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INITIAL BOUNDARY VALUE PROBLEM FOR ONE CLASS OF SYSTEM OF MULTIDIMENSIONAL INHOMOGENEOUS GBBM EQUATIONS

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Abstract

This paper studies the following initial-boundary value problem for the system of multidimensional inhomogeneous GBBM equations

$$u_t - \Delta u_t + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \operatorname{grad} \varphi(u) = f(u), \tag{1.1}$$

$$u|_{t=0} = u_0(x), \quad x \in \Omega, \tag{1.2}$$

$$u|_{\partial\Omega} = 0, \quad t \geqslant 0, \tag{1.3}$$

where $u(x, t) = (u_1(x, t), \cdots, u_N(x, t))$, $f(u) = (f_1(u_1, \cdots, u_N), \cdots, f_N(u_1, \cdots, u_N))$, $\varphi(u) = \varphi(u_1, \cdots, u_N)$. The existence and uniqueness of the global solution for the problem (1.1) (1.2) (1.3) are proved. The asymptotic behavior and "blow up" phenomenon of the solution for the problem (1.1) (1.2) (1.3) are investigated under certain conditions

§ 1. Introduction

BBM equation has been proposed and studied by Benjamin, Bona and Mahony in specific physical situations under longwave limit in nonlinear dispersive media. In [1, 2] J. A. Goldstein et al. have proposed and studied Generalized BBM equation in higher dimensions (GBBM). In this paper, by using the Galerkin approximation method, we prove the existence and uniqueness of the initial boundary value problem for the system of GBBM equations, and study the regularity and "blow up" of the solution for GBBM equations.

Here, we adopt the usual notation and convention. Let $H^m(\Omega)$ denote the Sobolev space with the norm $\|u\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \leqslant m} \|D^\alpha u\|^2_{L_2} \right)^{\frac{1}{2}}$ or simply $\|u\|_m$; $H_0^m(\Omega)$ denote the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$; $\|u\|_{L_\infty} = \operatorname{ess\,sup}_{x \in \Omega} |u(x)|$, and so on (see [3]).

We first consider the following initial-boundary value problem of multidimensional inhomogeneous GBBM equation

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$$u_t-\Delta u_t+\nabla\cdot\boldsymbol{\varphi}(u)=f(u), \tag{1.1}$$

$$u|_{t=0}=u_0(x), \tag{1.2}$$

$$u|_{\partial\Omega}=0, \tag{1.3}$$

where

$$\Delta\equiv\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2},\quad \nabla\equiv\left(\frac{\partial}{\partial x_1},\ \cdots,\ \frac{\partial}{\partial x_n}\right),$$

$$\boldsymbol{\varphi}(u)=(\varphi_1(u),\ \cdots,\ \varphi_n(u)),\ \nabla\cdot\boldsymbol{\varphi}(u)=\sum_{i=1}^n\frac{\partial\varphi_i(u)}{\partial x_i},$$

$\Omega\subset R^n$ is a bounded domain, $\partial\Omega$ is its boundary.

We construct an approximate solution of the problem (1.1)—(1.3) by the Galerkin method, and choose a basis $\{w_j\}\subset H_0^1\cap H^2$, where w_j are the eigen-functions of the problem:

$$-\Delta w_j=\lambda_j w_j,\ w_j|_{\partial\Omega}=0,\ j=1,\ 2,\ \cdots. \tag{1.4}$$

Obviously, if the domain Ω is suitably smooth, then there will exist such a special basis. In fact, if $\Omega\in C^2$, then the basis $\{w_j\}\in H^2(\Omega)\cap H_0^1(\Omega)\subset H_0^1(\Omega)$, and it is dense in $H_0^1(\Omega)$.

Now suppose that the approximate solution can be written as

$$u_m(x,\ t)=\sum_{j=1}^m\alpha_{jm}(t)w_j(x). \tag{1.5}$$

According to Galerkin's method, these coefficients $\alpha_{jm}(t)$ need to satisfy the following initial value problem of the system of the ordinary differential equations

$$(u_{mt}-\Delta u_{mt}+\nabla\cdot\boldsymbol{\varphi}(u_m)-f(u_m),\ w_s)=0,\ s=1,\ 2,\ \cdots,\ m, \tag{1.6}$$

$$u_m|_{t=0}=u_{0m}(x) \tag{1.7}$$

where

$$u_{0m}(x)\xrightarrow{H^2}u_0(x).$$

Under the conditions of Lemma and the a priori estimates in § 2, we know that there exists a global solution in the interval $[0,\ T]$ for the initial value problem (1.6) (1.7) of the system of nonlinear ordinary differential equations and it can approximate the solution of the problem (1.1)—(1.3). Furtermore, we obtain the global smooth solution of the problem (1.1)—(1.3) in § 2.

By a method similar to that we used in § 2, we consider in § 3 the following initial-boundary value problem for one class of system of multidimensional inhomogeneous GBBM equations:

$$\boldsymbol{u}_t-\Delta\boldsymbol{u}_t+\sum_{i=1}^n\frac{\partial}{\partial x_i}\operatorname{grad}\varphi(\boldsymbol{u})=\boldsymbol{f}(\boldsymbol{u}), \tag{1.8}$$

$$\boldsymbol{u}|_{t=0}=\boldsymbol{u}_0(x), \tag{1.9}$$

$$\boldsymbol{u}|_{\partial n}=0, \tag{1.10}$$

where $\boldsymbol{u}(x,\ t)=(u_1(x,\ t),\ \cdots,\ u_N(x,\ t))$, $\boldsymbol{f}(\boldsymbol{u})=(f_1(u_1,\ \cdots,\ u_N),\ \cdots,\ f_N(u_2,\ \cdots,\ u_N))$, $\varphi(\boldsymbol{u})=\varphi(u_1,\ \cdots,\ u_N)$. We also obtain the existence and uniqueness of the global solution for the problem (1.8)—(1.10).

At last, in § 4 we give the sufficient conditions of "blow up" of the solution for the problem of one class of system of generalized inhomogeneous BBM equations.

§ 2. The Initial Boundary Value Problem (1.1)—(1.3)

Now we make the a priori estimates for the solution of the problem (1.6)—(1.7).

Lemma 1. *If the following conditions are satisfied*

(i) $\boldsymbol{\varphi}(u)\in C^1$, $(f(u), u)\leqslant b(u, u)$, $b=\text{const.}$,

(ii) $\boldsymbol{u}_0(x)\in H_0^1(\Omega)$,

then for the solution $u_m(x, t)$ *of the problem* (1.6) (1.7) *there is the estimate*

$$\|u_m\|^2_{L_2\times L_\infty}+\|\nabla u_m\|^2_{L_2\times L_\infty}\leqslant E_0, \tag{2.1}$$

where the constant E_0 *is independent of* m

Proof Multiplying (1.6) by $\alpha_{sm}(t)$ and summing them up for s from 1 to m, we have

$$(u_{mt}(t)-\Delta u_{mt}(t)+\nabla\cdot\boldsymbol{\varphi}(u_m)-f(u_m), u_m(t))=0. \tag{2.2}$$

Since

$$(u_{mt}, u_m)=\frac{1}{2}\frac{d}{dt}\|u_m\|^2_{L_2}, -(\Delta u_{mt}(t), u_m(t))=\frac{1}{2}\frac{d}{dt}\|\nabla u_m(t)\|^2_{L_2},$$

$$(\nabla\cdot\boldsymbol{\varphi}(u_m), u_m(t))=\Big(\sum_{i=1}^n\frac{\partial\varphi_i(u_m)}{\partial x_i}, u_m(t)\Big)=\Big(\sum_{i=1}^n\frac{\partial}{\partial x_i}\Phi_i(u_m), 1\Big)=0,$$

where

$$\Phi_i(u_m)=\int_0^{u_m}\varphi_i'(z)z\,dz, \ (f(u_m), u_m(t))\leqslant b\|u_m(t)\|^2_{L_2},$$

from (2.2) it follows that

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2_{L_2}+\frac{1}{2}\frac{d}{dt}\|\nabla u_m(t)\|^2_{L_2}\leqslant b\|u_m(t)\|^2_{L_2}. \tag{2.3}$$

Integrating (2.3) with respect to t, yields

$$\|u_m(t)\|^2_{L_2}+\|\nabla u_m(t)\|^2_{L_2}\leqslant 2b\int_0^t\|u_m(\tau)\|^2_{L_2}d\tau+\|u_m(0)\|^2_{L_2}+\|\nabla u_m(0)\|^2_{L_2}$$

$$\leqslant 2b\int_0^t\|u_m(\tau)\|^2_{L_2}d\tau+C. \tag{2.4}$$

By using Gronwall's inequality, we obtain (2.1).

Lemma 2 (**Sobolev's estimates**). (i) *If* $u\in H_0^1(\Omega)$, *we have*

$$\|u\|_{L_q(\Omega)}\leqslant C(\Omega, n, q)\|u\|_{H^1(\Omega)}, \tag{2.5}$$

where $1\leqslant q\leqslant\frac{2n}{n-2}$, *as* $n>2$; $1\leqslant q<\infty$, *as* $n=2$.

(ii) *if* K *is a nonnegative integer, we have*

$$\|D^k u\|_{L_\infty} \leqslant C_1(k, \Omega)\|u\|_{H^{2+k}(\Omega)}, \text{ for } u \in H^{2+k}(\Omega), \text{ and } n \leqslant 3. \tag{2.6}$$

(iii) *Let* $D^m u \in L_q(\Omega)$, $u \in L_q(\Omega)$, $\Omega \subset R^k$, $1 \leqslant q$, $r < \infty$, $0 \leqslant j \leqslant k$, $j/m \leqslant a < 1$, $1 \leqslant p \leqslant \infty$. *Then there is a constant* C *such that*

$$\|D^j u\|_{L_p} \leqslant C \|D^m u\|_{L_r}^a \|u\|_{L_q}^{1-a}, \tag{2.7}$$

where

$$\frac{1}{p} = \frac{j}{k} + a\left(\frac{1}{p} - \frac{m}{k}\right) + (1-a)\frac{1}{q}.$$

Lemma 3 *Suppose that the conditions of Lemma 1 are satisfied, and assume that*

$$\max_{i=1,\cdots,n} |\varphi_i'(u)| \leqslant A|u|^{2/(n-2)} + B, \quad |f'(u)| \leqslant A|u|^{8/n+B}, \quad n \leqslant 3, \ A, B = \text{const.}, \tag{2.8}$$

and $u_0(x) \in H^2 \cap H_0^1$. *Then for the solution of the problem* (1.6)(1.7) *there is the estimate*

$$\|\Delta u_m\|_{L_2 \times L_\infty}^2 \leqslant E_1, \tag{2.9}$$

where the constant E_1 *is independent of* m.

Proof By $-\Delta w_j = \lambda_j w_j$ and (1.6), it follows that

$$(u_{mt} - \Delta u_{mn} + \nabla \cdot \boldsymbol{\varphi}(u_m) - f(u_m), \ -\Delta u_m) = 0. \tag{2.10}$$

Since

$$(u_{mt} - \Delta u_{mt}) = \frac{1}{2}\frac{d}{dt}\|\nabla u_m\|_{L_2}^2,$$

$$(-\Delta u_{mt}, \ -\Delta u_m) = \frac{1}{2}\frac{d}{dt}\|\Delta u_m\|_{L_2}^2,$$

$$|(\nabla \cdot \boldsymbol{\varphi}(u_m), -\Delta u_m)| = \left|\left(\sum_{i=1}^n \frac{\partial \varphi_i(u_m)}{\partial x_i}, -\Delta u_m\right)\right| = \left|\left(\sum_{i=1}^n \varphi_i'(u_m)\frac{\partial u_m}{\partial x_i}, -\Delta u_m\right)\right|$$

$$\leqslant \frac{1}{2}\|\Delta u_m\|_{L_2}^2 + \frac{1}{2}\left\|\sum_{i=1}^n \varphi_i'(u_m)\frac{\partial u_m}{\partial x_i}\right\|_{L_2}^2$$

$$\leqslant \frac{1}{2}\|\Delta u_m\|_{L_2}^2 + C\sum_{i=1}^n \|\varphi_i'(u_m)\|_{L_{2p}}^2 \left\|\frac{\partial u_m}{\partial x_i}\right\|_{L_{2p'}}^2 \quad \left(\frac{1}{p} + \frac{1}{p'} = 1\right)$$

$$\leqslant \frac{1}{2}\|\Delta u_m\|_{L_2}^2 + C\sum_{i=1}^n \|\varphi_i'(u_m)\|_{L_n}^2 \left\|\frac{\partial u_m}{\partial x_i}\right\|_{L_{2n/(n-2)}}^2 \quad \left(\text{Take } p = \frac{n}{2}, \ p' = \frac{n}{n-2}\right)$$

$$\leqslant \frac{1}{2}\|\Delta u_m\|_{L_2}^2 + C_1(\|u_m\|_{H^2}^2 + B_1) \quad \text{(By Sobolev inequalities)}$$

$$\leqslant C_2\|\Delta u_m\|_{L_2}^2 + C_3,$$

$$|(-f(u_m), -\Delta u_m)| \leqslant \|f'(u_m)\|_{L_\infty}\|\nabla u_m\|_{L_2}^2 \leqslant C_4\|u_m\|_{H^2}^2 + C_5 \leqslant C_6\|\Delta u_m\|_{L_2}^2 + C_7,$$

(By using inequality $\|\Delta u\|_{L_2}^2 \geqslant C\|u\|_{H^2(\Omega)}^2$, $u \in H_0^1 \cap H^2$, see [5])
by (2.10) it follows that

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_m\|_{L_2}^2 + \frac{1}{2}\frac{d}{dt}\|\Delta u_m\|_{L_2}^2 \leqslant C_8\|\Delta u_m\|_{L_2}^2 + C_9. \tag{2.11}$$

By using Gronwall's inequality, the estimate (2.9) is obtained.

Corollary.

$$\sup_{0 \leqslant t \leqslant T} \|u_m\|_{L_\infty} \leqslant E_2, \tag{2.12}$$

where the constant E_2 *is independent of* m.

Proof By Sobolev's inequality (2.6)

$$\|u_m\|_{L_\infty} \leqslant C_1\|u_m\|_{H^2}$$

and using the inequality

$$\|u_m\|_{H^2}\leqslant C\|\Delta u_m\|_{L_2}$$

and (2.9), the estimate (2.12) is derived immediately.

Lemma 4. *If the conditions of Lemma 3 are satisfied, then we have the estimate*

$$\|u_{mt}\|^2_{L_2\times L_\infty}+\|\nabla u_{mt}\|^2_{L_2\times L_\infty}\leqslant E_3, \tag{2.13}$$

where the constant E_3 is independent of m.

Proof Multiplying (1.6) by $\alpha'_{sm}(t)$, and summing them up for s from 1 to m, we have

$$(u_{mt}(t)-\Delta u_{mt}(t)+\nabla\varphi(u_m)-f(u_m),\ u_{mt}(t))=0. \tag{2.14}$$

From (2.14) it follows that

$$\|u_{mt}(t)\|^2_{L_2}+\|\nabla u_{mt}(t)\|^2_{L_2}\leqslant|(\nabla\cdot\varphi(u_m),\ u_{mt}(t))|+|(f(u_m),\ u_{mt})|. \tag{2.15}$$

By the conditions of this Lemma and Lemma 3, it follows that

$$|(f(u_m),\ u_{mt})|\leqslant\|f(u_m)\|_{L_\infty}(C_1+\|u_{mt}\|^2_{L_2})\leqslant C_2(1+\|u_{mt}\|^2_{L_2}),$$

$$\begin{aligned}&|(\nabla\cdot\varphi(u_m),\ u_{mt}(t)|\\ &\leqslant\frac{1}{2}\left\|\sum_{i=1}^{n}\varphi'_i(u_m)\frac{\partial u_m}{\partial x_i}\right\|^2_{L_2}+\frac{1}{2}\|u_{mt}(t)\|^2_{L_2}\\ &\leqslant\frac{n}{2}\operatorname*{Max}_{i=1,\cdots,n}\|\varphi'_i(u_m)\|^2_{L_\infty}\|\nabla u_m\|^2_{L_2}+\frac{1}{2}\|u_{mt}(t)\|^2_{L_2}\\ &\leqslant C_3(1+\|u_{mt}\|^2_{L_2}).\end{aligned}$$

Hence (2.15) implies

$$\frac{1}{2}\|u_{mt}(t)\|^2_{L_2\times L_\infty}+\frac{1}{2}\|\nabla u_{mt}(t)\|^2_{L_2\times L_\infty}\leqslant C.$$

Here constants C is independent of m. The Lemma has been proved.

Lemma 5. *If the conditions of Lemma 3 are satisfied, then we have*

$$\|\Delta u_{mt}\|^2_{L_2\times L_\infty}\leqslant E_4, \tag{2.16}$$

where the constant E_4 is independent of m.

Proof From (1.6) it follows that

$$(u_{mt}-\Delta u_{mt}+\nabla\cdot\varphi(u_m)-f(u_m),\ -\Delta u_{mt})=0. \tag{2.17}$$

Thus from (2.16) we have

$$\|\Delta u_{mt}\|^2_{L_2}\leqslant[\|u_{mt}\|^2_{L_2}+\|\nabla\cdot\varphi(u_m)\|_{L_2}+\|f(u_m)\|_{L_2}]\cdot\|\Delta u_{mt}\|_{L_2}\leqslant C\|\Delta u_{mt}\|_{L_2}. \tag{2.18}$$

Hence(2.18) implies (2.16).

Theorem 1. *Suppose the following conditions are satisfied:*

(i) $\varphi(u)\in C^2$ *and*

$$|\varphi'(u)|\leqslant A|u|^{2/n-2}+B,$$

(ii) $f(u)\in C^1$, *and*

$$(u,\ f(u))\leqslant b(u,\ u),\ |f'(u)|\leqslant A|u|^{8/n}+B,$$

(iii) $u_0(x)\in H^2(\Omega)\cap H^1_0(\Omega)$, $x=(x_1,\ \cdots,\ x_n)$, $n\leqslant3$,

where A, B and b are constants, which are independent of u. Then there exists a global generalized solution $u(x,\ t)\in L^\infty(0,\ T;\ H^2\cap H^1_0)$, $u_t(x,\ t)\in L^\infty(0,\ T;\ H^2\cap H^1_0)$ for the problem (1.1)—(1.3).

Proof By Lemma 1, Lemma 3, Lemma 4, Lemma 5 and compact argument, we can choose a subsequenece $\{u_\nu(x,t)\}$ from the sequence $\{u_m(x,t)\}$, such that

$$u_\nu(x,t)\to u(x,t) \text{ in } L^\infty(0,T;H^2) \text{ weakly star}, \nu\to\infty,$$
$$u_\nu(x,t)\to u(x,t) \text{ in strong topology of } H^1(\Omega\times[0,T]),\ \nu\to\infty,$$
$$u_{\nu t}(x,t)\to u_t(x,t) \text{ in } L^\infty(0,T;H^2) \text{ weakly star}, \nu\to\infty,$$
$$f(u_\nu(x,t))\to f(u(x,t)) \text{ in strong topology of } L_2(\Omega\times[0,T]),\ \nu\to\infty,$$

and

$$\nabla\cdot\boldsymbol{\varphi}(u_\nu)\to\nabla\boldsymbol{\varphi}(u) \text{ in strong topology of } L^\infty(0,T;L_2),\ \nu\to\infty.$$

In fact

$$\|\nabla\cdot\boldsymbol{\varphi}(u_\nu(x,t))-\nabla\cdot\boldsymbol{\varphi}(u(x,t))\|_{L_2}=\left\|\sum_{i=1}^n\left(\frac{\partial\varphi_i(u_\nu)}{\partial x_i}-\frac{\partial\varphi_i(u)}{\partial x_i}\right)\right\|_{L_2}$$
$$=\left\|\sum_{i=1}^n\varphi_i'(u_\nu)(u_{\nu x_i}-u_{x_i})+\sum_{i=1}^n(\varphi_i'(u_\nu)-\varphi_i'(u)u_{x_i}\right\|_{L_2}$$
$$\leqslant C[\|u_{\nu x_i}-u_{x_i}\|_{L_2}+\|u_\nu-u\|_{L_\infty}\|u_{x_i}\|_{L_2}]\leqslant C_1\|u_\nu-u\|_{H^1}\to 0,$$

where

$$\|u_0-u\|_{L_\infty}\leqslant C_2\|u_\nu-u\|_{L_2}^{1/4}\|u_\nu-u\|_{H^2}^{3/4}\leqslant C_3\|u_\nu-u\|_{L_2}^{1/4}\to 0\quad (n=3),$$
$$\|u_\nu-u\|_{L_\infty}\leqslant C_2\|u_\nu-u\|_{L_2}^{1/2}\|u_\nu-u\|_{H^2}^{1/2}\leqslant C_3\|u_\nu-u\|_{L_2}^{1/2}\to 0\quad (n=2).$$

Hence taking $m=\nu\to\infty$ from (1.6) we have

$$(u_t-\Delta u_t+\nabla\cdot\boldsymbol{\varphi}(u)-f(u),\ w_j)=0.$$

By using the density of $\{w_j(x)\}$ in $L_2(\Omega)$, it follows that

$$(u_t-\Delta u_t+\nabla\cdot\varphi(u)-f(u),\ \nu)=0,\ \forall v\in L_2. \tag{2.19}$$

Taking $m=\nu\to\infty$ in (1.7), it is known that $u(x,t)$ satisfies the initial condition (1.2). We complete this proof of this theorem.

Now we are going to consider the reqularities of the global generalized solution for the problem (1.1)—(1.3).

Lemma 6. *Suppose that the conditions of Lemma 3 are satisdied, and assume that*

(i) $\boldsymbol{\varphi}(u)\in C^{2p+1}(\Omega)$, $f(u)\in C^{2p}(\Omega)$, *where* $p\geqslant 1$ *is interger,*

(ii) $u_0(x)\in H^{2p+1}(\Omega)\cap H_0^1(\Omega)$.

Then for the solution $u_m(x,t)$ *of the problem* (1.6)(1.7) *we have*

$$\|\Delta^p u_m\|^2_{L_2\times L_\infty}+\|\nabla^{2p+1}u_m\|^2_{L_2\times L_\infty}\leqslant E_5, \tag{2.20}$$

where the constant E_5 *is independent of* m.

Proof From (1.6) it follows that

$$(u_{mt}-\Delta u_{mt}+\nabla\cdot\boldsymbol{\varphi}(u_m)-f(u_m),\ \Delta^{2p}u_m)=0. \tag{2.21}$$

Since

$$(\Delta^p u_{mt},\ \Delta^p u_m)=\frac{1}{2}\frac{d}{dt}\|\Delta^p u_m\|^2_{L_2},$$
$$(\nabla\cdot\boldsymbol{\varphi}(u_m),\ \Delta^{2p}u_m)=(\Delta^p\cdot\nabla\cdot\boldsymbol{\varphi},\ \Delta^p u_m),$$

$$(-\Delta u_{mt},\ \Delta^{2p}u_m)=-(\Delta^{p+1}u_{mt},\ \Delta^p u_m)=\frac{1}{2}\frac{d}{dt}\|\nabla^{2p+1}u_m\|^2_{L_2}.$$

By the Corollary to Lemma 3, $\|u_m\|_{L_\infty}\leqslant E_3$, and the hypothesis $\varphi(u)\in C^{2p+1}(\Omega)$, $f(u)\in C^{2p}(\Omega)$, it follows that

$$|(\Delta^p\nabla\cdot\varphi,\ \Delta^p u_m)|\leqslant C(\|\nabla^{2p+1}u_m\|^2_{L_2}+\|\Delta^p u_m\|^2_{L_2}),$$

$$|(-f(u_m),\ \Delta^{2p}u_m)|=|(-\Delta^p f(u_m),\ \Delta^p u_m)|\leqslant C(1+\|\Delta^p u_m\|^2_{L_2}).$$

Thus from (2.21) it follows that

$$\frac{1}{2}\frac{d}{dt}\|\Delta^p u_m\|^2_{L_2}+\frac{1}{2}\frac{d}{dt}\|\nabla^{2p+1}u_m\|^2_{L_2}\leqslant C(\|\nabla^{2p+1}u_m\|^2_{L_2}+\|\Delta^p u_m\|^2_{L_2}+1).$$

By using Gronwall's inequality, we obtain (2.20).

Lemma 7. *Suppose that the conditions of Lemma 3 are satisfied, and assume that $\varphi(u)\in C^{k+1}(\Omega)$, $f(u)\in C^k(\Omega)$, $k\geqslant 1$. Then for the solution $u_m(x,t)$ of the problem* (1.6) (1.7) *we have*

$$\|D_t^{k+1}u_m\|^2_{L_2\times L_\infty}+\|\nabla D_t^{k+1}u_m\|^2_{L_2\times L_\infty}\leqslant E_6, \tag{2.22}$$

where the constant E_6 is independent of m.

Proof As $k=1$, differentiating (1.6) with respect to t, multiplying the resulting relation by $\alpha''_{sm}(t)$ and summing up for s from 1 to m, we obtain

$$(u_{mtt}-\Delta u_{mtt}+(\nabla\cdot\varphi(u_m))_t-f'(u_m)u_{mt},\ u_{mtt})=0. \tag{2.23}$$

Hence

$$\begin{aligned}\|u_{mtt}\|^2_{L_2}+\|\nabla u_{mtt}\|^2_{L_2}&\leqslant|((\nabla\cdot\varphi(u_m))_t,\ u_{mtt})|+|(f'(u_m)u_{mt},\ u_{mtt})|\\&\leqslant\left|\left(\sum_{i=1}^n\varphi''_i(u_m)u_{mt}\cdot u_{mx_i},\ u_{mtt}\right)\right|\\&\quad+\left|\left(\sum_{i=1}^n\varphi'_i(u_m)u_{mx_it},\ u_{mtt}\right)\right|+|(f'(u_m)u_{mt},\ u_{mtt})|\\&\leqslant\operatorname*{Max}_i\|\varphi''_i(u_m)\|_{L_\infty}\cdot\|u_{mt}\|_{L_\infty}\cdot\sum_{i=1}^n\|u_{mx_i}\|_{L_2}\cdot\|u_{mtt}\|_{L_2}\\&\quad+\operatorname*{Max}_i\|\varphi'(u_m)\|_{L_\infty}\cdot\sum_{i=1}^n\|u_{mx_it}\|_{L_2}\|u_{mtt}\|_{L_2}\\&\quad+\|f'(u_m)\|_{L_\infty}\|u_{mt}\|_{L_2}\|u_{mtt}\|_{L_2}\\&\leqslant C\|u_{mtt}\|_{L_2}\ \text{(by using (2.13) (2.16))}\\&\leqslant\frac{1}{2}\|u_{mtt}\|^2_{L_2}+\frac{1}{2}C^2.\end{aligned}$$

(2.22) is true as $k=1$. Now suppose (2.22) holds as k. Differentiating (1.6) k times with respect to t, multiplying the resulting relation by $D_t^{k+1}\alpha_{sm}(t)$, and summing up for s from 1 to m, we obtain

$$(D_t^k u_{mt}-D_t^k\Delta u_{mt}+D_t^k(\nabla\cdot\varphi(u_m))-D_s^k f(u_m),\ D_t^{k+1}u_m)=0. \tag{2.24}$$

Since

$$(D_t^k u_{mt},\ D_t^{k+1}u_m)=\|D_t^{k+1}u_m\|^2_{L_2},$$

$$-(D_t^k\Delta u_{mt},\ D_t^{k+1}u_m)=\|\nabla D_t^{k+1}u_m)\|^2_{L_2},$$

$$|(D_t^k(\nabla\cdot\boldsymbol{\varphi}),\ D_t^{k+1}u_m)|\leqslant\frac{1}{4}\|D_t^{k+1}u_m\|_{L_2}^2+C\|\nabla D_t^k u_m\|_{L_2}^2\leqslant\frac{1}{4}\|D_t^{k+1}u_m\|_{L_2}^2+C_1,$$

$$|(D_t^k f(u_m),\ D_t^{k+1}u_m)|\leqslant\frac{1}{4}\|D_t^{k+1}u_m\|_{L_2}^2+C\|D_t^k u_m\|_{L_2}^2\leqslant\frac{1}{4}\|D_t^{k+1}u_m\|_{L_2}^2+C_2,$$

(2.22) is true.

Theorem 2. *Suppose that the conditions of Theorem 1 are satisfied, and assume that*

(i) $\boldsymbol{\varphi}(u)\in C^{k+1}(\Omega)$, $f(u)\in C^k(\Omega)$,

(ii) $u_0(x)\in H^{k+1}(\Omega)\cap H_0^1(\Omega)$,

then there exists the global smooth solution $u(x, t)$ for the problem (1.1)—(1.3),

$u(x, t)\in L^\infty(0, T;\ H^{k+1}\cap H_0^1)$, $D_t^j u\in L^\infty(0, T;\ H^{k+1-j}\cap H_0^1)$, $j=1, \cdots, k+1$.

Theorem 3 (Uniqueness Theorem). *Suppose that $\boldsymbol{\varphi}(u)\in C^2$, $f(u)\in C^1$, and $u_0(x)\in H_0^1(\Omega)$. Then the smooth solution of the problem* (1.1)—(1.2) *is unique.*

Proof Suppose that there are two solutions $u(x, t)$ and $v(x, t)$. Setting $\zeta=u-v$, we can derive the equation

$$\zeta_t-\Delta\zeta_t+\nabla\cdot\boldsymbol{\varphi}(u)-\nabla\cdot\boldsymbol{\varphi}(v)-(f(u)-f(v))=0. \tag{2.25}$$

Since

$$\begin{aligned}\nabla\cdot\boldsymbol{\varphi}(u)-\nabla\cdot\boldsymbol{\varphi}(v)&=\sum_{i=1}^n(\varphi_{iu}(u)-\varphi_{iu}(v))u_{x_i}+\sum_{i=1}^n\varphi_{iu}(v)(u_{x_i}-v_{x_i})\\&=\sum_{i=1}^n\varphi_{iuu}(\xi)(u-v)u_{x_i}+\sum_{i=1}^n\varphi_{iu}(v)(u_{x_i}-v_{x_i}),\end{aligned}$$

where

$$\min(u, v)\leqslant\xi\leqslant\operatorname{Max}(u, v),$$
$$f(u)-f(v)=f'(\eta)(u-v),$$

where

$$\min(u, v)\leqslant\eta\leqslant\operatorname{Max}(u, v),$$

multiplying (2.25) by ζ and taking the inner product, we obtain

$$(\zeta_t, \zeta)-(\Delta\zeta_t, \zeta)+(\nabla\cdot\boldsymbol{\varphi}(u)-\nabla\cdot\boldsymbol{\varphi}(v), \zeta)-(f(u)-f(v), \zeta)=0.$$

Since

$$(\zeta_t, \zeta)=\frac{1}{2}\frac{d}{dt}\|\zeta\|_{L_2}^2,\quad -(\Delta\zeta_t, \zeta)=\frac{1}{2}\frac{d}{dt}\|\nabla\zeta\|_{L_2}^2,$$
$$|(\nabla\cdot\boldsymbol{\varphi}(u)-\nabla\cdot\boldsymbol{\varphi}(v), \zeta)|\leqslant C(\|\zeta\|_{L_2}^2+\|\nabla\zeta\|_{L_2}^2),$$
$$|(f(u)-f(v), \zeta)|\leqslant C_1\|\zeta\|_{L_2}^2,$$

it yields

$$\frac{1}{2}\frac{d}{dt}\|\zeta\|_{L_2}^2+\frac{1}{2}\frac{d}{dt}\|\nabla\zeta\|_{L_2}^2\leqslant C_2(\|\zeta\|_{L_2}^2+\|\nabla\zeta\|_{L_2}^2).$$

By using Gronwall's inequality and $\zeta|_{t=0}=0$, $\nabla\zeta|_{t=0}=0$, the proof of the theoram is completed.

§ 3. The System of Multidimensional GBBM Equations

Now we consider the following system of multidimensional GBBM equations

$$\boldsymbol{u}_t-\Delta\boldsymbol{u}_t+\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\operatorname{grad}\varphi(\boldsymbol{u})=\boldsymbol{f}(\boldsymbol{u}) \tag{3.1}$$

with initial-boundary conditions

$$\boldsymbol{u}|_{t=0}=\boldsymbol{u}_0(x),\ \boldsymbol{u}|_{\partial\Omega}=0, \tag{3.2}$$

where $\boldsymbol{u}=(u_1,\cdots,u_N)$ is a vector valued function, $\varphi(\boldsymbol{u})=\varphi(u_1,\cdots,u_N)$ is a scalar function of variable vector $\boldsymbol{u}$, $\boldsymbol{f}(\boldsymbol{u})=(f_1(\boldsymbol{u}),\cdots,f_N(\boldsymbol{u}))$ is a vector valued function of variable vector $\boldsymbol{u}$. "grad" denotes the gradient operator for vector $\boldsymbol{u}$.

For the problem (3.1) (3.2), we also apply the Galerkin method to establish the existence and uniqueness of the global smooth solution. Let basis $\{w_j\}$ be the eigen-functions of the problem (1.4) and suppose that the approximate solution for the problem (3.1) (3.2) as follows:

$$\boldsymbol{u}_m(x,t)=\sum_{j=1}^{m}\boldsymbol{\alpha}_{jm}(t)w_j(x), \tag{3.3}$$

where $\boldsymbol{u}_m(x,t)=(u_{m1}(x,t),\cdots,u_{mN}(x,t))$, $\boldsymbol{\alpha}_{jm}(t)=(\alpha_{jm1}(t),\cdots,\alpha_{jmN}(t))$ are functional vectors. According to Galerkin's method, these coefficients need to satisfy the following initial velue problem of system of the ordinary differential equations

$$(u_{mlt},w_s)-(\Delta u_{mlt},w_s)+\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\operatorname{grad}\varphi(\boldsymbol{u}_m),w_s)-(f_l(\boldsymbol{u}_m),w_s)=0, \tag{3.4}$$

$$l=1,2,\cdots,N;\ s=1,2,\cdots,m,$$

$$u_{ml}|_{t=0}=u_{0ml}(x),\quad l=1,2,\cdots,N, \tag{3.5}$$

where $u_{0ml}(x)\xrightarrow{H^2}u_{0l}(x)$, $m\to\infty$, $l=1,2,\cdots,N$, and

$$u_{ml}(x,t)=\sum_{j=1}^{m}\alpha_{jml}(t)w_j(x).$$

In a way similar to what we have done in § 2, we may establish the a priori estimates for the problem (3.4) (3.5).

Lemma 7. *Suppose that the following conditions are satisfied:*

(i) $\varphi(\boldsymbol{u})\in C^2$, $\int_\Omega \boldsymbol{u}\cdot\boldsymbol{f}(\boldsymbol{u})dx\le b\int_\Omega \boldsymbol{u}\cdot\boldsymbol{u}\,dx$,

where

$$\boldsymbol{u}\cdot\boldsymbol{f}(u)=\sum_{l=1}^{N}u_l f_l(u_1,\cdots,u_N),\ b=\text{const}.$$

(ii) $\boldsymbol{u}_0(x)\in H_0^1(\Omega)$.

Then for the solution of problem (3.4) (3.5), *we have*

$$\|\boldsymbol{u}_m(x,t)\|_{L_2\times L_\infty}+\|\nabla\boldsymbol{u}_m\|_{L_2}\le E_0, \tag{3.6}$$

where the constant E_0 *is independent of* m, *and*

$$\|\nabla \boldsymbol{u}_m\|_{L_2\times L_\infty} = \sum_{l=1}^{N} \|\nabla u_{ml}\|^2_{L_2\times L_\infty}.$$

Proof Multiplying (3.4) by $\alpha_{sml}(t)$, and summing them for s from 1 to m and for l from 1 to N, we have

$$\sum_{l=1}^{N}\Big[(u_{mlt}, u_{ml}) - (\Delta u_{mlt}, u_{ml}) + \Big(\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\varphi_{u_{ml}}(\boldsymbol{u}_m) - f_l(\boldsymbol{u}_m), u_{ml}\Big)\Big] = 0. \quad (3.7)$$

Since

$$\sum_{l=1}^{N}(u_{mlt}, u_{ml}) = \frac{1}{2}\frac{d}{dt}\|u_m\|^2_{L_2},$$

$$\sum_{l=1}^{N}(-\Delta u_{mlt}, u_{ml}) = \frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}_m\|^2_{L_2},$$

$$\sum_{i=1}^{n}\sum_{l=1}^{N}\Big(\frac{\partial \varphi_{sml}}{\partial x_i}, u_{ml}\Big) = \sum_{i=1}^{n}\sum_{l=1}^{N}\int_\Omega (\varphi_{u_{ml}} u_{ml})_{x_i} dx - \sum_{i=1}^{n}\sum_{l=1}^{N}\int_\Omega \varphi_{u_{ml}} u_{mlx_i} dx$$

$$= \sum_{i=1}^{n}\sum_{l=1}^{N}\int_\Omega (\varphi_{u_{ml}} u_{ml})_{x_i} dx - \int_\Omega \frac{\partial \varphi(u_{m1}, \cdots, u_{mN})}{\partial x_i} dx = 0,$$

$$\sum_{l=1}^{N}(f_l(u_{m1}, \cdots, u_{ml}), u_{ml}) \leq b\sum_{l=1}^{N}(u_{ml}, u_{ml}) = b\|\boldsymbol{u}_m\|^2_{L_2},$$

from (3.7) it follows that

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_m\|^2_{L_2} + \frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}_m\|^2_{L_2} \leq b\|\boldsymbol{u}_m\|^2_{L_2}.$$

By using Gronwall's inequality, there is

$$\|\boldsymbol{u}_m(t)\|^2_{L_2} + \|\nabla \boldsymbol{u}_m(t)\|^2_{L_2} \leq e^{2bt}(\boldsymbol{u}_m(0)\|^2_{L_2} + \|\nabla \boldsymbol{u}_m(0)\|^2_{L_2}) = C.$$

Hence the estimate (3.3) holds.

Lemma 8. *Suppose that the conditions of Lemma 7 are satisfied, and assume that*

(i) $\varphi(\boldsymbol{u}) \in C^2$, $\boldsymbol{f}(\boldsymbol{u}) \in C^1$,

(ii) $\left|\frac{\partial^2 \varphi(\boldsymbol{u})}{\partial u_k \partial u_l}\right| \leq A|\boldsymbol{u}|^{\frac{2}{n-2}} + B,$ $\quad k, l = 1, \cdots, N, \; n \leq 3,$

$|\boldsymbol{f}'(\boldsymbol{u})| \leq A|\boldsymbol{u}|^{8/n} + B,$

where A and B are positive constants.

(iii) $\boldsymbol{u}_0(x) \in H^2 \cap H^1_0$, $x = (x_1, \cdots, x_n)$.

Then for the solution of the problem (3.1)(3.2), *there are the estimates*

$$\|\boldsymbol{u}_{mt}\|^2_{L_2\times L_\infty} + \|\nabla \boldsymbol{u}_{mt}\|^2_{L_2\times L_\infty} + \|\Delta \boldsymbol{u}_m\|^2_{L_2\times L_\infty} + \|\Delta \boldsymbol{u}_{mt}\|^2_{L_2\times L_\infty} \leq E_1,$$

where the constant E_1 is independent of m.

Proof It is similar to the proofs of Lemma 3, Lemma 4, and Lemma 5. Hence we have the following theorem.

Theorem 4. *Suppose that the following conditions are satisfied:*

(i) $\varphi(\boldsymbol{u}) \in C^3$, *and*

$$\left|\frac{\partial^2 \varphi(\boldsymbol{u})}{\partial u_l \partial u_k}\right| \leq A|\boldsymbol{u}|^{\frac{2}{n-2}} + B, \; k, l = 1, 2, \cdots, N, \; n \leq 3,$$

where A and B are positive constants.

(ii) $\boldsymbol{f}(\boldsymbol{u}) \in C^1$, *and*

$$\sum_{l=1}^{N}(u_l, f_l(u_1, \cdots, u_N)) \leqslant b\|\boldsymbol{u}\|_{L_2}^2, \ b=\text{const.} \qquad n\leqslant 3,$$
$$|\boldsymbol{f}'(\boldsymbol{u})| \leqslant A|\boldsymbol{u}|^{8/n}+B,$$

where A and B are positive constants.

Then there exists the global generalized solution $\boldsymbol{u}(x, t)$ for the problem (3.1)—(3.3)

$$\boldsymbol{u}(x, t) \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)),$$
$$\boldsymbol{u}_t(x, t) \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)).$$

Theorem 5. *If the conditions of Theorem 4 are satisfied, and*

(i) $\varphi(\boldsymbol{u}) \in C^{k+2}(\Omega)$, $\boldsymbol{f}(\boldsymbol{u}) \in C^k(\Omega)$,

(ii) $\boldsymbol{u}_0(x) \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$,

then there exists the global smooth solution $\boldsymbol{u}(x, t)$ for the problem (3.1) (3.2),

$$\boldsymbol{u}(x, t) \in L^\infty(0, T; H^{k+1} \cap H_0^1)$$
$$D^j\boldsymbol{u}(x, t) \in L^\infty(0, T; H^{k+2-j} \cap H_0^1), \quad j=1, 2, \cdots, k+1.$$

Theorem 6 (Uniqueness Theorem). *Suppose that $\varphi(\boldsymbol{u}) \in C^3$, $\boldsymbol{f}(\boldsymbol{u}) \in C^1$ and $\boldsymbol{u}_0(x) \in H_0^1(\Omega)$. Then the smooth solution of the problem* (3.1) (3.2) *is unique.*

§ 4. Asymptotic Behaviour and "Blow up" Problem

Now we are going to study the asymptotic behaviour as $t \to \infty$ and "blow up" problem of the solution for generalized GBBM equations.

We consider the following class of system of GBBM equations

$$\boldsymbol{u}_t - \Delta\boldsymbol{u}_t + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \operatorname{grad} \varphi(\boldsymbol{u}) = \boldsymbol{f}(x, t, \boldsymbol{u}) \tag{4.1}$$

with the initial-boundary conditions

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0(x), \tag{4.2}$$
$$\boldsymbol{u}|_{\partial\Omega} = 0. \tag{4.3}$$

Theorem 7. *Suppose that the generalized solution $\boldsymbol{u}(x, t)$ of the problem* (4.1) (4.2) (4.3) *exists for any $0<T<\infty$ and assume that the following conditions are satisfied:*

(i) *$N\times N$ Jacobi derivative matrix $\boldsymbol{f}_u(x, t, \boldsymbol{u})$ is nonpositive-semibounded, i. e., there is a constant $b<0$, such that*

$$\xi \cdot \boldsymbol{f}_u(x, t, \boldsymbol{u})\xi \leqslant b|\xi\|^2 \tag{4.4}$$

for any $\xi \in R^n$, where "$\cdot$" denotes the scalar product operator of two N-dimensional vectors and $|\xi|^2 = \xi\cdot\xi$.

(ii) *$\boldsymbol{f}(x, t, 0) \in L_2(Q_\infty)$, where $Q_\infty = \Omega \times (0, \infty)$.*

Then the generalized solution $\boldsymbol{u}(x, t)$ of the initial-boundary value problem (4.1)—(4.3) *has the asymptotic behaviour*

$$\lim_{t\to\infty}[\|\boldsymbol{u}(\cdot, t)\|_{L_2}+\|\nabla\boldsymbol{u}(\cdot, t)\|_{L_2}]=0. \tag{4.5}$$

Proof Taking the inner product (4.1) with $\boldsymbol{u}(x, t)$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}(\cdot, t)\|_{L_2}^2+\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{u}(\cdot, t)\|_{L_2}^2\\&=\int_\Omega \boldsymbol{u}\cdot\boldsymbol{f}_u(x, t, \tau\boldsymbol{u})\boldsymbol{u}\,dx+\int_\Omega \boldsymbol{f}(x, t, 0)\cdot\boldsymbol{u}\,dx\\&\leqslant b\|\boldsymbol{u}(\cdot, t)\|_{L_2}^2+\delta\|\boldsymbol{u}(\cdot, t)\|_{L_2}^2+\frac{1}{4\delta}\int_\Omega|\boldsymbol{f}(x, t, 0)|^2dx,\end{aligned} \tag{4.6}$$

where we take $\delta>0$ such that $b+\delta<0$. By using Gronwall's inequality, it follows that

$$\|\boldsymbol{u}(\cdot, t)\|_{L_2}^2+\|\nabla\boldsymbol{u}(\cdot, t)\|_{L_2}^2\leqslant e^{2(b+\delta)t}\left\{\|\boldsymbol{u}_0(x)\|_{L_2}^2+\|\nabla\boldsymbol{u}_0\|_{L_2}^2+\frac{1}{2\delta}\|\boldsymbol{f}(x, t, 0)\|_{L_2(Q_\infty)}\right\}.$$

The theorem is valid.

We consider the following quite wide class of system of GBBM equations

$$\boldsymbol{u}_t-\Delta\boldsymbol{u}_t=\boldsymbol{f}(\boldsymbol{u}, \nabla\boldsymbol{u}) \tag{4.7}$$

with the initial-boumdary conditions

$$\boldsymbol{u}|_{t=0}=\boldsymbol{u}_0(x), \tag{4.8}$$

$$\boldsymbol{u}|_{\partial\Omega}=0. \tag{4.9}$$

Theorem 8. *Suppose that the following conditions are satisfied:*

(i) *N-dimensional vector valued function $\boldsymbol{f}(\boldsymbol{u}, \nabla\boldsymbol{u})$ admits the property*

$$\int_\Omega \boldsymbol{u}\cdot\boldsymbol{f}(\boldsymbol{u}, \nabla\boldsymbol{u})dx\geqslant C_0\psi(\|\boldsymbol{u}/\cdot, t)\|_{L_2}^2+\|\nabla\boldsymbol{u}(\cdot, t)\|_{L_2}^2), \tag{4.10}$$

where $C_0>0$ and the integral

$$\int^\infty\frac{dz}{\psi(z)}<\infty. \tag{4.11}$$

(ii) *the norm $\|\boldsymbol{u}_0\|_{L_2(\Omega)}+\|\nabla\boldsymbol{u}_0\|_{L_2(\Omega)}$ is not zero.*

Then for the generalized solution $\boldsymbol{u}(x, t)$ of the initial-boundary value problem (4.7) (4.8) (4.9), the norm $\|\boldsymbol{u}(\cdot, t)\|_{L_2}^2+\|\nabla\boldsymbol{u}(\cdot, t)\|_{L_2}$ tends to infinity for a certain finite value of t.

Proof Taking the scalar product of the system (4.7) and the N-dimensional vector $\boldsymbol{u}$ and integrating the resulting relation over Ω, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}(\cdot, t)\|_{L_2}^2+\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{u}(\cdot, t)\|_{L_2}^2=\int_\Omega \boldsymbol{f}(\boldsymbol{u}, \nabla\boldsymbol{u})\cdot\boldsymbol{u}\,dx\\&\geqslant C_0\psi(\|\boldsymbol{u}(\cdot, t)\|_{L_2}^2+\|\nabla\boldsymbol{u}(\cdot, t)\|_{L_2}^2).\end{aligned} \tag{4.12}$$

Hence (4.12) becomes

$$\frac{d}{dt}w(t)\geqslant 2C_0\psi(w(t)),$$

where $w(t)=\|\boldsymbol{u}(\cdot, t)\|_{L_2}^2+\|\nabla\boldsymbol{u}(\cdot, t)\|_{L_2}^2$. This shows that when the initial value $w(0)=\|\boldsymbol{u}_0\|_{L_2}^2+\|\nabla\boldsymbol{u}_0\|_{L_2}^2\neq 0$, $w(t)$ tends to infinity for a certain finite value of t.

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