $g\in\Gamma_\varphi(B)$. Using the same method we have

$$\|g_k - g\|_{\Gamma_\varphi} \leqslant \varliminf_{l\to\infty}\|g_k - g_l\|_{\Gamma_\varphi} \to 0, \text{ as } k\to\infty.$$

This shows that $\Gamma_\varphi(B)$ is a Banach space under the norm (7).

(ii) Let $g\in\Gamma_\varphi(B)$. Since

$$\varphi(1-\rho)\int_S |g_r^{[1]}(\rho\zeta) - g^{[1]}(\rho\zeta)|\,d\sigma(\zeta) \leqslant 2\varphi(1-\rho)\int_S |g^{[1]}(\rho\zeta)|\,d\sigma(\zeta)$$

and

$$2\int_0^1 \varphi(1-\rho)\int_S |g^{[1]}(\rho\zeta)|\,d\sigma(\zeta)\,d\rho = 2\|g\|_{\Gamma_\varphi} < \infty,$$

we have

$$\lim_{r\to 1}\|g_r - g\|_{\Gamma_\varphi} = \lim_{r\to 1}\int_0^1 \varphi(1-\rho)\int_S |g_r^{[1]}(\rho\zeta) - g^{[1]}(\rho\zeta)|\,d\sigma(\zeta)\,d\rho = 0$$

by the Lebesgue dominant convergence theorem.

(iii) Choose $r_k\nearrow 1$ as $k\to\infty$, then $\int_S |g_{r_k}^{[1]}(\rho\zeta)|\,d\sigma(\zeta)$ is an increasing sequence for every $\rho\in(0, 1)$ by the monotonicity of the mean. Application of the Levi's lemma gives

$$\int_0^1 \varphi(1-\rho)\int_S |g^{[1]}(\rho\zeta)|\,d\sigma(\zeta)\,d\rho = \lim_{k\to\infty}\int_0^1 \varphi(1-\rho)\int_S |g_{r_k}^{[1]}(\rho\zeta)|\,d\sigma(\zeta)\,d\rho \leqslant M.$$

The proof is complete.

§ 4. The Bounded Linear Functionals on the Spaces $\lambda_\varphi(B)$ and $\Gamma_\varphi(B)$

Let f, g be holomorphic in B and

$$f(z)=\sum_{k=0}^{\infty}\sum_{|\alpha|=k}a_\alpha z^\alpha,\quad g(z)=\sum_{k=0}^{\infty}\sum_{|\alpha|=k}b_\alpha z^\alpha.$$

A calculation based on the formulas of [1, p.16] yields

$$\int_S f(\rho\zeta)\overline{g(r\rho^{-1}\zeta)}d\sigma(\zeta)=\sum_{k=0}^{\infty}\Big(\sum_{|\alpha|=k}a_\alpha\bar b_\alpha\omega_\alpha\Big)r^k, \tag{8}$$

where $0<r<\rho<1$ and

$$\omega_\alpha=\int_S|\zeta^\alpha|^2d\sigma(\zeta)=\frac{(n-1)!\alpha!}{(n-1+|\alpha|)!}.$$

We denote the limit of (8) as $r\to1$, if it exists, by

$$(f,g)=\lim_{r\to1}\sum_{k=0}^{\infty}\Big(\sum_{|\alpha|=k}a_\alpha\bar b_\alpha\omega_\alpha\Big)r^k=\lim_{r\to1}\int_S f(\rho\zeta)\overline{g(r\rho^{-1}\zeta)}d\sigma(\zeta). \tag{9}$$

Lemma 2. *If $f\in\Lambda_\varphi(B)$ and $g\in\Gamma_\varphi(B)$, then*

(i) *The limit of (8) exists and*

$$|(f,g)|\le C\|f\|_{\Lambda_\varphi}\|g\|_{\Gamma_\varphi}, \tag{10}$$

where the constant C is independent of f and g.

(ii) $$\lim_{r\to1}(f_r,g_\rho)=(f,g_\rho)$$

for any $\rho\in(0,1)$.

Proof (i) By the definition of the fractional derivative

$$\int_S f^{[1]}(\rho\zeta)\overline{g^{[1]}(r\rho\zeta)}d\sigma(\zeta)=\sum_{k=0}^{\infty}(k+1)^2r^k\rho^{2k}\sum_{|\alpha|=k}a_\alpha\bar b_\alpha\omega_\alpha. \tag{11}$$

Multiply both sides of (11) by $\rho\log\frac{1}{\rho}$, then integrate with respect to ρ on the interval $(0,1)$ and use the equality

$$\int_0^1\rho^{2k+1}\log\frac{1}{\rho}d\rho=\frac{1}{4(k+1)^2},$$

we obtain

$$\sum_{k=0}^{\infty}\Big(\sum_{|\alpha|=k}a_\alpha\bar b_\alpha\omega_\alpha\Big)r^k=4\int_0^1\rho\log\frac{1}{\rho}\int_S f^{[1]}(\rho\zeta)\overline{g^{[1]}(r\rho\zeta)}d\sigma(\zeta)d\rho. \tag{12}$$

Applying the inequality $\rho\log\frac{1}{\rho}\le1-\rho$, $(0<\rho\le1)$, to (12) gives

$$\Big|\sum_{k=0}^{\infty}\Big(\sum_{|\alpha|=k}a_\alpha\bar b_\alpha\omega_\alpha\Big)r^k\Big|\le4\|f\|_{\Lambda_\varphi}\int_0^1\int_S\varphi(1-r\rho)|g^{[1]}(r\rho\zeta)|d\sigma(\zeta)d\rho$$
$$\le\frac{4}{r}\|f\|_{\Lambda_\varphi}\|g\|_{\Gamma_\varphi}. \tag{13}$$

Letting $r\to1$ in (13) gives (10) if the limit of (8) exists.

We now prove that the limit of (8) exists. By the equalit y (12) and Theorem 3 (ii)

$$\Big|\sum_{k=0}^{\infty}\Big(\sum_{|\alpha|=k}a_\alpha\bar b_\alpha\omega_\alpha\Big)r_1^k-\sum_{k=0}^{\infty}\Big(\sum_{|\alpha|=k}a_\alpha\bar b_\alpha\omega_\alpha\Big)r_2^k\Big|$$
$$\le4\int_0^1\rho\log\frac{1}{\rho}\int_S|f^{[1]}(\rho\zeta)||g^{[1]}(r_1\rho\zeta)-g^{[1]}(r_2\rho\zeta)|d\sigma(\zeta)d\rho$$
$$\le4\|f\|_{\Lambda_\varphi}\{\|g_{r_1}-g\|_{\Gamma_\varphi}+\|g_{r_2}-g\|_{\Gamma_\varphi}\}\to0$$

as $r_1\to 1$, $r_2\to 1$. This completes the proof.

(ii) It follows from the Lebesgue bounded convergence theorem.

The following two theorems are the main result of the present paper.

Theorem 4. (i) *For every $T\in\Gamma_\varphi^*(B)$, there exists a unique $f\in\Lambda_\varphi(B)$ such that*

$$T(g)=(g,f)$$

for every $g\in\Gamma_\varphi(B)$, and

$$C'\|f\|_{\Lambda_\varphi}\leqslant\|T\|_{\Gamma_\varphi^*}\leqslant C\|f\|_{\Lambda_\varphi}\tag{14}$$

where the constants C, C' are independent of f and T.

(ii) *Conversely, for every $f\in\Lambda_\varphi(B)$,*

$$T_f(g)=(g,f)$$

defines a bounded linear functional on $\Gamma_\varphi(B)$ and

$$\|T_f\|_{\Gamma_\varphi^*}\leqslant C\|f\|_{\Lambda_\varphi},$$

where the constant C is independent of f.

Proof (i) Suppose $T\in\Gamma_\varphi^*(B)$, and define

$$f(z)=\sum_{k=0}^{\infty}\sum_{|\alpha|=k}\overline{T(z^\alpha/\omega_\alpha)}z^\alpha.\tag{15}$$

A simple estimate gives

$$\|z^\alpha\|_{\Gamma_\varphi}=\int_0^1\varphi(1-r)\int_S(|\alpha|+1)r^{|\alpha|}|\zeta^\alpha|d\sigma(\zeta)dr\leqslant\varphi(1)\sqrt{\omega_\alpha}.\tag{16}$$

Using (16) and Schwarz inequality, we obtain

$$\begin{aligned}\Big|\sum_{|\alpha|=k}\overline{T(z^\alpha/\omega_\alpha)z^\alpha}\Big|&\leqslant\|T\|_{\Gamma_\varphi^*}\sum_{|\alpha|=k}\frac{1}{\omega_\alpha}\|z^\alpha\|_{\Gamma_\varphi}|z^\alpha|\\&\leqslant\varphi(1)\ \|T\|_{\Gamma_\varphi^*}\Big(\frac{(n+k-1)!}{k!(n-1)!}\Big)^{\frac12}\Big(\sum_{|\alpha|=k}\frac{1}{\omega_\alpha}|z^\alpha|^2\Big)^{\frac12}\\&\leqslant\varphi(1)\ \|T\|_{\Gamma_\varphi^*}\frac{(n+k-1)!}{k!(n-1)!}|z|^k.\end{aligned}$$

This shows that the series (15) converges uniformly on any compact subset of B and so f is holomorphic in B. We now prove that $f\in\Lambda_\varphi(B)$. Fix $\eta\in S$ and let

$$h(z)=\sum_{k=0}^{\infty}(k+1)\sum_{|\alpha|=k}(\overline{\eta^\alpha}/\omega_\alpha)z^\alpha.$$

By the continuity of T,

$$T(h_r)=\sum_{k=0}^{\infty}(k+1)\Big(\sum_{|\alpha|=k}T(z^\alpha/\omega_\alpha)\,\overline{\eta^\alpha}\Big)r^k=\overline{f^{[1]}(r\eta)},\quad 0<r<1$$

and

$$|f^{[1]}(r\eta)|\leqslant\|T\|_{\Gamma_\varphi^*}\|h_r\|_{\Gamma_\varphi}.\tag{17}$$

Since $\{z^\alpha/\sqrt{\omega_\alpha}\}$ is a complete orthogonal system in B and orthonormal on S, we see that $\sum\limits_{k=0}^{\infty}\sum\limits_{|\alpha|=k}(z^\alpha/\sqrt{\omega_\alpha})(\overline{\eta^\alpha}/\sqrt{\omega_\alpha})$ is the Cauchy-Szego kernel of B, i.e.

$$\sum_{k=0}^{\infty}\sum_{|\alpha|=k}(z^\alpha\overline{\eta^\alpha})/\omega_\alpha=(1-\langle z,\eta\rangle)^{-n}=C(z,\eta).$$

By the formula of [1, p. 18]

$$\int_S |C^{[2]}(r\rho\zeta, \eta)|\,d\sigma(\zeta) \leqslant A\int_S |1-\langle r\rho\zeta, \eta\rangle|^{-(n+2)}\,d\sigma(\zeta) \leqslant A(1-r\rho)^{-2},$$

where A is a constant. Thus

$$\|h_r\|_{\Gamma_\varphi} = \int_0^1 \varphi(1-\rho)\int_S |C^{[2]}(r\rho\zeta, \eta)|\,d\sigma(\zeta)\,d\rho$$

$$\leqslant A\int_0^1 \frac{\varphi(1-\rho)}{(1-r\rho)^2}\,d\rho \leqslant \frac{A}{r}\,\frac{\varphi(1-r)}{1-r} \tag{18}$$

by the property (iii) of φ. Combining (17) and (18) shows that $f\in\Lambda_\varphi(B)$ and

$$\|T\|_{\Gamma_\varphi^*} \geqslant C'\|f\|_{\Lambda_\varphi}. \tag{19}$$

Now let $g\in\Gamma_\varphi(B)$ and $g(z)=\sum_{k=0}^{\infty}\sum_{|\alpha|=k} b_\alpha z^\alpha$,

$$T(g_r) = \sum_{k=0}^{\infty}\Big(\sum_{|\alpha|=k} b_\alpha T(z^\alpha/\omega_\alpha)\omega_\alpha\Big)r^k. \tag{20}$$

Because of the continuity of T and $\|g_r-g\|_{\Gamma_\varphi}\to 0$ as $r\to 1$, letting $r\to 1$ in (20) implies

$$T(g) = \lim_{r\to 1}\sum_{k=0}^{\infty}\Big(\sum_{|\alpha|=k} b_\alpha T(z^\alpha/\omega_\alpha)\omega_\alpha\Big)r^k = (g, f).$$

On the other hand, by Lemma 2

$$|T(g)| \leqslant C\|g\|_{\Gamma_\varphi}\|f\|_{\Lambda_\varphi},$$

hence

$$\|T\|_{\Gamma_\varphi^*} \leqslant C\|f\|_{\Lambda_\varphi}. \tag{21}$$

Combining (19) and (21) gives (14).

The proof of the uniqueness of f is easy. If there exists another $\tilde{f}\in\Lambda_\varphi(B)$ with $T(g)=(g,\tilde{f})$ for every $g\in\Gamma_\varphi(B)$, then

$$(g, f-\tilde{f}) = 0 \tag{22}$$

for every $g\in\Gamma_\varphi(B)$. Taking $g=z^\alpha$ in (22) gives $f-\tilde{f}=0$.

(ii) Conversely, by Lemma 2, (g, f) exists for $g\in\Gamma_\varphi(B)$ and $f\in\Lambda_\varphi(B)$ and

$$|(g, f)| \leqslant C\|g\|_{\Gamma_\varphi}\|f\|_{\Lambda_\varphi}.$$

This shows that $T_f(g)=(g, f)$ is a bounded linear functional on $\Gamma_\varphi(B)$, and

$$\|T_f\|_{\Gamma_\varphi^*} \leqslant C\|f\|_{\Lambda_\varphi}.$$

This completes the proof.

Theorem 5. (i) *For every $F\in\lambda_\varphi^*(B)$, there exists a unique $g\in\Gamma_\varphi(B)$, such that*

$$F(f) = (f, g)$$

for every $f\in\lambda_\varphi(B)$, and

$$C'\|g\|_{\Gamma_\varphi} \leqslant \|F\|_{\lambda_\varphi^*} \leqslant C\|g\|_{\Gamma_\varphi}. \tag{23}$$

(ii) *Conversely, for every $g\in\Gamma_\varphi(B)$,*

$$F_g(f) = (f, g)$$

defines a bounded linear functional on λ_φ(B) and

$$\|F_g\|_{\lambda_\varphi^*} \leqslant C\|g\|_{\Gamma_\varphi}.$$

Proof (i) Let $F\in\lambda_\varphi^*$(B) and define

$$g(z)=\sum_{k=0}^{\infty}\sum_{|\alpha|=k}\overline{F(z^\alpha/\omega_\alpha)}\,z^\alpha.$$

We first prove that g is holomorphic in B. For given $\zeta\in S$, a not hard computation gives

$$\sum_{k=0}^{\infty}\Big(\sum_{|\alpha|=k}\frac{|\zeta^\alpha|^2}{\omega_\alpha}\Big)r^k=(1-r)^{-\frac{n}{2}}. \tag{24}$$

A more general formula about (24) may be found in [2, Theorem 4.5.1]. Thus

$$|\zeta^\alpha|\leqslant\sqrt{\omega_\alpha}\,r^{-\frac{k}{2}}(1-r)^{-n},\ |\alpha|=k$$

and

$$\|z^\alpha\|_{\Lambda_\varphi}=\sup_{z\in B}\frac{1-|z|}{\varphi(1-|z|)}(|\alpha|+1)|z^\alpha|\leqslant\frac{k+1}{\varphi(1)}\sqrt{\omega_\alpha}\,r^{-\frac{k}{2}}(1-r)^{-\frac{n}{2}}$$

by the increasing property of $t/\varphi(t)$ and the maximum principle. Let $|z|\leqslant r<1$, then

$$|z^\alpha|=r^{|\alpha|}|(z/r)^\alpha|\leqslant r^k\sup_{\zeta\in S}|\zeta^\alpha|\leqslant\sqrt{\omega_\alpha}\,r^{\frac{k}{2}}(1-r)^{-n}$$

so that

$$\frac{1}{k+1}|F(z^\alpha/\omega_\alpha)z^\alpha|\leqslant\|F\|_{\lambda_\varphi^*}(\varphi(1))^{-1}(1-r)^{-\frac{n}{2}}.$$

This shows that the function

$$\sum_{k=0}^{\infty}\frac{1}{k+1}\sum_{|\alpha|=k}\overline{F(z^\alpha/\omega_\alpha)z^\alpha} \tag{25}$$

is holomorphic in rB and so is holomorphic in B since the arbitrariness of $r\in(0,1)$. Therefore g, the fractional derivative of order 1 of (25), is holomorphic in B.

Now let $f\in\lambda_\varphi(B)$ and $f(z)=\sum_{k=0}^{\infty}\sum_{|\alpha|=k}a_\alpha z^\alpha$, then

$$F(f_r)=\sum_{k=0}^{\infty}\Big(\sum_{|\alpha|=k}a_\alpha F(z^\alpha/\omega_\alpha)\omega_\alpha\Big)r^k. \tag{26}$$

Let $r\to1$ in (26), we obtain

$$F(f)=\lim_{r\to1}\sum_{k=0}^{\infty}\Big(\sum_{|\alpha|=k}a_\alpha F(z^\alpha/\omega_\alpha)\omega_\alpha\Big)r^k=(f,g) \tag{27}$$

by the continuity of F and Theorem 2 (iv).

To prove $g\in\Gamma_\varphi(B)$, we note that the norm of $h\in\Gamma_\varphi(B)$ can be written as

$$\|h\|_{\Gamma_\varphi}=\sup_{T\in\Gamma_\varphi^*,\,T\neq0}\frac{|T(h)|}{\|T\|_{\Gamma_\varphi^*}}$$

by a corollary of Hahn–Banach theorem, and

$$\|h\|_{\Gamma_\varphi}\leqslant C\sup_{f\in\Lambda_\varphi,\,f\neq0}\frac{|(h,f)|}{\|f\|_{\Lambda_\varphi}} \tag{28}$$

by Theorem 4. Now $f_r\in\lambda_\varphi(B)$, $g_\rho\in\Gamma_\varphi(B)$ for any $f\in\Lambda_\varphi(B)$ and r, $\rho\in(0,1)$, by (27) and Theorem 2 (iii), we have

$$|(f_r,g_\rho)|=|(f_{r\rho},g)|=|F(f_{r\rho})|\leqslant\|F\|_{\lambda_\varphi^*}\|f_{r\rho}\|_{\Lambda_\varphi}\leqslant\|F\|_{\lambda_\varphi^*}\|f\|_{\Lambda_\varphi}$$

and

$$|(f,g_\rho)|\leqslant\|F\|_{\lambda_\varphi^*}\|f\|_{\Lambda_\varphi}$$

by Lemma 2 (ii). Thus $\|g_\rho\|_{\Gamma_\varphi}\leqslant C\|F\|_{\lambda_\varphi^*}$ by (28). This gives $g\in\Gamma_\varphi(B)$ and

$$\|g\|_{\Gamma_\varphi} \leqslant C\|F\|_{\lambda_\varphi^*} \tag{29}$$

by Theorem 3 (iii). Using Lemma 2 again yields

$$\|F\|_{\lambda_\varphi^*} \leqslant C\|g\|_{\Gamma_\varphi}. \tag{30}$$

Combining (29) and (30) gives (23).

The proofs of the uniqueness of g and the second part of Theorem 5 are the same as that of Theorem 4.

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