Chin. Ann. of Math.
8B (2) 1987

DARBOUX TRANSFORMATIONS ASSOCIATED WITH BOITI-TU EIGENVALUE PROBLEM*

LI YISHEN (李翊神)** GU XINSHEN (顾新身)**

Abstract

To the hierarchy of nonlinear evolution equations which associate with Boiti-Tu eigenvalue problem, an explicit and universal form of Bäcklund transformations and auniversal proof are presented. It is called Darboux transformation. By this method, to ask for a new solution of every system of equations of the hierarchy, it is sufficient to solve some linear problems. Here the constraints at the boundary for the potentials (for example, at $x=\pm\infty$) are removed.

§ 0. Introduction

In [1] Boiti and G. Z. Tu have introduced the following eigenvalue problem:

$$\varphi_x=U\varphi,\ U=-i\lambda\sigma_3+u\sigma_1+\lambda^{-1}(is\sigma_3-v\sigma_2), \tag{0.1}$$

where σ_j are Pauli matrices

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

We call (0.1) Boiti-Tu eigenvalue problem. In [2] the auxiliary eigenvalue problem have been introduced:

$$\varphi_t=V\varphi,\ V=\sum_{j=0}^{n}V_j\lambda^{n-j}+\sum_{l=0}^{p}W_l\lambda^{l-p-1}. \tag{0.2}$$

From the integrability condition of (0.1) and (0.2), a hierarchy of nonlinear evolution equations can be obtained. We call these equations evolution equations associated with Boiti-Tu eigenvalue problem.

In this paper, to every system of evolution equations of the hierarchy, we give a uniform method to ask for new solutions from a given system of solutions of the same system of equations. We call it Darboux Transformation (D.T.). D.T. has two outstanding merits: (1) It does not need any confined conditions at $x=\pm\infty$. (2) The process does not need to solve any nonlinear problem. It needs to solve some linear ordinary differential equations and linear algebric equations only.

Manuscript received December 12, 1984.
* Projects supported by the Science Fund of the Chinese Academy of Sciences
** Department of Mathematics, University of Science and Technology of China, Hefei, China.

In part one, we quote some results of [3], introduce the concept of D.T. and its merits, and point out that D.T. consists of two parts: one is gauge transformation and the other is the relations between the old solutions and the new solutions. In part two, by the integrability condition of (0.1) and (0.2):

$$U_t - V_x + UV - VU = 0, \tag{0.3}$$

we solve V from (0.3) for given U. And we give the general solution V by recursion fomulas, in which the integral operation does not appear. So we do not need to assume any confined conditions at $|x| = \infty$ for potentials. In part three, the methods of getting gauge transformation and the D.T. formulas are given, and the properties of gauge transformation are discussed. In addition, the decomposable and permutable properties of N-degree D.T. are proved. In part four, we prove that the new potentials which is defined by D.T. satisfy the same system of equations as the old solutions. We give a new method of the proof. In part five, we discuss the relation between D.T. and Bäcklund transformation (B.T.). We indicate that the new potentials defined by D.T. satisfy the B.T. not only in the x part but also in the t part. The new potentials are just the solutions of B.T.

§ 1. Darboux Transformation and its Merits

In [3] the following problem was discussed

$$\left.\begin{array}{l}\varphi_x = U\varphi,\ U = -i\lambda\sigma_3 + u\sigma_1 + \lambda^{-1}(is\sigma_3 - v\sigma_2),\\ \varphi_t = V\varphi,\ V = -i\lambda\sigma_3 + u\sigma_1 + \lambda^{-1}(-is\sigma_3 + v\sigma_2).\end{array}\right\} \tag{1.1}$$

The integrability condition of (1.1) is

$$U_t - V_x + UV - VU = 0. \tag{1.2}$$

Rewrite (1.2) in its components as follows

$$\left.\begin{array}{r}S_t + S_x + 4uv = 0,\\ v_t + v_x + 4su = 0,\\ u_t - u_x - 4v = 0.\end{array}\right\} \tag{1.3}$$

Consider

$$\left.\begin{array}{l}\bar\varphi_x = \bar U\bar\varphi,\ \bar U = -i\lambda\sigma_3 + \bar u\sigma_1 + \lambda^{-1}(i\bar s\sigma_3 - \bar v\sigma_2),\\ \bar\varphi_t = \bar V\bar\varphi,\ \bar V = -i\lambda\sigma_3 + \bar u\sigma_1 + \lambda^{-1}(-i\bar s\sigma_3 + \bar v\sigma_2),\end{array}\right\} \tag{1.4}$$

where $\bar U$, $\bar V$ can be obtained from the expressions of U, V by replacing u, v, s with $\bar u$, $\bar v$, $\bar s$. The integrability condition of (1.4) is

$$\bar U_t - \bar V_x + \bar U\bar V - \bar V\bar U = 0. \tag{1.5}$$

From (1.5) we know that $\bar u$, $\bar v$, $\bar s$ satisfy the nonlinear equation (1.3).

In [3] suppose T is a 1-degree polynomial of λ. By solving the following differential equations directly

$$\left.\begin{aligned} T_x&=\bar{U}T-TU,\\ T_t&=\bar{V}T-TV,\end{aligned}\right\}\tag{1.6}$$

the gauge transformation T from φ to $\bar{\varphi}$ is obtained

$$\left.\begin{aligned} &\bar{\varphi}=T\varphi,\\ &T=\lambda I+a\sigma_3+b\sigma_2 \text{ (where } I \text{ is an idential matrix).}\end{aligned}\right\}\tag{1.7}$$

Suppose φ is a fundamental solution matrix of (1.1) whose components are $\varphi_{jl}(j, l=1, 2)$. Given a pair of parameters (λ_1, k_1), a and b can be obtained by the following formulas:

$$\left.\begin{aligned} \phi_1(x, \lambda_1, k_1)&=\varphi_{11}(x, \lambda_1)+k_1\varphi_{12}(x, \lambda_1),\\ \phi_2(x, \lambda_1, k_1)&=\varphi_{21}(x, \lambda_1)+k_1\varphi_{22}(x, \lambda_1),\end{aligned}\right\}\tag{1.8}$$

$$\left.\begin{aligned} a&=\lambda_1[\phi_1^2(x, \lambda_1, k_1)+\phi_2^2(x, \lambda_1, k_1)]/[\phi_2^2(x, \lambda_1, k_1)-\phi_1^2(x, \lambda_1, k_1)],\\ b&=[-2i\lambda_1\phi_1(x, \lambda_1, k_1)\phi_2(x, \lambda_1, k_1)]/[\phi_2^2(x, \lambda_1, k_1)-\phi_1^2(x, \lambda_1, k_1)].\end{aligned}\right\}\tag{1.9}$$

And the following formulas are obtained:

$$\left.\begin{aligned} \bar{u}&=2b+u,\\ \bar{v}&=-\frac{1}{\lambda_1^2}[2isab-vb^2+va^2],\\ \bar{s}&=-\frac{1}{\lambda_1^2}[-2ivab+sb^2-sa^2].\end{aligned}\right\}\tag{1.10}$$

In [3] (1.7) and (1.10) is called Darboux transformation.

From above process we can see that when we do D.T., the only confined conditions are $\lambda_1\neq 0$ and $\phi_2^2(x, \lambda_1, k_1)-\phi_1^2(x, \lambda_1, k_1)\neq 0$, and there is no need of any confined conditions to u, v, s at $x=\pm\infty$. In addition, to ask for new solutions of (1.3), it is sufficient to solve some linear problems and to do some algebraic operations, there is no need for solving any nonlinear problems. These are the outstanding merits of D.T.

§ 2. The Nonlinear Evolution Equations (N.E.E.) Associated with Boiti-Tu Eigenvalne Problem

Suppose that in the auxilary eigenvalue problem (0.2) V has the following expansion

$$\left.\begin{aligned} &V=\sum_{j=0}^{k}V_{2j}\lambda^{2k+1+2j}+\sum_{j=0}^{k}V_{2j+1}\lambda^{2k-2j}+\sum_{l=0}^{2p}W_l\lambda^{l-2p-1},\\ &V_{2j}=d_{2j}\sigma_3-\frac{1}{2}f_{2j}\sigma_2,\quad V_{2j+1}=\frac{1}{2}e_{2j+1}\sigma_1\quad (0\leqslant j\leqslant k),\\ &W_{2l}=a_{2l}\sigma_3-\frac{1}{2}c_{2l}\sigma_2\quad (0\leqslant l\leqslant p),\\ &\text{as } p\geqslant 1,\ W_{2l+1}=\frac{1}{2}b_{2l+1}\sigma_1\quad (0\leqslant l\leqslant p-1).\end{aligned}\right\}\tag{2.1}$$

It is easy to verify that as V has above expansion, (0.1) and (0.2) can be compatible. Comparing the coefficients of different powers of λ in (0.3), we get the following recursive differential equations

$$\left.\begin{aligned} & e_{-1}=f_{-2}=d_{-2}=0, \\ & f_{2j}=sf_{2j-2}+2ivd_{2j-2}+\frac{1}{2}e_{2j-1x} \quad (0\leqslant j\leqslant k+1). \\ & d_{2jx}=-iuf_{2j}+iue_{2j-1} \quad (0\leqslant j\leqslant k), \\ & e_{2j+1}=2iud_{2j}+se_{2j-1}-\frac{1}{2}f_{2jx} \quad (0\leqslant j\leqslant k), \end{aligned}\right\} \tag{2.2}$$

$$\left.\begin{aligned} & b_{-1}=a_{-2}=c_{-2}=0, \\ & sb_{2l+1}=b_{2l-1}-2iua_{2l}+\frac{1}{2}c_{2lx} \quad (0\leqslant l\leqslant p), \\ & vb_{2l+1}=uc_{2l}-ia_{2lx} \quad (0\leqslant l\leqslant p), \\ & sc_{2l}+2iva_{2l}=c_{2l-2}-\frac{1}{2}b_{2l-1x} \quad (0\leqslant l\leqslant p). \end{aligned}\right\} \tag{2.3}$$

The evolution equations are

$$\left.\begin{aligned} u_t &= f_{2k+2}-c_{2p}, \\ v_t &= -se_{2k+1}+sb_{2p+1}, \\ s_t &= -ve_{2k+1}+vb_{2p+1}, \end{aligned}\right\} \tag{2.4}$$

where we introduce $b_{-1}=a_{-2}=c_{-2}=e_{-1}=d_{-2}=f_{-2}=0$ for keeping the recursion formulas in uniformity, and we introduce f_{2k+2}, b_{2p+1} by (2.2) and (2.3) for convenience. When a_{2l}, c_{2l}, $b_{2l+1}(0\leqslant l\leqslant p)$ are all equal to zero, (2.4) is reduced to

$$\left.\begin{aligned} u_t &= f_{2k+2}, \\ v_t &= -se_{2k+1}, \\ s_t &= -ve_{2k+1}. \end{aligned}\right\} \tag{2.5}$$

When d_{2j}, e_{2j+1}, f_{2j} are all equal to zero, (2.4) is reduced to

$$\left.\begin{aligned} u_t &= -c_{2p}, \\ v_t &= sb_{2p+1}, \\ s_t &= vb_{2p+1}. \end{aligned}\right\} \tag{2.6}$$

We call (2.5), (2.6) and (2.4) the evolution equations corresponding to positive expansion, negative expansion and mixed expansion of λ respectively. Equation (1.3) is just an equation corresponding to a mixed expansion of λ (where $k=0$, $l=0$).

By (2.4), or (2.5), or (2.6), we get

$$(v^2-s^2)_t=0. \tag{2.7}$$

That is v^2-s^2 must be independent of t.

Formula (2.3) here is just the formulas (2.2)—(2.6) under condition (2.17) in [2]. Formula (2.2) is just (2.7)—(2.10) in [2]. Formulas (2.5)—(2.7) are (2.11a)—(2.11d) in [2].

We regard (2.2) and (2.3) as a system of recursion differential equations. From (2.2) we can solve f_0, d_0, e_1, f_2, d_2, e_3, $\cdots$, f_{2k}, d_{2k}, e_{2k+1}, f_{2k+2} step by step. We notice that to solve d_{2j}, by the second formula of (2.2)

$$d_{2jx}=-iuf_{2j}+ive_{2j-1},$$

it is needed to do integral operation. But we know that $-iud_{2j}+ive_{2j-1}$ is just a derivative of a function of x. So we can directly write d_{2j} by explicit formulas. In the following Lemma 1 general expressions of d_{2j} are given. Similarly in the following Lemma 2, from (2.3), general expressions of a_{2l}, b_{2l+1}, and c_{2l} are given, which are some recursion formulas and in which the integral operation does not appear.

Lemma 1. *Suppose d_0 is not equal to zero. Then the general integral of the recursion differential equations (2.2) are given by the following recursion formulas:*

$$\left.\begin{aligned}
&e_{-1}=d_{-2}=f_{-2}=0,\\
&f_{2j}=sf_{2j-2}+2ivd_{2j-2}+\frac{1}{2}e_{2j-1x}\quad(0\leqslant j\leqslant k+1),\\
&d_0=\alpha_0,\\
&\text{as } k\geqslant 1,\ \sum_{j=0}^{m+1}\left(d_{2j}d_{2m+2-2j}+\frac{1}{4}f_{2j}f_{2m+2-2j}\right)+\frac{1}{4}\sum_{j=0}^{m}e_{2j+1}e_{2m-2j+1}=\alpha_{2m+2}\quad(0\leqslant m\leqslant k-1),\\
&e_{2j+1}=2iud_{2j}+se_{2j-1}-\frac{1}{2}f_{2jx}\quad(0\leqslant j\leqslant k),
\end{aligned}\right\}\tag{2.8}$$

where α_0, $\alpha_{2m+2}(0\leqslant m\leqslant k-1)$ are integral constants (which are independent of x).

(Proof is omitted.)

Lemma 2. *Suppose $s^2-v^2\neq 0$ for every x, and suppose $|a_0|+|c_0|\neq 0$. Then the general integral of the recursion differential equations (2.3) are given by the following recursion formulas*

$$\left.\begin{aligned}
&b_{-1}=a_{-2}=c_{-2}=0,\\
&a_0^2+\frac{1}{4}c_0^2=\beta_0^2,\\
&sc_{2l}+2iva_{2l}=c_{2l-2}-\frac{1}{2}b_{2l-1x}\quad(0\leqslant l\leqslant p),\\
&\text{as } p\geqslant 1\quad(0\leqslant l\leqslant p-1),\\
&\sum_{m=0}^{l+1}\left(a_{2m}a_{2l+2-2m}+\frac{1}{4}c_{2m}c_{2l+2-2m}\right)+\frac{1}{4}\sum_{m=0}^{l}b_{2m+1}b_{2l+1-2m}=\beta_{2(l+1)},\\
&b_{2l+1}=\frac{s\left(b_{2l-1}-2iua_{2l}+\frac{1}{2}c_{2lx}\right)-v\left(uc_{2l}-ia_{2lx}\right)}{s^2-v^2}\quad(0\leqslant l\leqslant p),
\end{aligned}\right\}\tag{2.9}$$

where $\beta_{2l}(0\leqslant l\leqslant p)$ are integral constants (which are independent of x).

Proof It is sufficient to verify the second and the third formulas of (2.3).

By $s^2-v^2\neq 0$ and the second and the third equalities of (2.9) we get

$$a_0=\frac{s\beta_0}{\sqrt{s^2-v^2}},\quad c_0=\frac{-2iv\beta_0}{\sqrt{s^2-v^2}}. \tag{2.10}$$

Since $|a_0|+|c_0|\neq 0$ and $s^2-v^2\neq 0$, we have $\beta_0\neq 0$. The third formulaof (2.9) at $l=l'$ and the fourth formula of (2.9) at $l+1=l'(1\leqslant l'\leqslant p)$ form a system of linear algebric equations of $a_{2l'}$ and $c_{2l'}$. Because the determinant of coefficients is

$$2a_0s-ivc_0=2\beta_0\sqrt{s^2-v^2}\neq 0,$$

$a_{2l'}$, $c_{2l'}(1\leqslant l'\leqslant p)$ can be solved.

By (2.10) the second and the third formulas of (2.3) are equavalent to the following

$$\left.\begin{aligned} &b_{2l+1}=\left[s\left(b_{2l-1}-2iua_{2l}+\frac{1}{2}c_{2lx}\right)-v(uc_{2l}-ia_{2lx})\right]\Big/(s^2-v^2),\\ &2a_0a_{2lx}+\frac{1}{2}c_0c_{2lx}=-c_0b_{2l-1}+2iuc_0a_{2l}-2iua_0c_{2l}. \end{aligned}\right\} \tag{2.11}$$

The first equality of (2.11) is just the last equality of (2.9). Differentiating the second and the fourth equalities of (2.9) to x, by induction we can get the second equality of (2.11) from (2.9). That is, every a_{2l}, c_{2l}, b_{2l+1} from (2.9) satisfies (2.11). Because $\beta_{2l}(0\leqslant l\leqslant p)$ are arbitrary constants (which are independent of x), the fourth formula of (2.9) is just the general integral of the second equation of (2.11). Thus, Lemma 2 is proved.

It is easy to get the following corollary from Lemma 1 and Lemma 2.

Corollary. *The nonlinear equations (2.4), (2.5) and (2.6) are all purely differential equations.*

By (2.1), (2.8) and (2.9) the following two lemmas can be proved.

Lemma 3. *Suppose that U is given by (0.1), V is given by (2.1) and $d_0\neq 0$. Then*$-\det V$ *is an even polynomial of λ and λ^{-1}, and the sum of all the terms whose power is not less than λ^{2k+2} is*

$$\alpha_0^2\lambda^{4k+2}+\sum_{m=0}^{k-1}\alpha_{2m+2}\lambda^{4k-2m}. \tag{2.12}$$

Lemma 4. *Suppose that U is given by (0.1), V is given by (2.1), $s^2-v^2\neq 0$ and $|c_0|+|a_0|\neq 0$. Then* $-\det V$ *is an even polynomial of λ and λ^{-1}, and the sum of all the terms whose power is not greater than λ^{-2p-2} is*

$$\beta_0^2\lambda^{-4p-2}+\sum_{l=0}^{p-1}\beta_{2l+2}\lambda^{2l-4p}. \tag{2.13}$$

By Lemma 3 and Lemma 4, we get the following corollary.

Corollary. *Suppose (0.3) is valid and V has the expansion of (2.1). Then the matrixes V_{2j}, $V_{2j+1}(0\leqslant j\leqslant k)$ are determined by α_0^2, $\alpha_{2j}(1\leqslant j\leqslant k)$ and α_0,* i.e., *by α_0 and the k coefficients of k highereven-degree terms in the expansion of* $-\det V$*. If V does not contain any terms of negative power, the evolution equation (2.5) is just determined by these integral constants $\alpha_{2j}(0\leqslant j\leqslant k)$ (which are independent of x). Similarly, all the* *matrixes* $W_l(0\leqslant l\leqslant 2p)$ *in expansion* (2.1) *of* V *are determined by* β_0^2, $\beta_{2l}(1\leqslant l\leqslant p)$ *and* β_0, i. e., *by* β_0 *and* p *coefficients of* p *lower even-degree terms in the expansion of* $-\det V$. *If there are negative power terms only in the expansion of* V, *then the evolution equation* (2.6) *is just determined by these integral constants* $\beta_{2l}(0\leqslant l\leqslant p)$, *which are not dependent on* x. *In general, the evolution equation is determined by* α_{2j} $(0\leqslant j\leqslant k)$ *and* $\beta_{2l}(0\leqslant l\leqslant p)$.

In this part, we have used the idea of [4].

§ 3. N-degree Gauge Transformations and N-degree Darboux Transformations

Comparing with part one, we see that to get an N-degree gauge transformation is equavalent to to get a matrix $T^{(N)}$ such that $T^{(N)}$ is an N-degree polynomial of λ and $T^{(N)}$ satisfies the following equations

$$\left.\begin{aligned} T_x^{(N)}&=\tilde{U}T^{(N)}-T^{(N)}U,\\ T_t^{(N)}&=\tilde{V}T^{(N)}-T^{(N)}V,\end{aligned}\right\}\tag{3.1}$$

where U is defined by (0.1), V is defined by (2.1), U and V are compatible, i. e., U and V satisfy the equation

$$U_t-V_x+UV-VU=0\tag{3.2}$$

and the dependent relation from $\tilde{u}$, $\tilde{v}$, $\tilde{s}$ to $\tilde{U}$, $\tilde{V}$ is the same as the dependent relation from u, v, s to U, V. In [3], $T^{(1)}$ has been directly solved. But the more large k, p and N, the more difficult to solve $T^{(N)}$ by (3.1) directly. The central problem of this paper is to solve gauge transformation $T^{(N)}$. We explicit that an N-degree gauge transformation depends on $2N$ parameters. We give the solvable condition of $T^{(N)}$ (fundamental hypothesis). And finally the problem to solve $T^{(N)}$ is turned to some linear problems. For convenience, at first we give the procedure of solving $T^{(N)}$ as definition, then we go to prove by definition that $T^{(N)}$ is really a gauge transformation, i.e., we go to prove that the dependent relation from $\tilde{u}$, $\tilde{v}$, $\tilde{s}$ to $\tilde{U}$, $\tilde{V}$ is the same as the dependent relation from u, v, s to U, V, and $T^{(N)}$ satisfies (3.1).

1. Definition of N-degree gauge transformation

Suppose φ is a fundamental matrix solution of equations:

$$\left.\begin{aligned} &\varphi_x=U\varphi,\ U=-i\lambda\sigma_3+u\sigma_1+\lambda^{-1}(is\sigma_3-v\sigma_2),\\ &\varphi_t=V\varphi,\ V \text{ has the expansion } (2.1).\end{aligned}\right\}\tag{3.3}$$

And suppose i) $\varphi(x, t, \lambda)$ has some analytic property to λ as we need afterward; ii) φ satisfies

$$\varphi(x, t, -\lambda)=\sigma_1\varphi(x, t, \lambda).\tag{3.4}$$

Hypothesis (3.4) is admissible because

$$[\sigma_1\varphi(x, t, -\lambda)]_x=U(\lambda)[\sigma_1\varphi(x, t, -\lambda)],$$

$$[\sigma_1\varphi(x,\ t,\ -\lambda)]_t=V(\lambda)[\sigma_1\varphi(x,\ t,\ -\lambda)].$$

Suppose $\lambda_j(1\leqslant j\leqslant N)$ are N constants which satisfy

$$\lambda_j\neq 0\quad(1\leqslant j\leqslant N),$$
$$\lambda_j\neq\lambda_l\quad(j\neq l,\ 1\leqslant j,\ l\leqslant N),$$

$\mu_j,\ \nu_j(1\leqslant j\leqslant N)$ are constants which satisfy

$$|\mu_j|+|\nu_j|\neq 0.$$

Let

$$\left.\begin{aligned}\phi_j(x,\ t)&=\begin{pmatrix}\phi_j^1(x,\ t)\\ \phi_j^2(x,\ t)\end{pmatrix}=\varphi(x,\ t,\ \lambda_j)\begin{pmatrix}\mu_j\\ \nu_j\end{pmatrix},\\ \psi_j(x,\ t)&=\begin{pmatrix}\psi_j^1(x,\ t)\\ \psi_j^2(x,\ t)\end{pmatrix}=\varphi(x,\ t,\ -\lambda_j)\begin{pmatrix}\mu_j\\ \nu_j\end{pmatrix},\end{aligned}\right\}\tag{3.5}$$

and

$$\left.\begin{aligned}&T^{(N)}=\sum_{j=0}^{N}T_j^{(N)}\lambda^{N-j},\\ &T_0^{(N)}=\mathrm{I}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\\ &T_j^{(N)}=\begin{pmatrix}T_{j,11}^{(N)}&T_{j,12}^{(N)}\\ T_{j,21}^{(N)}&T_{j,22}^{(N)}\end{pmatrix}\quad(1\leqslant j\leqslant N).\end{aligned}\right\}\tag{3.6}$$

Make equations

$$\left.\begin{aligned}&T^{(N)}(\lambda_j)\phi_j=0,\\ &T^{(N)}(-\lambda_j)\psi_j=0\end{aligned}\quad(1\leqslant j\leqslant N).\right\}\tag{3.7}$$

This is a system of linear algebraic equations, which contains $4N$ equations and $4N$ unknown functions $T_{j,11}^{(N)},\ T_{j,12}^{(N)},\ T_{j,21}^{(N)},\ T_{j,22}^{(N)}(1\leqslant j\leqslant N)$. Rewrite it as follows

$$\begin{pmatrix}\phi_1^1&\phi_1^2&\lambda_1\phi_1^1&\lambda_1\phi_1^2\cdots\lambda_1^{N-1}\phi_1^1&\lambda_1^{N-1}\phi_1^2\\ \psi_1^1&\psi_1^2&-\lambda_1\psi_1^1&-\lambda_1\psi_1^2\cdots(-\lambda_1)^{N-1}\psi_1^1&(-\lambda_1)^{N-1}\psi_1^2\\ \cdots&\cdots&\cdots&\cdots&\cdots\\ \phi_N^1&\phi_N^2&\lambda_N\phi_N^1&\lambda_N\phi_N^2\cdots\lambda_N^{N-1}\phi_N^1&\lambda_N^{N-1}\phi_N^2\\ \psi_N^1&\psi_N^2&-\lambda_N\psi_N^1&-\lambda_N\psi_N^2\cdots(-\lambda_N)^{N-1}\psi_N^1&(-\lambda_N)^{N-1}\psi_N^2\end{pmatrix}\begin{pmatrix}T_{N,l1}^{(N)}\\ T_{N,l2}^{(N)}\\ \vdots\\ T_{1,l1}^{(N)}\\ T_{1,l2}^{(N)}\end{pmatrix}$$
$$=-\begin{pmatrix}\lambda_1^N\phi_1^l\\ (-\lambda_1)^N\psi_1^l\\ \vdots\\ \lambda_N^N\phi_N^l\\ (-\lambda_N)^N\psi_N^l\end{pmatrix},\tag{3.8}$$

where $l=1,\ 2.$

To solve $T^{(N)}$ from (3.7) or (3.8), we suppose the following fundamental hypothesis is valid.

Fundamental hypothesis: Tne coeficient determinant of (3.8) is not equal to zero.

Under the fundamental hypothesis $T^{(N)}$ can be obtained uniquely from (3.8).

Because $\lambda_j \neq 0$, $\lambda_j \neq \lambda_l$ $(j \neq l,\ 1 \leqslant j,\ l \leqslant N)$, by linear algebra it is easy to prove that under the fundamental hypothesis, for any integer $k (1 \leqslant k \leqslant N)$ and for any permutation

$$\begin{pmatrix} 1, & 2, & \cdots, & k \\ i_1, & i_2, & \cdots, & i_k \end{pmatrix},$$

we have

$$\det \begin{pmatrix} \phi^1_{i_1} & \phi^2_{i_1} & \lambda_{i_1}\phi^1_{i_1} & \lambda_{i_1}\phi^2_{i_1} \cdots \lambda^{k-1}_{i_1}\phi^1_{i_1} & & \lambda^{k-1}_{i_1}\phi^2_{i_1} \\ \psi^1_{i_1} & \psi^2_{i_1} & -\lambda_{i_1}\psi^1_{i_1} & -\lambda_{i_1}\psi^2_{i_1} & (-\lambda_{i_1})^{k-1}\psi^1_{i_1} & (-\lambda_{i_1})^{k-1}\psi^2_{i_1} \\ & & \cdots\cdots & \cdots\cdots & \cdots\cdots & \\ \phi^1_{i_k} & \phi^2_{i_k} & \lambda_{i_k}\phi^1_{i_k} & \lambda_{i_k}\phi^2_{i_k} \cdots \lambda^{k-1}_{i_k}\phi^1_{i_k} & & \lambda^{k-1}_{i_k}\phi^2_{i_k} \\ \psi^1_{i_k} & \psi^2_{i_k} & -\lambda_{i_k}\psi^1_{i_k} & -\lambda_{i_k}\psi^2_{i_k} \cdots & (-\lambda_{i_k})^{k-1}\psi^1_{i_k} & (-\lambda_{i_k})^{k-1}\psi^2_{i_k} \end{pmatrix} \neq 0. \tag{3.9}$$

When we replace ϕ_j by $\rho_j \phi_j (\rho_j \neq 0)$ (by (3.4) and (3.5), ψ_j is replaced by $\rho_j \psi_j$), the equation (3.8) is invariant. So, under the fundamental hypothesis, the solution $T^{(N)}$ of (3.8) depends on $2N$ parameters $(\lambda_1, k_1; \lambda_2, k_2; \cdots; \lambda_N, k_N)$ where

$$k_j = \frac{\mu_j}{\nu_j} \quad (j = 1, 2, \cdots, N).$$

We define N-degree gauge transformatitn $\mathscr{T}$ as a transformation which acts on φ as follows

$$\mathscr{T}(\lambda_1, k_1; \lambda_2, k_2; \cdots; \lambda_N, k_N; \varphi)\varphi = T^{(N)}\varphi, \tag{3.10}$$

where $(\lambda_1, k_1; \lambda_2, k_2; \cdots; \lambda_N, k_N)$ expresses systems of parameters, 1-degree gauge transformation depends on one system of parameter, (λ_1, k_1), and N-degree gauge transformation depends on N systems of parameters. $T^{(N)}$ is called the matrix of N-degree gauge transformation. By definition it is easy to know by (3.8) that for any permutation

$$\begin{pmatrix} 1, & 2, & \cdots, & N \\ i_1, & i_2, & \cdots, & i_N \end{pmatrix}$$

the matrix of $\mathscr{T}(\lambda_1, k_1; \lambda_2, k_2; \cdots; \lambda_N, k_N; \varphi)$ is equal to the matrix of $\mathscr{T}(\lambda_{i_1}, k_{i_1}; \lambda_{i_2}, k_{i_2}; \cdots; \lambda_{i_N}, k_{i_N}; \varphi)$. So the N-degree gauge transformation is invariant under any change of the order of N system of parameters.

2. Properties and theorem

By the definition of D.T. and the fundamental hypothesis we get the following properties,

Property 1. The matrix of 1-degree gauge transformation $\mathscr{T}(\lambda_1, k_1; \varphi)$ is $T^1(\lambda_1, k_1; \varphi)$:

$$T^{(1)}(\lambda_1, k_1; \varphi) = \lambda I + a\sigma_3 + b\sigma_2,$$

$$\left.\begin{aligned} a&=\frac{\lambda_1[(\phi_1^2)^2+(\phi_1^1)^2]}{(\phi_1^2)^2-(\phi_1^1)^2}, \\ b&=\frac{-2i\lambda_1\phi_1^1\phi_1^2}{(\phi_1^2)^2-(\phi_1^1)^2}, \end{aligned}\right\} \tag{3.11}$$

and we have

$$\left.\begin{aligned} &a^2+b^2=\lambda_1^2, \\ &\det T^{(1)}=(\lambda-\lambda_1)(\lambda+\lambda_1), \end{aligned}\right\} \tag{3.12}$$

Property 2. Suppose $T^{(1)}$ is the matrix of $\mathscr{T}(\lambda_1, k_1, \varphi)$ (i.e., (3.11)). Let

$$\tilde{\varphi}=T^{(1)}\varphi.$$

Then we have

$$\left.\begin{aligned} &\tilde{\varphi}_x=\tilde{U}\tilde{\varphi}, \\ &\tilde{U}=-i\lambda\sigma_3+\tilde{u}\sigma_1+\lambda^{-1}(i\tilde{s}\sigma_3-\tilde{v}\sigma_2), \end{aligned}\right\} \tag{3.13}$$

and

$$\left\{\begin{aligned} \tilde{u}&=u+2b, \\ \tilde{v}&=\frac{-2iabs-(a^2-b^2)v}{\lambda_1^2}, \\ \tilde{s}&=\frac{(a^2-b^2)s+2iabv}{\lambda_1^2}. \end{aligned}\right. \tag{3.14}$$

Proof It is sufficient to prove the first equality of (3.13) in the case that $\tilde{\varphi}=T^{(1)}\varphi$, (3.14) and the second equality of (3.13) hold. Consider

$$D=T_x^{(1)}+T^{(1)}U-\tilde{U}T^{(1)}.$$

Using the expressions of U and $\tilde{U}$, (3.11), (3.14) and $T^{(1)}(\lambda_1)\phi_1=0$, we get that D is independent of λ and $D\phi_1=0$. Similarly, $D\psi_1=0$. By hypothesis, $\det(\phi_1, \psi_1)\neq 0$, so $D=0$, i.e.,

$$T_x^{(1)}+T^{(1)}U-\tilde{U}T^{(1)}=0. \tag{3.15}$$

Because of $\det T^{(1)}=(\lambda-\lambda_1)(\lambda+\lambda_1)$, $T^{(1)}$ is invertible as $\lambda\neq\pm\lambda_1$. If φ has some proper analytic property, then letting $\lambda\to\lambda_1$ (or $\lambda\to-\lambda_1$) in $\tilde{\varphi}_x=\tilde{U}\tilde{\varphi}$ ($\lambda\neq\pm\lambda_1$) we get that $\tilde{\varphi}_x=\tilde{U}\tilde{\varphi}$ is also valid at $\lambda=\pm\lambda_1$, that is, for all λ, $\tilde{\varphi}_x=\tilde{U}\tilde{\varphi}$. The property is proved.

Definition. (3.11) *together with* (3.14) *is called a* 1-*degree Darboux Transformation.*

Comparing with part 1, we see that the 1-degree D.T. here is the same as (1.9) and (1.10) in part 1.

Property 3. If

$$\tilde{\varphi}(x, t, \lambda)=\mathscr{T}(\lambda_1, k_1, \varphi)\varphi(x, t, \lambda), \tag{3.16}$$

then

$$\tilde{\varphi}(x, t, -\lambda)=-\sigma_1\tilde{\varphi}(x, t, \lambda), \tag{3.17}$$

$$\tilde{v}^2-\tilde{s}^2=v^2-s^2. \tag{3.18}$$

Proof It can be directly verified.

Property 4. Suppose the matrix of $\mathscr{T}(\lambda_1, k_1; \lambda_2, k_2; \cdots; \lambda_j, k_j; \varphi)$ is

$$T^{(j)}=\lambda^j I+\sum_{l=1}^{j} T_l^{(j)}\lambda^{j-l} \quad (j\geqslant 1),$$

the matrix of $\mathscr{T}(\lambda_1, k_1; \lambda_2, k_2; \cdots; \lambda_{j+1}, k_{j+1}; \varphi)$ is

$$T^{(j+1)}=\lambda^{j+1}I+\sum_{l=1}^{j+1} T_l^{(j+1)}\lambda^{j+1-l} \quad (j+1\leqslant N)$$

and the fundamental hypothesis is valid. Then there is a matrix R which does not depend on λ such that

$$T^{(j+1)}=(\lambda I+R)T^{(j)}$$

and $\lambda I+R$ is just the matrix of $\mathscr{T}(\lambda_{j+1}, k_{j+1}; T^{(j)}\varphi)$.

Proof By the definition of gauge transformation and the fundamental hypothesis, $T^{(j)}$ and $T^{(j+1)}$ exist and have the expansions of this property. Let

$$R=T_1^{(j+1)}-T_1^{(j)},$$
$$D=T^{(j+1)}-(\lambda I+R)T^{(j)}.$$

It is obvious that D is a polynomial of λ whose degree is not greater than $j-1$. We suppose

$$D=\sum_{l=0}^{j-1} D_l\lambda^{j-1-l}.$$

Using the symbols of (3.5) by the definition of gauge transformation we have

$$\begin{matrix} T^{(j)}\phi_m=0, \ T^{(j)}\psi_m=0 \\ T^{(j+1)}\phi_m=0, \ T^{(j+1)}\psi_m=0 \end{matrix} \quad (1\leqslant m\leqslant j).$$

So

$$D\phi_m=0, \ D\psi_m=0 \quad (1\leqslant m\leqslant j).$$

That is

$$\sum_{l=0}^{j-1} D_l\lambda_m^{j-1-l}\phi_m=0 \quad (1\leqslant m\leqslant j),$$

$$\sum_{l=0}^{j-1} D_l(-\lambda_m)^{j-1-l}\psi_m=0 \quad (1\leqslant m\leqslant j).$$

This is a system of $4N$ homogeneous linear algebraic equations in which there are $4N$ unknown variables. By the fundamental hypothesis and (3.9) we get $D_l=0 (0\leqslant l\leqslant j-1)$, i.e., $D=0$. So

$$T^{(j+1)}=(\lambda I+R)T^{(j)}.$$

By the definition of $T^{(j+1)}$, $T^{(j+1)}$ satisfies

$$T^{(j+1)}(\lambda_{j+1})\phi_{j+1}=0, \ T^{(j+1)}(-\lambda_{j+1})\psi_{j+1}=0.$$

So

$$(\lambda_{j+1}I+R)T^{(j)}(\lambda_{j+1})\phi_{j+1}=0,$$
$$(-\lambda_{j+1}I+R)T^{(j)}(-\lambda_{j+1})\psi_{j+1}=0.$$

Thus $\lambda I+R$ is the matrix of $\mathscr{T}(\lambda_{j+1}, k_{j+1}; T^{(j)}\varphi)$. In other hand, if $\lambda I+R'$ is also the matrix of $\mathscr{T}(\lambda_{j+1}, k_{j+1}; T^{(j)}\varphi)$, then $(\lambda I+R')T^{j}$ must be the matrix of $\mathscr{T}(\lambda_1, k_1; \cdots; \lambda_{j+1}, k_{j+1}; \varphi)$. By the uniqueness of the matrix of gauge transformation we get $T^{(j+1)} = (\lambda I+R')T^{(j)}$. Comparing the coefficients of λ^j we get $R'=T_1^{(j+1)}-T_1^{(j)}$. Remembering that we have let $R=T_1^{(j+1)}-T_1^{(j)}$, we get $R'=R$. That is, $\lambda I+R$ is the unique matrix of $\mathscr{T}(\lambda_{j+1}, k_{j+1}; T^{(j)}\varphi)$.

Theorem 1 (decomposition theorem). *Suppose $\lambda_j (1\leqslant j\leqslant N)$ are complex constants which satisfy $\lambda_j\neq 0$ $(1\leqslant j\leqslant N)$, $\lambda_j\neq\lambda_l$ $(j\neq l,\ 1\leqslant j,\ l\leqslant N)$, $k_j=\frac{\mu_j}{\nu_j}$ are complex constants $(|\mu_j|+|\nu_j|\neq 0)$. Suppose $\varphi(x, t, \lambda)$ is the fundamental solution matrix of (3.3), φ satisfies (3.4) and the fundamental hypothesis is valid. Given any permutation*

$$\begin{pmatrix} 1, & 2, & \cdots, & N \\ i_1, & i_2, & \cdots, & i_N \end{pmatrix},$$

let

$$\varphi_j=\mathscr{T}(\lambda_{i_1}, k_{i_1}; \lambda_{i_2}, k_{i_2}; \cdots; \lambda_{i_j}; k_{i_j}; \varphi)\varphi. \tag{3.19}$$

Then N-degree gauge transformation can be decomposed as the product of N 1-degree gauge transformations

$$\mathscr{T}(\lambda_1, k_1;\lambda_2, k_2; \cdots; \lambda_N, k_N; \varphi)\varphi$$
$$=\mathscr{T}(\lambda_{i_N},k_{i_N}; \varphi_{N-1})\mathscr{T}(\lambda_{i_{N-1}}, k_{i_{N-1}}; \varphi_{N-2})\cdots\mathscr{T}(\lambda_{i_1}, k_{i_1}; \varphi)\varphi. \tag{3.20}$$

If we represent the matrix of $\mathscr{T}(\lambda_1, k_1; \lambda_2, k_2, \cdots; \lambda_N, k_N; \varphi)$ by $T^{(N)}$, represent the matrix of $\mathscr{T}(\lambda_{i_j}, k_{i_j}; \varphi_{j-1})$ by M_j, then (3.20) can be rewritten as follaws

$$T^{(N)}=M_N M_{N-1}\cdots M_1. \tag{3.21}$$

n addition, the highest power term in the expansion of $T^{(N)}$ is $\lambda^N I$, and the highest power term in the expansion of M_j is equal to λI $(1\leqslant j\leqslant N)$. (3.21) says that matrix $T^{(N)}$ can be rewritten as the product of N 1-degree polynomial matrixes of λ.

Summarily. *N-degree gauge transformation can be decomposed as the product of N 1-degree gauge transformations and the decomposition is independent of the order of systems of parameters.*

Proof By definition

$$\mathscr{T}(\lambda_1, k_1; \lambda_2, k_2; \cdots; \lambda_N, k_N; \varphi) =\mathscr{T}(\lambda_{i_1}, k_{i_1}; \lambda_{i_2}, k_{i_2}; \cdots; \lambda_{i_N}, k_{i_N}; \varphi).$$

Using Property 4 once and again we get (3.20). (3.21) is the matrix' representation of (3.20). Theorem 1 is proved.

Corollary. *The determinent of matrix $T^{(N)}$ of N-degree gauge transformation is*

$$\det T^{(N)}=\prod_{j=1}^{N}(\lambda-\lambda_j)(\lambda+\lambda_j).$$

Note. Suppose that in the theorem $T^{(N)}$ is

$$T^{(N)}=\lambda^N I+\sum_{j=1}^{N}T_j^{(N)}\lambda^{n-j}$$

and

$$T^{(N)}=M_N M_{N-1}\cdots M_1.$$

By Property 1 we know that all the matrixes $M_j-\lambda I\,(1\leqslant j\leqslant N)$ are independent of λ, and in the decomposition on basis $(I,\ \sigma_1,\ \sigma_2,\ \sigma_3)$ there are only the terms of σ_2 and σ_3. It can be verified by induction that when j is even,

$$T_j^{(N)}=T_{j,11}^{(N)}I+T_{j,12}^{(N)}\sigma_1,$$

and when j is odd,

$$T_j^{(N)}=T_{j,11}^{(N)}\sigma_3+iT_{j,12}^{(N)}\sigma_2.$$

Property 5. Suppose

$$\varphi_x=U\varphi,\ U=-i\lambda\sigma_3+u\sigma_1+\lambda^{-1}(is\sigma_3-v\sigma_2)$$

and suppose the matrix of N-degree gauge transformation $\mathscr{T}(\lambda_1,\ k_1;\ \cdots;\ \lambda_N,\ k_N;\ \varphi)$ is

$$T^{(N)}=\lambda^N I+\sum_{j=1}^{N}T_j^{(N)}\lambda^{N-j}$$

and

$$\tilde{\varphi}=T\varphi.$$

Then we have

$$\tilde{\varphi}_x=\tilde{U}\tilde{\varphi},\ \tilde{U}=-i\lambda\sigma_3+\tilde{u}\sigma_1+\lambda^{-1}(i\tilde{s}\sigma_3-\tilde{v}\sigma_2)\tag{3.22}$$

and

$$\begin{aligned}\tilde{u}&=u+2iT_{1,12}^{(N)},\\ \tilde{s}&=\frac{[(T_{N,11}^{(N)})^2+(T_{N,12}^{(N)})^2]s-2T_{N,11}^{(N)}T_{N,12}^{(N)}v}{(T_{N,11}^{(N)})^2-(T_{N,12}^{(N)})^2},\\ \tilde{v}&=\frac{2T_{N,11}^{(N)}T_{N,12}^{(N)}s-[(T_{N,11}^{(N)})^2+(T_{N,12}^{(N)})^2]v}{(T_{N,11}^{(N)})^2-(T_{N,12}^{(N)})^2}\cdot(-1)^{N-1}.\end{aligned}\tag{3.23}$$

Proof By use of Theorem 1, N-degree gauge transformation can be rewritten as the product of N 1-degree gauge transformations. To every one of 1-degree gauge transformations using Property 2 once and again, we finally get that $\tilde{\varphi}_x=\tilde{U}\tilde{\varphi}$ and $\tilde{U}$ has an expansion of (3.22). In the expansion of $T_x+TU-\tilde{U}T=0$, comparing the coefficients of λ^{-1} and λ^N and using the note of Theorem 1, we can obtain (3.23).

Definition. (3.23) *together with N-degree gauge transfomation is called N-degree Darboux Transformation.*

Corollary to Theorem 1 and Property 5. N-degree D.T. depends on N systems of parameters $(\lambda_1,\ k_1;\ \lambda_2,\ k_2;\ \cdots;\ \lambda_N,\ k_N)$. And it can be decomposed as N 1-degree D.T. every one of which corresponds one system of parameters $(\lambda_j,\ k_j)$ respectively. If the systems of $2N$ parameters are invariant, then the N 1-degree D.T. in the decomposition are commutative.

By calculating we get the following property.

Property 6 (superposition theorem). Suppose the matrix of $\mathscr{T}(\lambda_1,\ k_1;\ \varphi)$ is

$$\lambda I+a\sigma_3+b\sigma_2,$$

the matrix of $\mathscr{T}(\lambda_2,\ k_2;\ \varphi)$ is

$$\lambda I+a'\sigma_3+b'\sigma_2.$$

Then the matrix of $\mathscr{T}(\lambda_1,\ k_1;\ \lambda_2,\ k_2;\ \varphi)$ has the decompositions

$$T^{(2)}=(\lambda I+R)(\lambda I+a\sigma_3+b\sigma_2)=(\lambda I+R')(\lambda I+a'\sigma_3+b'\sigma_2), \tag{3.24}$$

where

$$R=\frac{1}{(b'-b)^2+(a'-a)^2}\{[a'(a'-a)^2+(b'-b)(a'b+a'b'-2ab')])\sigma_3 \\ +(b'(b'-b)^2+(a'-a)(b'a+b'a'-2ba')])\sigma_2\},$$

$$R'=\frac{1}{(b-b')^2+(a-a')^2}\{[a(a-a')^2+(b-b')(ab'+ab-2a'b)]\,\sigma_3 \\ +[b(b-b')^2+(a-a')(ba'+ba-2b'a)]\sigma_2\}.$$

If

$$(v,\ v,\ s)\ \begin{cases} \xrightarrow{\mathscr{T}(\lambda_1,\ k_1;\ \varphi)} (u_1,\ v_1\ s_1) \\ \xrightarrow{\mathscr{T}(\lambda_1,\ k_1;\ \lambda_2,\ k_2;\ \varphi)} (\tilde{u},\ \tilde{v},\ \tilde{s}), \\ \xrightarrow[\mathscr{T}(\lambda_2,\ k_2;\ \varphi)]{} (u_2,\ v_2,\ s_2) \end{cases}$$

then

$$\left.\begin{aligned}
&\tilde{u}=u_1+2r_2,\\
&\tilde{s}s_1+\tilde{v}v_1=(s^2-v^2)\left(1-\frac{2r_2^2}{\lambda_2^2}\right),\\
&\tilde{s}s_2+\tilde{v}v_2=(s^2-v^2)\left(1-\frac{2r_2'^2}{\lambda_1^2}\right),\\
&r_2=\frac{1}{(b'-b)^2+(a'-a)^2}[b'(b'-b)^2+(a'-a)(b'a+b'a'-2ba')],\\
&r_2'=\frac{1}{(b-b')^2+(a-a')^2}[b(b-b')^2+(a-a')(ba'+ba-2b'a)],\\
&a=\frac{\lambda_1^2(sv_1+vs_1)}{i(u_1-u)(v^2-s^2)},\quad a'=\frac{\lambda_2^2(sv_2+vs_2)}{i(u_2-u)(v^2-s^2)},\\
&b=\frac{u_1-u}{2},\quad b'=\frac{u_2-u}{2},
\end{aligned}\right\} \tag{3.26}$$

$\tilde{u},\ \tilde{v},\ \tilde{s}$ can be got from (3.26) directly.

§ 4. The Nonlinear Evolution Equations Satisfied by the Potentials after D.T.

In this part we go to prove that through D.T.

$$\tilde{\varphi}=T^{(N)}\varphi$$

$\tilde{\varphi}$satisfies

$$\tilde{\varphi}_t=\tilde{V}\tilde{\varphi},$$

where $\tilde{V}$ can be obtained from the expression (2.1) of V by replacing V_j with $\tilde{V}_j$, W_l with $\tilde{W}_l$, u, v, s with $\tilde{u}$, $\tilde{v}$, $\tilde{s}$. And we go to prove that the dependent relations from $\tilde{u}$, $\tilde{v}$, $\tilde{s}$ to $\tilde{V}$ is the same as the relation from u, v, s to V. On the other hand we go to prove that $T^{(N)}$ satisfies

$$T_t^{(N)}=\tilde{V}T^{(N)}-T^{(N)}V$$

which is the second equality of (3.1). Thus D.T. gives a method to ask for a new system of solutions $\tilde{u}$, $\tilde{v}$, $\tilde{s}$ from a given system of solutions u, v, s of the same system of evolution equations. Since N-degree D.T. can be decomposed as the product of N 1-degree D.T., to complete above proof, it is sufficient to prove above conclusions under 1-degree D.T..

Lemma 5. *Suppose that $T^{(1)}$ is the matrix of 1-degree gauge transformation, V has the expansion (2.1). Let*

$$\tilde{V}=(T_t^{(1)}+T^{(1)}V)(T^{(1)})^{-1}. \tag{4.1}$$

Then $\tilde{V}$ is also a polynomial of λ and λ^{-1} in which the highest power of λ is λ^{2k+1} and the lowest power of λ is λ^{-2p-1}.

Proof By the definition of $T^{(1)}$, $T^{(1)}(\lambda_1)\phi_1=0$. Differentiating it with respect to t we get $[T_t^{(1)}(\lambda_1)+T^{(1)}(\lambda_1)V(\lambda_1)]\phi_1=0$. Since $\phi_1\neq\begin{pmatrix}0\\0\end{pmatrix}$, by virtue of linear algebra the equality $(T_t^{(1)}(\lambda_1)+T^{(1)}(\lambda_1)V(\lambda_1))(T^{(1)}(\lambda_1))^*=0$ can be proved, where $(T^{(1)}(\lambda_1))^*$ is the companion matrix of $T^{(1)}(\lambda_1)$). So at $\lambda=\lambda_1$,

$$(T_t^{(1)}+T^{(1)}V)(T^{(1)})^*|_{\lambda=\lambda_1}=0.$$

It is similar to prove that at $\lambda=-\lambda_1$,

$$(T_t^{(1)}+T^{(1)}V)(T^{(1)})^*|_{\lambda=-\lambda_1}=0.$$

So every componant of $(T_t^{(1)}+T^{(1)}V)(T^{(1)})^*$ has the factor $(\lambda-\lambda_1)(\lambda+\lambda_1)$. Noticing (3.12) we know that

$$(T_t^{(1)}+T^{(1)}V)(T^{(1)})^{-1}=(T_t^{(1)}+T^{(1)}V)(T^{(1)})^*/\det T^{(1)}$$

is also a polynomial of λ and λ^{-1}. Comparing the highest and lowest power of λ, we know that in the expansion of $\tilde{V}$, λ^{2k+1} is the highest power and λ^{-2p-1} is the lowest power.

Lemma 6. *Suppose $T^{(1)}$ is the matrix of 1-degree gauge transformation, V has the expansion (2.1), and $\tilde{V}=(T_t^{(1)}+T^{(1)}V)(T^{(1)})^{-1}$. Then $\tilde{V}$ has an analogic expansion as V. More precisely, we have*

$$\left.\begin{aligned}
\tilde{V}&=\sum_{j=0}^{k}\tilde{V}_{2j}\lambda^{2k+1-2j}+\sum_{j=0}^{k}\tilde{V}_{2j+1}\lambda^{2k-2j}+\sum_{l=0}^{2p}\tilde{W}_l\lambda^{l-2p-1},\\
\tilde{V}&=\tilde{d}_{2j}\sigma_3-\frac{1}{2}\tilde{f}_{2j}\sigma_2,\quad \tilde{V}_{2j+1}=\frac{1}{2}\tilde{e}_{2j+1}\sigma_1\quad(0\leqslant j\leqslant k),\\
\tilde{W}_{2l}&=\tilde{a}_{2l}\sigma_3-\frac{1}{2}\tilde{c}_{2l}\sigma_2\quad(0\leqslant l\leqslant p),\\
\text{as } p\geqslant 1,\ \tilde{W}_{2l+1}&=\frac{1}{2}\tilde{b}_{2l+1}\sigma_1\quad(0\leqslant l\leqslant p-1).
\end{aligned}\right\} \tag{4.2}$$

Proof By (4.1)

$$\tilde{V}T^{(1)}=T_t^{(1)}+T^{(1)}V.$$

By Lemma 1, we can suppose that the first equality of (4.2) is valid and we go to prove the other equality of (4.2). By (3.11)

$$T^{(1)}=\lambda I+a\sigma_3+b\sigma_2.$$

Substituting the first equality of (4.2) to the equality

$$\tilde{V}T^{(1)}=T_t^{(1)}+T^{(1)}V$$

and comparing coefficients of different power, we get

$$\left.\begin{array}{l}\tilde{V}_0=V_0,\\ \tilde{V}_{j+1}+\tilde{V}_j(a\sigma_3+b\sigma_2)=V_{j+1}+(a\sigma_3+b\sigma_2)V_j \quad (0\leqslant j\leqslant 2k),\\ \tilde{V}_{2k+1}(a\sigma_3+b\sigma_2)+\tilde{W}_{2p}=(a\sigma_3+b\sigma_2)_t+(a\sigma_3+b\sigma_2)V_{2k+1}+W_{2p},\\ \tilde{W}_{l-1}+\tilde{W}_l(a\sigma_3+b\sigma_2)=W_{l-1}+(a\sigma_3+b\sigma_2)W_l \quad (1\leqslant l\leqslant 2p),\\ \tilde{W}_0(a\sigma_3+b\sigma_2)=(a\sigma_3+b\sigma_2)W_0.\end{array}\right\} \tag{4.3}$$

By (4.2) and (2.1), (4.3) can be proved by induction. Lemma 6 is proved.

By the way, from the last equality of (4.3) and the following equality

$$(a\sigma_3+b\sigma_2)(a\sigma_3+b\sigma_2)=(a^2+b^2)I\overset{(3.12)}{=\!=}\lambda_1^2 I$$

we get

$$\left.\begin{array}{l}\tilde{a}_0=[(a^2-b^2)a_0-abc_0]/\lambda_1^2,\\ \tilde{c}_0=[-4aba_0-(a^2-b^2)c_0]/\lambda_1^2.\end{array}\right\} \tag{4.4}$$

Lemma 7. *If* (3.1) *is valid, then*

$$(\tilde{U}_t-\tilde{V}_x+\tilde{U}\tilde{V}-\tilde{V}\tilde{U})T^{(N)}=T^{(N)}(U_t-V_x+UV-VU). \tag{4.5}$$

Proof Direct calculus.

Lemma 8. *Under 1-degree gauge transformation,* $\tilde{d}_0$ *and* d_0 *is independent of potentials and* $\tilde{d}_0=d_0$, $\tilde{a}_0$, $\tilde{c}_0$ *depend on* $\tilde{u}$, $\tilde{v}$, $\tilde{s}$ *and the dependent relation from* $\tilde{u}$, $\tilde{v}$, $\tilde{s}$ *to* $\tilde{a}_0$, $\tilde{c}_0$ *is the same as the dependent relation from* u, v, s *to* a_0, c_0.

Proof By the first equality of (4.3) $\tilde{d}_0=d_0$ which is independent of potentials. By (4.1) and (3.15) we know that as $N=1$, (3.1) is valid. By (3.2), (4.5) and the fact that $T^{(1)}$ is inversible when $\lambda\neq\pm\lambda_1$, we see that $\tilde{U}$, $\tilde{V}$ satisfy the following equation

$$\tilde{U}_t-\tilde{V}_x+\tilde{U}\tilde{V}-\tilde{V}\tilde{U}=0. \tag{4.6}$$

By (4.2) $\tilde{V}$ has an analogic expansion as V does. By (4.6) and by (2.10) of Lemma 1 and Lemma 2, we have

$$\tilde{a}_0=\frac{\tilde{s}\tilde{\beta}_0}{\sqrt{\tilde{s}^2-\tilde{v}^2}},\qquad \tilde{c}_0=\frac{-2i\tilde{v}\tilde{\beta}_0}{\sqrt{\tilde{s}^2-\tilde{v}^2}}. \tag{4.7}$$

By (3.18)

$$\tilde{v}^2-\tilde{s}^2=v^2-s^2.$$

By (3.14)

$$\tilde{v}=\frac{-2iabs-(a^2-b^2)v}{\lambda_1^2}$$

$$\tilde{s}=\frac{(a^2-b^2)s+2iabv}{\lambda_1^2}.$$

By (3.12)

$$a^2+b^2=\lambda_1^2.$$

Using these equalities and comparing the right hands of (4.7) and (4.4), by $v^2-s^2\neq 0$ and (2.10) we get

$$\beta_0=\tilde{\beta}_0.$$

By (4.7) and (2.10), we know that the dependent relation from $\tilde{u}$, $\tilde{v}$, $\tilde{s}$ to $\tilde{a}_0$, $\tilde{c}_0$ is the same as the dependent relation from u, v, s to a_0, c_0. The lemma is proved.

Lemma 9. *Suppose that $T^{(1)}$ is the matrix of 1-degree gauge transformation, and V has the expansion (2.1). Let*

$$\tilde{V}=(T_t^{(1)}+T^{(1)}V)(T^{(1)})^{-1}.$$

Then in the expansion of $\det\tilde{V}$ the sum of the terms whose power is not higher than λ^{-2p-2} or whose power is not lower than λ^{2k+2} is equal to the corresponding sum in the expansion of $\det V$. That is

$$\left.\begin{array}{l}\tilde{\alpha}_0^2=\alpha_0^2,\ \tilde{\beta}_0^2=\beta_0^2,\\ \text{as } k\geqslant 1,\ \tilde{\alpha}_{2j}=\alpha_{2j}\quad (1\leqslant j\leqslant k),\\ \text{as } l\geqslant 1,\ \tilde{\beta}_{2l}=\beta_{2l}\quad (1\leqslant l\leqslant p).\end{array}\right\}\tag{4.8}$$

Proof By hypothesis

$$\tilde{V}T^{(1)}=T_t^{(1)}+T^{(1)}V.\tag{4.9}$$

Because V has the expansion (2.1) and $T^{(1)}$ is a linear polynomial of λ, the highest power of the expansion of $T^{(1)}V$ is λ^{2k+2} and the lowest power is λ^{-2p-1}. Because $T_t^{(1)}$ is independent of λ, in the expansion of $\det(T_t^{(1)}+T^{(1)}V)$ the sum of all terms whose power is not less than λ^{2k+3} or whose power is not higher than λ^{-2p-2} is equal to the sum of the corresponding terms of $\det T^{(1)}V$, that is, the sum is equal to

$$\alpha_0^2\lambda^{4k+4}+\sum_{j=0}^{k-1}(\alpha_{2j+2}-\alpha_{2j}\lambda_1^2)\lambda^{4k+2-2j}-\beta_0^2\lambda_1^2\lambda^{-4p-2}+\sum_{l=1}^{p}(\beta_{2l-2}-\beta_{2l}\lambda_1^2)\lambda^{2l-4p-2}.\tag{4.10}$$

On the other hand, by Lemma 6, $\tilde{V}$ has the same form of expansion as V does. Using equation (4.6) and Lemma 1 to Lemma 4 and noticing $\det T^{(1)}=\lambda^2-\lambda_1^2$, we see that, in the expansion of $-\det(\tilde{V}T^{(1)})$, the sum of all terms whose power is not lower than λ^{2k+3} or whose power it not higher than λ^{-2p-2} is

$$\tilde{\alpha}_0^2\lambda^{4k+4}+\sum_{j=0}^{k-1}(\tilde{\alpha}_{2j+2}-\tilde{\alpha}_{2j}\lambda_1^2)\lambda^{4k+2-2j}-\tilde{\beta}_0^2\lambda_1^2\lambda^{-4k-2}+\sum_{l=1}^{p}(\tilde{\beta}_{2l-2}-\tilde{\beta}_{2l}\lambda_1^2)\lambda^{2l-4p-2}.\tag{4.11}$$

Calculating determinants of both sides of (4.9), by above discussion, (4.10) is equal to (4.11). Comparing the coefficients of different powers, we get (4.8).

Theorem 2. *Under N-degree Darboux transformation, the new potentials $\tilde{u}$, $\tilde{v}$, $\tilde{s}$ satisfy the same evolution equations as u, v, s satisfy.*

Proof Because N-degree D.T. can be decomposed as the product of N 1-degree

D.T., to prove the theorem, it is sufficient to prove it under 1-degree D.T..

By (4.2) we know that the expansion of $\tilde{V}$ can be obtained from the expansion of V by replacing a_{2l}, b_{2l+1}, c_{2l}, d_{2j}, e_{2j+1}, f_{2j} with $\tilde{a}_{2l}$, $\tilde{b}_{2l+1}$, $\tilde{c}_{2l}$, $\tilde{d}_{2j}$, $\tilde{e}_{2j+1}$, $\tilde{f}_{2j}$. By (3.13) we know that the expression of $\tilde{U}$ can be obtained from the expansion of U by replacing u, v, s with $\tilde{u}$, $\tilde{v}$, $\tilde{s}$. Using equation (4.6), Lemma 1 and Lemma 2, we see that the recursion formulas from $\tilde{u}$, $\tilde{v}$, $\tilde{s}$ to $\tilde{a}_{2l}$, $\tilde{b}_{2l+1}$, $\tilde{c}_{2l}$, $\tilde{d}_{2j}$, $\tilde{e}_{2j+1}$, $\tilde{f}_{2j}$ are the same as the recursion formulas from u, v, s to a_{2l}, b_{2l+1}, c_{2l}, d_{2j}, e_{2j+1}, f_{2j}. Because in the recursion formulas (2.8) and (2.9) there is not any integral operation. To prove the dependent relation from $\tilde{u}$, $\tilde{v}$, $\tilde{s}$ to $\tilde{a}_{2l}$, $\tilde{b}_{2l+1}$, $\tilde{c}_{2l}$, $\tilde{d}_{2j}$, $\tilde{e}_{2j+1}$, $\tilde{f}_{2j}$ is the same as the dependent relation from u, v, s to a_{2l}, b_{2l+1}, c_{2l}, d_{2j}, e_{2j+1}, f_{2j}, it is sufficient to prove that the corresponding integral constants by which the dependent relation can be uniquely determined are equal to each other resrectively. These have been proved in Lemma 8 and Lemma 9. Thus we assert tnat the dependent relation from $\tilde{u}$, $\tilde{v}$, $\tilde{s}$ to $\tilde{V}$, $\tilde{U}$ is just the same as the dependent relation from u, v, s to V, U. So $\tilde{u}$, $\tilde{v}$, $\tilde{s}$ satisfy the same evolution equations as u, v, s does. Theorem 2 is proved.

§ 5. The Relation of D.T. and B.T.

In [2], the Bäcklund gauge is obtained from (3.1). We have proved adove that the matrix of D.T. satisfies (3.1). So the old and new potentials also satisfy the B.T. formulas. That is, giving a system of solutions u, v, s, the new potentials $\tilde{u}$, $\tilde{v}$ and $\tilde{s}$ are just a system of solutions of B.T. equations (in general, they are some nonlinear equations). Analysing above solving proceeding we notice that in our case only it is necessary to solve some linear equations, that is, to solve equation

$$d\phi = U\phi\, dx + V\phi\, dt$$

and to solve the linear algebraic equations (3.8). And by (3.23) we can get the new potentials directly. Summarily, when using D.T. to ask for new solutions, it is sufficient to solve some linear problems. This is our motive to discuss the D.T..

References

[1] Boiti, M., and Tu, G. Z., A simple apporoach to the Hamilton structure of soliton equation, *Nuovo cimento*, **75B** (1983), 145.

[2] Boiti, M., Jorome J. P. Leon and Fempindli, F., A recursive generation of local higher-order Sine-Godon equations and their Bäcklund transformation, *J. Math. Phys.*, **25**: 6 (1984), 1725—1734.

[3] Li Yi-Shen, Darboux transformation of an eigenvalue problem and soliton solutions of Sine-Godon equation which contain an exterior field, submitted to Acta Mathematicae Applicatae Sinica (in Chinese).

[4] Li Yi-Shen, Some relation formulas deduced from structure equation of the SL(2, R) group, *Journal of China University of Science and Technology* (*Math. Issue*) (in Chinese).