Chin. Ann. of Math.
8B (2) 1987

GLOBAL CLASSICAL SOLUTIONS OF THE CAUCHY PROBLEMS FOR NONLINEAR VORTICITY EQUATIONS AND ITS APPLICATIONS

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Abstract

In this paper, the definition of a vorticity equation on a compact Riemannian manifold without boundary is given. Some accurate a priori estimates are obtained with the use of the peculiarity of the equation. Then, the existence and "uniqueness" of global classical solution of the Cauchy problem for the equation is proved. As its application, the existence and "uniqueness" of Cauchy problem for barotropic nondivergent model is obtained. The model is a fundamental one in atmospheric dynamics.

§ 1. Introduction

Barotropic nondivergent model is a fundamental one in atmospheric dynamics. In 1950, with the use of electronic computer, the first successful numerical weather prediction was made by Charney by use of this model. The following Cauchy problem for this model which corresponds to the global weather prediction is frequently investigated[1].

$$\begin{cases} \dfrac{\partial}{\partial t}\Delta\psi+\dfrac{1}{a^2}J(\psi,\Delta\psi)+J(\psi,2\omega\cos\theta)=0, \text{ in } S^2\times(0,+\infty), & (1.1)\\ \Delta\psi|_{t=0}=\Delta\psi_0, & (1.2)\end{cases}$$

where ψ is unknown function (stream function) of independent variables (λ, θ, t); λ is longitude, $0\leqslant\lambda\leqslant 2\pi$; θ is colatitude (i.e. the North Pole corresponds to the point $\theta=0$), $0\leqslant\theta\leqslant\pi$; t is time; a is radius of the earth; ω is angular speed of the rotation of the earth; $S^2=\{(\lambda,\theta)\,|\,0\leqslant\lambda\leqslant 2\pi;\ 0\leqslant\theta\leqslant\pi\}$ is the unit sphere of $\mathbb{R}^3$;

$$\Delta=\frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial^2}{\partial\lambda^2}\right)$$

is the laplacian on S^2;

$$J(F,G)=\frac{1}{\sin\theta}\left(\frac{\partial F}{\partial\theta}\frac{\partial G}{\partial\lambda}-\frac{\partial F}{\partial\lambda}\frac{\partial G}{\partial\theta}\right).$$

Manuscript received December 2, 1984.

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Equation (1.1) known as vorticity equation is the most simple model in atmospheric dynamics. It describes the general character of atmospheric motion caused by Coriolis force and gravity. Up to now Equations (1.1), (1.2) still play an important role in the theory of atmospheric dynamics. The well-posedness of it was considered by many authors. In 1959, the existence and the "uniqueness" of the classical solution of the Cauchy problem for the linearized equation was proved[2]. In 1964, Tan and others[3] proved the existence and "uniqueness" of generalized solution of the Cauchy problem for the quasilinearized equation. In 1979, Zeng[1] proved that the local weak solution of Equations (1.1), (1.2) exists anb and two classical solutions of Equations (1.1), (1.2) merely differ by a function depending on time t only. In this paper, the existence and "uniqueness" of global classical solutoin of Equations (1.1), (1.2) is given (The precise meaning of uniqueness will be explained in the following sections).

For convenience's sake of mathematical treatment, we now introduce the definition of vorticity equation on a Riemannian manifold. Equation (1.1) is one of this type.

Let M be an n-dimensional, compact, oriented C^∞-Riemannian manifold without boundary. $\{\Omega_j, \varphi_j\}_{j=1}^J$ is a C^∞-atlas on it. $\widetilde{\Omega}_j=\varphi_j(\Omega_j)\subset R^n$. $x^j=(x_1^j, \cdots, x_n^j)$ denotes an arbitrary point in $\widetilde{\Omega}_j$. (P, t) is a point in $M\times[0, +\infty)$. If F is a function defined in $M\times[0, +\infty)$, we write $F_j=F\circ\varphi_j^{-1}$ defined in $\Omega_j\times[0, +\infty)$.

Let $T(P, t)$ be a given 2-order contravariant tensor field on M which depends on parameter t smoothly. In each $\widetilde{\Omega}_j\times[0, +\infty)$, $T(P, t)$ can be designated by functions $\{r_{\alpha\beta}^j(x^j, t)\}_{\alpha,\beta=1}^n$. We also assume that $\boldsymbol{R}(P, t)$, $\widetilde{\boldsymbol{R}}(P, t)$, $\boldsymbol{R}_0(P, t)$ are three given C^∞-vector fields on M, which also depend on parameter t smoothly,

$$\varphi_{j*}(\boldsymbol{R})=\{r_\alpha^j(x^j, t)\},\ \varphi_{j*}(\widetilde{\boldsymbol{R}})=\{\tilde{r}_\alpha^j(x^j, t)\}_{\alpha=1}^n,\ \varphi_{j*}(\boldsymbol{R}_0)=\{r_{0\alpha}^j(x^j, t)\}_{\alpha=1}^n.$$

For any differentiable function $v(P, t)$ defined on $M\times[0, +\infty)$, we define on M a family of vector fields $\boldsymbol{W}(T, \boldsymbol{R}, \widetilde{\boldsymbol{R}}, v)$ varying with parameter t as follows:

$$\boldsymbol{W}(T, \boldsymbol{R}, \widetilde{\boldsymbol{R}}, v)(P, t)=(\varphi_j^{-1})*\left\{\sum_{\alpha=1}^n r_{\alpha\beta}^j(x^j, t)\frac{\partial v_j}{\partial x_\alpha^j}+r_\beta^j(x^j, t)v_j+\tilde{r}_\beta^j(x^j, t\right\}_{\beta=1}^n,$$

when $$(P, t)=\varphi_j^{-1}(x^j, t)\in\Omega_j\times[0, +\infty).$$

We will always abbreviate $\boldsymbol{W}(T, \boldsymbol{R}, \widetilde{\boldsymbol{R}}, v)$ to $\boldsymbol{W}(v)$.

Let b and d be two given smooth functions defined on $M\times[0, +\infty)$. We define first-order linear partial differential operators $L_{(\boldsymbol{W}(v),b)}$ and $L_{(\boldsymbol{R}_0,d)}$ as follows:

$$(L_{(\boldsymbol{W}(v),b)}u)\circ\varphi_j^{-1}(x^j, t)$$
$$=\sum_{\beta=1}^n\left[\sum_{\alpha=1}^n r_{\alpha\beta}^j(x^j, t)\frac{\partial v_j}{\partial x_\alpha^j}+r_\beta^j(x^j, t)v_j+\tilde{r}_\beta^j(x^j, t)\right]\frac{\partial u_j}{\partial x_\beta^j}+b_ju_j,$$

$$(L_{(\boldsymbol{R}_0,d)}u)\circ\varphi_j^{-1}(x^j, t)=\sum_{\beta=1}^n r_{0\beta}^j(x^j, t)\frac{\partial u_j}{\partial x_\beta^j}+d_ju_j,$$

when $$(P, t)=\varphi_j^{-1}(x^j, t)\in\Omega_j\times[0, +\infty).$$

Here and afterwards, u is a differentiable function defined on $M\times[0, +\infty)$.

Lastly, let A be a second-order linear elliptic operator on M with smooth coefficients not depending on t. We define

$$L(v, u)=\left(\frac{\partial}{\partial t}+L_{(W(v),b)}\right)Au+L_{(R_0,d)}u,$$

and call $L(u, u)$ a vorticity operator on M.

In the paper, we consider the following Cauchy problem

$$\begin{cases}\left(\dfrac{\partial}{\partial t}+L_{(W(u),b)}\right)Au+L_{(R_0,d)}u=f, \text{ in } M\times[0, +\infty), & (1.3)\\ Au|_{t=0}=Au_0, & (1.4)\end{cases}$$

where u is an unknown function, u_0 and f are given functions.

Generally speaking, $\lambda=0$ may be an eigenvalue of operator A. In Equations (1.1), (1.2), $\lambda=0$ is an eigenvalue of Laplacian Δ on the unit sphere S^2. It is just the difficulty in solving the Cauchy problem. Observing Equation (1.1) carefully, we are inspired to make the following assumptions in this paper.

(H1). A is a second-order self-adjoint elliptic operator.

(H2). Let $N=\{u|Au=0 \text{ in } M\}$. Then, for any differentiable function $u(P, t)$, $U(P, t)$, $v(P, t)$,

$$L_{(W(v),b)}u(t)\perp N,\ L_{(R_0,d)}U(t)\perp N,\ \forall t\geqslant 0, \tag{1.5}$$

hold, where $f\perp N$ means $(f, g)_{L_2(M)}=0$, $\forall g\in N$.

It is easy to verify that Equations (1.1), (1.2) satisfiy (H1), (H2). In fact, now $A=\Delta$ is a self-adjoint operator. $N=\{u|u=\text{constant}\}$ and

$$L_{(W(v),b)}u(t)=\frac{1}{a^2}J(v(t), u(t)),$$

$L_{(R_0,d)}U(t)=J(U(t), 2\omega\cos\theta)$. Because $\int_{S^2}J(F, G)\,ds=0$ for any differentiable functions F, G, the equality (1.5) holds. Thus, (H2) is also satisfied.

In the paper, all functions and all functional spaces are real ones.

§ 2. Elliptic Problem and A Priori Estimates

At first, we introduce Sobolev space $W^{k,p}(M)$ and Hörlder space as usual, here integer $k\geqslant 0$, $1<p<+\infty$, $0<\alpha<1$. Let $\{\alpha_j\}_{j=1}^J$ be a C^∞-partition of unity being subordinate to $\{\Omega_j\}_{j=1}^J$. For any $u\in C^{k+1}(M)$, let

$$|||u(x)|||_{C^k(M)}=\sum_{j=1}^{J}|||(\alpha_j u_j)(x^j)|||_{C^k(\bar\Omega_j)},$$

where

$$|||(\alpha_j u_j)(x^j)|||_{C^k(\bar\Omega_j)}=\|\alpha_j u_j\|_{C^k(\bar\Omega_j)}+\sup_{\substack{x,y\in\bar\Omega_j\\|x-y|<1}}\sum_{|\alpha|=k}\frac{|D^\alpha(\alpha_j u_j)(x))-D^\alpha(\alpha_j u_j)(y))|}{\chi(|x-y|)}$$

$$\chi(s)=s(1-l_n s),\ 0<s<1.$$

It is wellknown from the theory of elliptic equation that the null space of operator A denoted by N belongs to $C^\infty(M)$ and is a finite dimensional space. Let $\{e_i\}_{i=1}^I$ be an orthonormal base of N in $L_2(M)$.

Lemma 2.1. *Let A be a second-order linear self-adjoint elliptic operator on M with smooth coefficients. $f(t)\in C^\infty(M\times[0,+\infty))$ such that for any $t\geqslant 0$, $f(t)\perp N$. Then, there exists a unique $u\in C^\infty(M\times[0,+\infty))$ such that*

$$\{Au=f,\ in\ M\times[0,+\infty);\ u(t)\perp N,\ \forall t\geqslant 0\}.$$

Proof Existence. For any fixed $t\in[0,+\infty)$, since A is a self-adjoint operator and $f(t)\perp N$, there exists a $u_1(t)\in C^\infty(M)$ such that $Au_1(t)=f(t)$. Writing $c_i(t)=(u_i(t),e_i)_{L_2(M)}$ and setting $u(t)=u_1-\sum_{i=1}^I c_ie_i$, we have $Au(t)=f(t)$ and $u(t)\perp N$ for any $t\geqslant 0$.

Smoothness. Note that $\{A(u(t+\Delta t)-u(t))=f(t+\Delta t)-f(t)$, in M; $(u(t+\Delta t)-u(t))\perp N\}$. Thus, by the theory of elliptic equation, we have

$$\|u(t+\Delta t)-u(t)\|_{C^{k+1}(M)}\leqslant C\|f(t+\Delta t)-f(t)\|_{C^k(M)},\quad \text{integer } k\geqslant 1.$$

Setting $\Delta t\to 0$, it follows that $u(t)\in C^0([0,+\infty);C^{k+1}(M))$.

It is easy to verify that for any $t\geqslant 0$, $g(t)=\dfrac{\partial f}{\partial t}\perp N$. By the existence proved above, there exists a $v(t)\in C^0([0,+\infty);C^k(M))$ such that

$$\{Av(t)=g(t),\ \text{in } M\times[0,+\infty);\ v(t)\perp N \text{ for } \forall t\geqslant 0\}.$$

On the other hand

$$A\left(\frac{u(t+\Delta t)-u(t)}{\Delta t}-v(t)\right)=\frac{f(t+\Delta t)-f(t)}{\Delta t}-\frac{\partial f}{\partial t},$$

hence

$$\left\|\frac{u(t+\Delta t)-u(t)}{\Delta t}-v(t)\right\|_{C^k(M)}\leqslant C\left\|\frac{f(t+\Delta t)-f(t)}{\Delta t}-\frac{\partial f}{\partial t}\right\|_{C^{k-1}(M)}.$$

Setting $\Delta t\to 0$, we have $\dfrac{\partial u}{\partial t}=v(t)\in C^0([0,+\infty);C^k(M))$. Continuing this process, we can prove $u\in C^j([0,+\infty);C^k(M))$ for each integer j and k. It follows that $u\in C^\infty(M\times[0,+\infty))$.

Uniqueness. Let $u_i(t)\in C^\infty(M\times[0,+\infty))$ such that $Au_i=f$ and $u_i(t)\perp N$ for any $t\geqslant 0$, $i=1,2$. Then, $(u_1(t)-u_2(t))\in N$ for any $t\geqslant 0$. This implies $u_1\equiv u_2$.

Lemma 2.2. *Suppose A satisfies the assumptions in Lemma 2.1. Then there exists a constant C, which is independent of u, such that*

$$\|u\|_{C^1(M)}\leqslant C\|Au\|_{C^0(M)},\quad u\in C^\infty(M)\ and\ u\perp N.$$

Proof. Let α_j be an element of the partition of unity. We have

$$A(\alpha_ju)=\alpha_jAu+A_1u,$$

where A_1 is a first-order partial differential operator. Noting $\alpha_ju_j\in C_c^\infty(\tilde{\Omega}_j)$, we have by Lemma 1.4 of [4]

$$\|(\alpha_j u_j)(x^j)\|_{C^2(\tilde{\Omega}_j)} \leqslant C(\|Au\|_{C^0(M)} + \|u\|_{C^0(M)}).$$

Summing the above inequalities with respect to j and noting $\|u\|_{C^0(M)} \leqslant C\|Au\|_{C^0(M)}$, since $u \in N$, we complete the proof.

§3. Linearized Problem

In the following, $T>0$ is a constant assigned arbitrarily.

Lemma 3.1. *Suppose* $v, u, U, f \in C^\infty(M \times [0, T])$, $U_0 \in C^\infty(M)$. *Moreover*

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + L_{(W(v), b)}\right)AU + L_{(R_0, d)}u = f, \text{ in } M \times [0, T], & (3.1) \\ AU|_{t=0} = AU_0 & (3.2) \end{cases}$$

and

$$U(t) \perp N, \ \forall t \in [0, T].$$

Then

$$\|U(t)\|^2_{W^{k,p}(M)} \leqslant C\left(\|U_0\|^2_{W^{k,p}(M)} + \int_0^t (\|u(\tau)\|^2_{W^{k-1,p}(M)} + \|f(t)\|^2_{W^{k-2,p}(M)})\, d\tau\right) \cdot \exp\left\{C\int_0^t (\|v(\tau)\|_{W^{k-1,p}(M)} + 1)\, d\tau\right\}, \tag{3.3}$$

Particularly, if $U = u$ *in* (3.1), *then*

$$\|U(t)\|^2_{W^{k,p}(M)} \leqslant C\left(\|U_0\|^2_{W^{k,p}(M)} + \int_0^t \|f(\tau)\|^2_{W^{k-2,p}(M)} d\tau\right) \times \exp\left\{C\int_0^t (\|v(\tau)\|_{W^{k-1,p}(M)} + 1)\, d\tau\right\}, \tag{3.4}$$

$$\|U(t)\|^2_{W^{3,p}(M)} \leqslant C\left(\|U_0\|^2_{W^{3,p}(M)} + \int_0^t \|f(\tau)\|^2_{W^{1,p}(M)} d\tau\right) \times \exp\left\{C\int_0^t (\|v(\tau)\|_{C^2(M)} + 1)\, d\tau\right\}. \tag{3.5}$$

In above inequalities, $0 \leqslant t \leqslant T$, integer $k \geqslant 4$, $n < p < +\infty$. We also denote by C the different constants depending on k, p, T and M only, not depending on U, u, v, U_0, f.

Proof Let $AU = W$. Multiplying Equation (3.1) by α_j, we have in $\tilde{\Omega}_j \times (0, T)$,

$$\left(\frac{\partial}{\partial t} + \sum_{\beta=1}^n \left[\sum_{\alpha=1}^n r^j_{\alpha\beta} \frac{\partial v_j}{\partial x^j_\alpha} + r^j_\beta v_j + \tilde{r}^j_\beta\right] \frac{\partial}{\partial x^j_\beta} + b_j\right)(\alpha_j W_j)$$
$$- \sum_{\beta=1}^n \left[\sum_{\alpha=1}^n r^j_{\alpha\beta} \frac{\partial v_j}{\partial x^j_\alpha} + r^j_\beta v_j + \tilde{r}^j_\beta\right] \frac{\partial \alpha_j}{\partial x^j_\beta} W_j$$
$$+ \alpha_j\left(\sum_{\beta=1}^n r^j_{0\beta}(x^j, t) \frac{\partial}{\partial x^j_\beta} + d_j\right) v_j = \alpha_j f_j.$$

The corresponding initial condition is

$$\alpha_j W_j|_{t=0} = \alpha_j (AU_0)_j.$$

Since $\alpha_j W_j \in C_c^\infty(\tilde{\Omega}_j)$, by the methods of [5] it is easy to prove that

$$\frac{d}{dt}\|\alpha_j W_j(t)\|^p_{W^{k-2,p}(\tilde{\Omega}_j)}\leq C\|W(t)\|^p_{W^{k-2,p}(M)}(\|v(t)\|_{W^{k-1,p}(M)}+1)$$
$$+C(\|f(t)\|^p_{W^{k-2,p}(M)}+\|u(t)\|^p_{W^{k-1,p}(M)}).$$

We sum the above inequalities with respect to j, and then integrate it with respect to t in $[0, T]$. Because of $U(t)\perp N$ for any $t\geq 0$, it follows that

$$\|U(t)\|_{W^{k,p}(M)}\leq C\|AU(t)\|_{W^{k-2,p}(M)}.$$

Using this fact and Gronwall's inequality, we obtain (3.3). The inequalities (3.4) and (3.5) can be proved similarly.

In the following, we denote by $C^0([0, T]; W^{k,p}(M))$ the completion of function space $C^\infty(M\times[0, T])$ normed by $\sup\limits_{0\leq t\leq T}\|u(t)\|_{W^{k,p}(M)}$. If u and $\frac{\partial u}{\partial t}$ both belong to $C^0([0, T]; W^{k,p}(M))$, we write $u\in C^1([0, T]; W^{k,p}(M))$. If $u\in C^0([0, T]; W^{k,p}(M))$ for any $T>0$, we write $u\in C^0([0, +\infty); W^{k,p}(M))$, etc.

Using Lemma 3.1. we have

Theorem 3.1. *Suppose (H1), (H2) hold. $v\in C^\infty(M\times[0, +\infty))$ is a function given arbitrarily. $f(t)\in C^0([0, +\infty); W^{k-2,p}(M))$ such that $f(t)\perp N$ for any $t\geq 0$. $u_0\in W^{k,p}(M)$, $k\geq 4$, $n<p<+\infty$. Then, there exists a unique solution u of the following Cauchy problem for the linearized equation*

$$\begin{cases}\left(\frac{\partial}{\partial t}+L_{(W(v),b)}\right)Au+L_{(R_0,d)}u=f, \text{ in } M\times(0, +\infty). & (3.6)\\ Au|_{t=0}=Au_0, & (3.7)\end{cases}$$

such that $u\in C^0([0, +\infty), W^{k,p}(M))\cap C^1([0, +\infty); W^{k-1,p}(M))$ and $u(t)\perp N$, $\forall t\geq 0$.

Proof At first, we assign a constant $T>0$ arbitrarily and assume $u_0\in C^\infty(M)$, $f\in C^\infty(M\times[0, T])$ such that $f(t)\perp N$ for any $t\in[0, T]$. For any given $U\in C^\infty(M\times[0, T])$, solving the hyperbolic Cauchy problem

$$\begin{cases}\left(\frac{\partial}{\partial t}+L_{(W(v),b)}\right)\tilde{W}+L_{(R_0,d)}U=f, \text{ in } M\times(0, T), & (3.8)\\ \tilde{W}|_{t=0}=Au_0, & (3.9)\end{cases}$$

we obtain $\tilde{W}\in C^\infty(M\times[0, T])$. Let us prove that $\tilde{W}(t)\perp N$ for any $t\in[0, T]$. By (3.8), (3.9) and (H2)

$$\frac{\partial}{\partial t}\int_M\tilde{W}(t)e\,dM=0,\ \forall t\in[0, T],\ \forall e\in N \tag{3.10}$$

holds. Since A is a self-adjoint opertor,

$$\int_M\tilde{W}(0)e\,dM=\int_M(Au_0)e\,dM=\int_M u_0Ae\,dM=0.$$

Thus

$$\int_M\tilde{W}(t)e\,dM=0$$

for any $t\in[0, T]$. By Lemma 2.1, there exists a $W\in C^\infty(M\times[0, T])$ such that

$\{AW=\widetilde{W}$, in $M\times[0, T]$; $\forall t\in[0, T]$, $W(t)\perp N\}$.

If $u_0\in W^{k,p}(M)$, $f\in C^0([0, T]; W^{k-2,p}(M))$ and $f(t)\perp N$ for any $t\geqslant 0$, there exists a sequence $\{f_n(t)\}\subset C^\infty(M\times[0, T])$ such that

$$f_n\to f, \text{ in } C^0([0, T]; W^{k-2,p}(M));\ \forall n,\ \forall t\in[0, T],\ f_n(t)\perp N.$$

Then, using the methods of [5], we complete the proof of the theorem.

§ 4. Nonlinear Problem

In this section, using the method[6] of integrating and estimating along the characteristic curve lying on manifold, we will prove further estimates by the results given above. Then, the existence and "uniqueness" of global classical solution of Equations (1.3), (1.4) will be proved by the methods of [5]. Lastly, we will prove the existence and "uniqueness" of global classical solution of Equations (1.1), (1.2).

Since M is a smooth compact manifold without boundary, for any $v\in C^\infty(M\times[0, +\infty))$, $\forall s\in[0, T]$, $P^*\in M$, the characteristic curve $l(t; s, P^*)$ of operator $\frac{\partial}{\partial t}+L_{(W(v))}$ (i.e. the curve satisfying $\{l'(t)=W(v);\ l(t)|_{t=s}=P^*\}$ exists on $[0, T]$ globally according to the theory of ordinary differential equation[7].

For the sake of technical need, we construct two equivalent atlases $\{\Omega;\ \varphi_i\}_{i=1}^I$ and $\{\omega_i,\ \varphi_i\}_{i=1}^I$ such that $\omega_i\subset\subset\Omega_i$ and $\varphi_i(\omega_i)=B_1$, $\varphi_i(\Omega_i)=B_2$. Here B_j is an open ball of radius j, with center at the origin of $\mathbb{R}^n$, $j=1, 2$.

Suppose vector fields $\boldsymbol{R}(P, t)$ depend on t smoothly. We define

$$\|L_{(\boldsymbol{R})}\|_{C^0(M\times[0,T])}=\sup_{0\leqslant t\leqslant T}\sum_{\alpha=1}^{n}\sum_{i=1}^{I}\|r^i_\alpha(x^i, t)\|_{C^0(\bar{\Omega}_i)}.$$

Lemma 4.1. *Suppose $P^*\in\omega_k$, $\|L_{(\boldsymbol{R})}\|_{C^0(M\times[0,T])}\leqslant D$ (a positive constant). Then, curve $l(t; s, P^*)$ lies in chart Ω_k when $|t-s|<\frac{1}{\sqrt{n}\,D}$.*

Proof Let $x^k_i(t; s, P^*)$ be the i-th component of $\varphi_k(l(t; s, P^*))$. Then

$$\begin{cases}\dfrac{dx^k_i(t; s, P)}{dt}=r^k_i(x^k(t; s, P^*), t),\\ x^k_i|_{t=s}=\varphi^i_k(P^*),\end{cases}\quad i=1, 2, \cdots, n,$$

hold in $\widetilde{\Omega}_k$. Hence

$$x^k_i(t; s, P^*)=\varphi^i_k(P^*)+\int_s^t r^k_i(x^k(\tau; s, P^*), \tau)d\tau.$$

If follows that

$$|x^k(t; s, P^*)|<2, \text{ when } |t-s|<\frac{1}{\sqrt{n}\,D}.$$

This completes the proof.

In the following, $C_i(F_1, \cdots, F_k)$ denote the constants which only depend on

$F_1, \cdots, F_k$. The different subscripts "i" mean different dependent relations.

The following lemmas are extensions of the corresponding lemmas in section 5 of [5].

Lemma 4.2. *Suppose $u, v, f \in C^\infty(M\times[0, T])$, $u_0\in C^\infty(M)$ such that*

$$\begin{cases}\left(\dfrac{\partial}{\partial t}+L_{(W(v),b)}\right)Au+L_{(R_0,a)}u=f, \text{ in } M\times[0, T]. & (4.1)\\ Au|_{t=0}=Au_0. & (4.2)\end{cases}$$

Then

$$\|u(s)\|_{C^1(M)}\leqslant C_1(T, \|u_0\|_{C^2(M)}, \|f\|_{C^0(M\times[0,T])}), \ 0\leqslant s\leqslant T \qquad (4.3)$$

holds for all u satisfying $u(t)\perp N$ for any $t\in[0, T]$.

It should be pointed out that the right side of (4.3) does not depend on v. We define constant $\tilde{D}$ by

$$\tilde{D}=C_1(T, \|u_0\|_{C^2(M)}, \|f(t)\|_{C^0(M\times[0,T])}). \qquad (4.4)$$

If $\sup\limits_{0\leqslant t\leqslant T}\|v(t)\|_{C^1(M)}\leqslant E_1$, it is easy to verify that $\|L_{(W(v))}\|_{C^0(M\times[0,T])}\leqslant C_2(E_1)$. We define the constant D_1 by

$$D_1=C_2(\tilde{D}). \qquad (4.5)$$

Lemma 4.3. *Suppose $(H1)$, $(H2)$ and all assumptions in Lemma 4.2 hold. D_1 is given by (4.5). We assume $\|L_{(W(v))}\|_{C^0(M\times[0,T])}\leqslant D_1$. α and β are two constants such that $0<\alpha<\beta<1$, $\alpha=\beta e^{-2TD_1J_0}$, where $J_0=1+[2T\sqrt{n}D_1]$ ($[x]$ denotes the largest integer such that $[x]\leqslant x$). Then*

$$\|u(t)\|_{C^{2,\alpha}(M)}\leqslant C_3(T, D_1, \|u_0\|_{C^{2,\beta}(M)}, \|f(t)\|_{C^0([0,T];C^{0,\beta}(M))}), \ 0\leqslant t\leqslant T. \qquad (4.6)$$

Here we only give the outline for the proof of Lemma 4.2 and Lemma 4.3.

At first, via "localization", the proof is reduce to a problem in chart $\tilde{\Omega}_k\times[0,T]$. By the method of [5], we get corresponding estimates. In the intersection of two different charts, we can link the estimates by the method of [6, 8]. From Lemma 4.1, the times of this link we need to do is finite (at most J_0 times). Noting $\|u(t)\|_{C^1(M)}\leqslant C\|Au\|_{C^0(M)}$ holds for all $u(t)$ satisfying $u(t)\perp N$, we obtain (4.3), (4.6) by Lemma 2.2 and Gronwall's inequality.

Using Lemma 4.2, Lemma 4.3, Lemma 3.1, Theorem 3.1 and Schauder fixed point theorem, we can prove the following theorem (The technical details can be found in [5]).

Theorem 4.1. (i) *Suppose $f\in C^0([0, +\infty); W^{k-2,p}(M))$ and $f(t)\perp N$ for any $t\in[0, +\infty)$, $u_0\in W^{k,p}(M)$, integer $k\geqslant5$, $n<p<+\infty$. Then, there exists a unique solution u of Equations (1.3), (1.4) such that*

$$u\in C^0([0, +\infty); W^{k,p}(M))\cap C^1([0, +\infty); W^{k-1,p}(M))$$

and $u(t)\perp N$ for any $t\geqslant0$. (ii) *If $f\in C^\infty(M\times[0, +\infty))$ and $u(t)\perp N$ for any $t\geqslant0$, $u_0\in C^\infty(M)$, there exists a unique solution u of Equations (1.3), (1.4) such that $u\in C^\infty(M\times[0, +\infty))$ and $u(t)\perp N$ for any $t\geqslant0$.*

In section 1 we have pointed out that Equations (1.1), (1.2) satisfy (H1), (H2). From Theorem 4.1, we obtain the following theorem.

Theorem 4.2. (i) *Suppose* $\psi_0 \in W^{k,p}(S^2)$, *integer* $k \geqslant 5$, $2 < p < +\infty$. *Then, there exists a unique solution* ψ *of Equations* (1.1), (1.2) *such that*

$$\psi \in C^0([0, +\infty); W^{k,p}(S^2)) \cap C^1([0, +\infty); W^{k-1,p}(S^2))$$

and

$$\int_{S^2} \psi(t) ds = 0$$

for any $t \geqslant 0$. (ii) *Suppose* $\psi_0 \in C^\infty(S^2)$. *Then, there exists a unique solution* ψ *of Equations* (1.1), (1.2) *such that* $\psi \in C^\infty(S^2 \times [0, +\infty))$ *and*

$$\int_{S^2} \psi(t) ds = 0$$

for any $t \geqslant 0$.

By Theorem 4.1, there exists a global classical solution of Equations (1.1), (1.2). Zeng Qingcun proved in [1] that any two classical solutions of Equations (1.1), (1.2) differ merely by a function which depends on time t only, so they correspond to the same field of velocity. Thus, Problem (1.1), (1.2) is "well-posed" in view of physics.

The author is much indebted to his advisors Professor Gu Chaohao andProfessor Li Taqian for their guidance.

References

[1] Zeng Qingcun, Physical-mathematical basis of numerical weather prediction, Vol. 1, Science Press, Beijing, 1979, Chinese edition.

[2] The group of PDE of Institute of Mathematics, Academia Sinica, An initial value problem for a composite type equation relating to medium-long-range weather forecast, *Acta Meteorologica Sinica*, **3** (1959).

[3] Tan Yongji, et al., The generalized solution of the initial value problem for an equation of composite type, Collected papers of Mathematics, Institute of Mathematics of Fudan University, 1964.

[4] Kato, T., On the classical solutions of the two-dimensional non-stationary Euler equations, *Arch. Rational Mech. Anal.*, **25** (1967), 188—200.

[5] Mu Mu, Global classical solutions of nonlinear vorticity equations and its applications. *Acta Mathematica Scientia*, **6**:2 (1986), 193—210.

[6] Gu Chaohao, On the initial-boundary value problem for harmonic maps from the 2-dimensional Minkowski space, *Manuscripta Mathematica*, **33** (1980), 51—58.

[7] Arnold, V. I., Ordinary differential equations, The Massachusettes Institute of Technology, 1973.

[8] Mu Mu, Global classical solutions of the initial-boundary value problems for generalized vorticity equations and its applications, *Scientia Sinica* Series A, **30** (1987), 25—37.