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GLOBAL MULTI-HÖLDER ESTIMATE OF SOLUTIONS TO ELLIPTIC EQUATIONS OF HIGHER ORDER

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Abstract

In this paper the global multi-Hölder estimate of solutions to general boundary value problem of elliptic equations of higher order is discussed. Let u be the solution of $Pu=f$ of m-th order elliptic equation with Dirichlet conditions

$$D_n^j u=g_j,\quad 0\leqslant j\leqslant\frac{m}{2}-1,$$

where $f\in C^{\gamma,\delta}(\Omega)$, $g_j\in C^{m-j+\gamma,\delta}(\partial\Omega)$ with $\{0<\gamma<1,\ \delta\in R^1\}$ or $\{\gamma=0,\ \delta>1\}$ or $\{\gamma=1,\ \delta\leqslant 0\}$. Then $u\in C^{m+\tilde\gamma,\tilde\delta}$, where $(\tilde\gamma,\tilde\delta)=(\gamma,\delta)$ if $0<\gamma<1$ and $\delta\in R^1$, $(\tilde\gamma,\tilde\delta)=(\gamma,\delta-1)$ if $\gamma=0$, $\delta>1$ or $\gamma=1$, $\delta\leqslant 0$. Moreover, in the case $\gamma=0$ and $0\leqslant\delta<1$, $u\in C^{(m-1)+1,\delta-1}$.

In this paper we will discuss the global multi-Hölder estimate of solutions to general boundary value problems of elliptic equations of higher order. The multi-Hölder norm of a continuous function f in a given domain Ω is defined as

$$\|f\|_{\gamma,\delta}=\|f\|_0+\sup_{x,y\in\Omega}\frac{|f(x)-f(y)|}{|x-y|^{\gamma}(1+|\ln|x-y||)^{-\delta}}\tag{1}$$

where $0\leqslant\gamma\leqslant 1$, $\delta\in R^1$, and $\delta\leqslant 0$ if $\gamma=1$, $\delta>0$ if $\gamma=0$. For any integer m,

$$\|f\|_{m+\gamma,\delta}=\sum_{|\alpha|\leqslant m}\|\partial^\alpha f\|_{\gamma,\delta}.$$

Correspondingly, we denote by $C^{\gamma,\delta}(\Omega)$ the set of functions with finite norm $\|\cdot\|_{\gamma,\delta}$, it forms a Banach space equipped with this norm. If P is an elliptic operator of m^{th} order on domain Ω, then it is proved in [1] that $Pu=f\in C^{\gamma,\delta}(\Omega)$ implies $u\in C^{m+\tilde\gamma,\tilde\delta}(\Omega')$, where $\Omega'\subset\subset\Omega$ and

$$(\tilde\gamma,\tilde\delta)=\begin{cases}(\gamma,\delta), & \text{if } 0<\gamma<1,\ \delta\in R^1,\\ (\gamma,\delta-1), & \text{if } \gamma=0,\ \delta>1, \text{ or } \gamma=1,\ \delta\leqslant 0.\end{cases}\tag{2}$$

The aim of this paper is to extend this estimate to boundary value problems. More precisely, we will prove the following theorem.

Theorem 1. *Let $P(x, D)$ be an elliptic operator with C^∞ coefficients on domain Ω, whose boundary $\partial\Omega$ is smooth. Function u satisfies*

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$$\begin{cases} P(x, D)u=f, \\ D_t^j u=g_j, \quad 0\leqslant j\leqslant \frac{m}{2}-1. \end{cases} \tag{3}$$

If $f\in C^{\gamma,\delta}(\Omega)$, $g_j\in C^{m-j+\gamma,\delta}(\partial\Omega)$, and $\{0<\gamma<1, \delta\in R^1\}$ or $\{\gamma=0, \delta>1\}$ or $\{\gamma=1, \delta\leqslant 0\}$, then $u\in C^{m+\tilde{\gamma},\tilde{\delta}}(\Omega)$, where $(\tilde{\gamma}, \tilde{\delta})$ is determihed according to (2).

§ 1. Estimate of Singularities of Fundamental Solutions to Elliptic Equations of Higher Order

Since the proof of Theorem 1 can be localized and the boundary can be flattened for a bounded domain, we may assume that Ω is the upper half plane and the support of u is bounded. Moreover, the problem can be reduced to the case of constant coefficients by means of freezing coefficients, and the lower order can be omitted. Then we may also assume that P is an elliptic operator of homogeneuos order m with constant coefficients.

It is well known (e.g. see [3]) that $P(D)$ has fundamental solutions E with the following form

$$E=E_0-Q(x)\log|x|, \tag{4}$$

where E_0 is a homogeneous function of degree $m-n$, of C^∞ in $R^n\backslash 0$, and Q is a polynomial, which vanishes in the case $n>m$, and

$$Q(x)=\int_{|\xi|=1}\frac{\langle ix, \xi\rangle^{m-n}}{P(\xi)}\,d\xi\cdot\frac{(2\pi)^{-n}}{(m-n)!}.$$

Therefore, we have the estimate in the neighborhood of origin as follows:

$$|D^\alpha E|=\begin{cases} O(|x|^{m-n-|\alpha|}), & \text{if } m-n-|\alpha|\neq 0, \\ O(\log|x|), & \text{if } m-n-|\alpha|=0. \end{cases} \tag{5}$$

The solution (4) is not suitable to giving boundary estimates of solutions to boundary value problems. Instead, we will use Green's function——the fundamental solution satisfying Dirichlet boundary conditions. Besides, by increasing the number of variables we may assume that n is large enough such that $Q(x)$ in (4) can be regarded as 0. This treatment can save us much trouble, it will be explained in more detailes in the end of section 3 (see also [7]).

Lemma 1*. *Suppose $\omega(\xi)$ is a positively homogeneous function of degree 1 with respect to $\xi\in R^n$,* Re $\omega(\xi)<-c_0|\xi|$. *Then*

$$\left|\int e^{y\omega(\xi)}e^{ix\xi}\,d\xi\right|\leqslant\frac{Cy}{(|x|^2+y^2)^{\frac{n+1}{2}}},\quad y>0. \tag{6}$$

Proof Denote

* The author would like to thank Prof. A. Nagel, who led me to this simple proof.

$$Q_y(x)=\int e^{y\omega(\xi)}e^{ix\xi}d\xi,\quad P_y(x)=\int e^{-y|\xi|}e^{ix\xi}d\xi.$$

Then

$$P_y(x)=\frac{C_n y}{(x^2+y^2)^{\frac{n+1}{2}}},$$

where

$$C_n=2^n\pi^{\frac{n-1}{2}}\Gamma\left(\frac{n+1}{2}\right).$$

We have

$$\begin{aligned}|Q_y(x)-P_y(x)|&=\left|\int(e^{y\omega(\xi)}-e^{-y|\xi|})e^{ix\xi}\,d\xi\right|\\&=y^{-n}\left|\int(e^{\omega(\xi)}-e^{-|\xi|})e^{\frac{ix\xi}{y}}\,d\xi\right|\\&=y^{-n}\left(\left|\int_{|\xi|<\frac{y}{|x|}}(e^{\omega(\xi)}-e^{-|\xi|})e^{\frac{ix\xi}{y}}\,d\xi\right|\right.\\&\quad\left.+\left|\int_{|\xi|>\frac{y}{|x|}}\left(e^{\omega(\xi)}-e^{-|\xi|}\right)e^{\frac{ix\xi}{y}}\,d\xi\right|\right)\end{aligned}$$

Let us estimate the terms inside the parentheses. If $|x|\geqslant y$, then

the first term $\leqslant\int_{|\xi|<\frac{y}{|x|}}|\xi|\,d\xi=C\dfrac{y^{n+1}}{|x|^{n+1}}$,

the second term $\leqslant\int_{|\xi|>\frac{y}{|x|}}(e^{-c_0|\xi|}+e^{-|\xi|})d\xi\leqslant Ce^{-\delta\frac{y}{|x|}}\leqslant C\dfrac{y^{n+1}}{|x|^{n+1}}$,

therefore,

$$|Q_y(x)-P_y(x)|\leqslant C\frac{y}{|x|^{n+1}}\leqslant\frac{Cy}{(|x|^2+y^2)^{\frac{n+1}{2}}}.$$

And if $|x|\leqslant y$, it is obvious that

$$\begin{aligned}|Q_y(x)-P_y(x)|&\leqslant y^{-n}\left|\int(e^{\omega(\xi)}-e^{-|\xi|})\,e^{\frac{ix\xi}{y}}\,d\xi\right|\\&\leqslant y^{-n}\int(e^{\omega(\xi)}+e^{-|\xi|})d\xi\leqslant Cy^{-n}\leqslant\frac{Cy}{(|x|^2+y^2)^{\frac{n+1}{2}}}\end{aligned}$$

whence the lemma.

Lemma 2. *If v is the solution to*

$$\begin{cases}(D_t-\lambda(D_x))v=0,\\ v|_{t=0}=g_{t_0}(x),\end{cases}\tag{7}$$

where $\lambda(\xi)$ has positive imaginary part, and is a positively homogeneous function of degree 1, $|g_{t_0}(x)|\leqslant C(t_0^2+|x|^2)^{-a}$ with $\frac{n}{2}\geqslant a\geqslant 0$, then

$$|v(t,x)|\leqslant C((t-t_0)^2+|x|^2)^{-a}.\tag{8}$$

Proof Denote $\omega(\xi)=i\lambda(\xi)$. We have

$$v(t,x)=\int e^{ix\xi}e^{i\lambda(\xi)t}\hat{g}_{t_0}(\xi)\,d\xi=Q_t(x)*g_{t_0}(x)=\int Q_t(x')g_{t_0}(x-x')dx',$$

which implies by Lemma 1

$$|v(t,x)|\leqslant C\cdot\int\frac{t}{(t^2+|x'|^2)^{\frac{n+1}{2}}}\,\frac{dx'}{(t_0^2+|x-x'|^2)^{a}}. \tag{9}$$

Next we estimate the integral (9) in different cases:

i) If $t_0/2\leqslant t\leqslant 2t_0$, then

$$\begin{aligned}
&\int\frac{t}{(t^2+|x'|^2)^{\frac{n+1}{2}}}\,\frac{dx'}{(t_0^2+|x-x'|^2)^{a}}\\
&\leqslant\int_{|x-x'|\geqslant\frac{|x|}{2}}\frac{t}{(t^2+|x'|^2)^{\frac{n+1}{2}}}\,\frac{dx'}{\left(\left(t_0^2+\frac{|x|^2}{2}\right)\right)^{a}}\\
&\quad+\int_{|x-x'|\leqslant\frac{|x|}{2}}\frac{Ct}{(t^2+|x|^2)^{\frac{n+1}{2}}}\,\frac{dx'}{(t_0^2+|x-x'|^2)^{a}}\\
&\leqslant\frac{C}{(t_0^2+|x|^2)^{a}}\int\frac{t\,dx'}{(t^2+|x'|^2)^{\frac{n+1}{2}}}+\frac{C}{(t^2+|x|^2)^{\frac{n+1}{2}}}\\
&\quad\times\int_{|x-x'|\leqslant\frac{|x|}{2}}\frac{t_0\,dx'}{(t_0^2+|x-x'|^2)^{a}}\leqslant\frac{C}{(t_0^2+|x|^2)^{a}},
\end{aligned}$$

which can be controlled by $\frac{C}{((t-t_0)^2+|x|^2)^{a}}$ in view of $|t-t_0|\leqslant t_0$.

ii) If $t<t_0/2$, then we can obtain

$$\int_{|x-x'|\geqslant\frac{|x|}{2}}\frac{t}{(t^2+|x'|^2)^{\frac{n+1}{2}}}\,\frac{dx'}{(t_0^2+|x-x'|^2)^{a}}\leqslant\frac{C}{((t-t_0)^2+|x|^2)^{a}}$$

as in the case i). Moreover

$$\begin{aligned}
&\int_{|x-x'|\leqslant\frac{|x|}{2}}\frac{t}{(t^2+|x'|^2)^{\frac{n+1}{2}}}\,\frac{dx'}{(t_0^2+|x-x'|^2)^{a}}\\
&\leqslant\frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}}\int_{|x-x'|\leqslant\frac{|x|}{2}}\frac{dx'}{(t_0^2+|x-x'|^2)^{a}}\\
&\leqslant\frac{Ctt_0^{n-2a}}{(t^2+|x|^2)^{\frac{n+1}{2}}}\int_{r<\frac{|x|}{2t_0}}\frac{r^{n-1}\,dr}{(1+r^2)^{a}}\\
&\leqslant\begin{cases}\dfrac{Ctt_0^{n-2a}}{(t^2+|x|^2)^{\frac{n+1}{2}}}\,\dfrac{|x|^n}{t_0^n}\leqslant\dfrac{C}{t_0^{2a}}\leqslant\dfrac{C}{((t-t_0)^2+|x|^2)^{a}}, & \text{if } t_0>\dfrac{|x|}{2},\\[2ex]
\dfrac{Ctt_0^{n-2a}}{(t^2+|x|^2)^{\frac{n+1}{2}}}\displaystyle\int_{r<\frac{|x|}{2t_0}}r^{n-1-2a}\,dr\leqslant\dfrac{Ct|x|^{n-2a}}{(t^2+|x|^2)^{\frac{n+1}{2}}}\leqslant\dfrac{C}{((t-t_0)^2+|x|^2)^{a}}, & \text{if } t_0<\dfrac{|x|}{2}.\end{cases}
\end{aligned}$$

iii) If $t>2t_0$, we can discuss the cases $|x|>t$ and $|x|\leqslant t$ separately. Then the similar argument verifies (8).

Lemma 3. *If v is the solution to*

$$\begin{cases}(D_t-\lambda(D_x))v=f_{t_0}(t,x),\\ v|_{t=0}=g_{t_0}(x),\end{cases} \tag{10}$$

where λ and g_{t_0} are the same as in Lemma 2, and

$$|f_{t_0}(t, x)| \leqslant C[(t-t_0)^2+|x|^2]^{-a-\frac{1}{2}}, \quad \frac{n-1}{2} \geqslant a \geqslant 0,$$

then the estimate (8) *still holds.*

Proof Obviously, we may assume $g_{t_0}=0$. In order to obtain the desired estimate, we consider

$$I=\int_0^t \int_{R_{x'}} Q_{t-s}(x') f_{t_0}(s, x-x') dx' ds = \int_0^t * ds. \tag{11}$$

When $t_0>t$, we write

$$I=I_1+I_2=\int_0^{2t-t_0} * ds + \int_{2t-t_0}^t * ds$$

(In the case $2t-t_0 \leqslant 0$, we take $I_1=0$, $I_2=I$), where

$$|*| \leqslant \int \frac{C(t-s)}{((t-s)^2+|x'|^2)^{\frac{n+1}{2}}} \frac{dx'}{((t_0-s)^2+|x-x'|^2)^{a+\frac{1}{2}}}$$
$$\leqslant C[(t-t_0)^2+|x|^2]^{-a-\frac{1}{2}}.$$

Therefore

$$|I_2| \leqslant C(t-t_0)[(t-t_0)^2+|x|^2]^{-a-\frac{1}{2}} \leqslant C[(t-t_0)^2+|x|^2]^{-a}.$$

To estimate I_1, we notice that $s<2t-t_0$ and then $t-s<t_0-s<2(t-s)$. Hence the argument in the case i) of Lemma 2 shows

$$|*| \leqslant C[(t_0-s)^2+|x|^2]^{-a-\frac{1}{2}}$$

which implies

$$|I_1| \leqslant C\int_0^{2t-t_0} (t_0-s)[(t_0-s)^2+|x|^2]^{-a-1} ds$$
$$\leqslant C[(t_0-s)^2+|x|^2]^{-a}\Big|_0^{2t-t_0} \leqslant C[(t_0-t)^2+|x|^2]^{-a}.$$

When $t>t_0$, we may rewrite I as $\int_0^{2t_0-t}+\int_{2t_0-t}^t$, and the similar consideration still works.

Lemma 4. *If u satisfies*

$$\begin{cases} P_+(D)u=0, \\ \left.\dfrac{\partial^j u}{\partial t^j}\right|_{t=0} = g^j_{t_0}(x), \quad j=0, \cdots, m_+-1, \end{cases} \tag{12}$$

where the symbol $P_+(\tau, \xi)$ of $P(D)$ has factorization $\prod_{j=1}^{m_+}(\tau-\lambda_j(\xi))$ and each $\lambda_j(\xi)$ has positive imaginary part, assume for any pseudodifferential operator $\psi_l(D_x)$ with positively homogeneous symbol of degree l, and $0 \leqslant a \leqslant \frac{1}{2}(n-m_+)$,

$$|\psi_l g^j_{t_0}(x)| \leqslant C(t_0^2+|x|^2)^{-a-(j+l)/2}, \tag{13}$$

then the solution u must satisfy

$$|u(t, x)| \leqslant C[(t-t_0)^2+|x|^2]^{-a}. \tag{14}$$

Proof Set $u_1=(D_t-\lambda_2(D_x))\cdots(D_t-\lambda_{m_+}(D_x))u$. Then u_1 satisfies

$$\begin{cases}(D_t-\lambda_1(D_x))u_1=0,\\ u_1|_{t=0}=h^1_{t_0},\end{cases}\tag{15}$$

where

$$h^1_{t_0}=g^{m_+-1}_{t_0}-(\lambda_2(D_x)+\cdots+\lambda_{m_+}(D_x))g^{m_+-2}_{t_0}+\cdots$$
$$+(-1)^{m_+-1}\lambda_2(D_x)\cdots\lambda_{m_+}(D_x)g^0_{t_0},$$

which implies

$$|h^1_{t_0}|\leqslant C(t_0^2+|x|^2)^{-a-(m_+-1)/2}.$$

By virture of Lemma 2, we have

$$|u_1(t,x)|\leqslant C[(t-t_0)^2+|x|^2]^{-a-(m_+-1)/2}.\tag{16}$$

Moreover, we set $u_2=(D_t-\lambda_3(D_x))\cdots(D_t-\lambda_{m_+}(D_x))u$, which satisfies

$$(D_t-\lambda_2(D_x))u_2=u_1(t,x),$$
$$u_2|_{t=0}=h^2_{t_0},$$

where

$$h^2_{t_0}=g^{m_+-2}_{t_0}-(\lambda_3(D_x)+\cdots+\lambda_{m_+}(D_x))g^{m_+-3}_{t_0}+\cdots+(-1)^{m_+-2}\lambda_3(D_x)\cdots\lambda_{m_+}(D_x)g^0_{t_0},$$

and $h^2_{t_0}$ satisfies

$$|h^2_{t_0}|\leqslant C(t_0^2+|x|^2)^{-a-(m_+-2)/2}.$$

Using (16) and Lemma 3, we have

$$|u_2(t,x)|\leqslant C[(t-t_0)^2+|x|^2]^{-a-(m_+-2)/2}.\tag{17}$$

Successively proceed in this way, (14) is obtained.

Remark. If $P(D)$ can be factorized to $P_+(D)P_-(D)$, where $P_+(D)\,(P_-(D))$ has only roots with positive (negative) imaginary part, then u certainly satisfies $P(D)u=0$. It means that the solution to (12) also satisfies

$$\begin{cases}P(D)u=0,\\ \left.\dfrac{\partial^j u}{\partial t^j}\right|_{t=0}=g^j_{t_0}(x),\end{cases}\tag{18}$$

moreover, the solution is unique in the class of bounded functions. Therefore, if we replace $P_+(D)$ in Lemma 4 by $P(D)$, then the estimate (14) remains valid.

Lemma 5. *If G is the Green's function at x_0 of real elliptic operator $P(D)$ of order m in R^n, satisfying conditions $\partial^j_{x_n}G=0$ $\left(j=0,\cdots,\dfrac{m}{2}-1\right)$ on $x_n=0$, then the estimate*

$$|D^\alpha G(x,x_0)|\leqslant C|x-x_0|^{m-n-|\alpha|}\tag{19}$$

holds, where n is large according to the meaning in the beginning of this section.

Proof Let $x_{0,1}=\cdots=x_{0,n-1}=0$, and E be a fundamental solution at x_0 of $P(D)$, determined by (4), when n is large, we have

$$|\psi_l E|\leqslant C|x-x_0|^{m-n-l},\tag{20}$$

where ψ_l is a pseudodifferential operator with positively homogeneous symbol of degree l. Now we denote x_n by t, and define

$$g^j_{t_0}(x)=\left.\frac{\partial^j E}{\partial t^j}\right|_{t=0}$$

on $t=0$. Then $g^j_{t_0}$ satisfy (13) with $a=\frac{1}{2}(n-m)$. Let F be the solution to

$$\begin{cases} P_+(D)F=0, \\ \left.\dfrac{\partial^j F}{\partial t^j}\right|_{t=0}=g^j_{t_0}, \end{cases} \tag{21}$$

then F satisfies

$$|F|\leqslant C|x-x_0|^{m-n}.$$

As for the estimate of derivatives of F, it can also be deduced from (21), the conditions for $g^j_{t_0}$ and Lemma 4. Finally, taking $G=E-F$, we obtain the lemma.

§ 2. Estimate of Solution to Homogeneous Dirichlet Problems

The solution to $P(D)=f$ in $\{x_n\geqslant 0\}$ with homogeneous boundary conditions $D^j_{x_n}u=0\ \left(0\leqslant j\leqslant\frac{m}{2}-1\right)$ can be expressed by Green's function in Lemma 5, that is

$$u(x)=\int_{y_n>0} G(x,y)f(y)dy. \tag{22}$$

Now we are going to deduce the conclusion $u\in C^{m+\tilde{\gamma},\tilde{\delta}}$ from $f\in C^{\gamma,\delta}$ by means of the estimate (19).

Lemma 6. *Denote by $B_h(x)$ the ball in R^n with center at x and radius $h(h<1)$. If (γ,δ) satisfies*

$$0<\gamma<1,\ \delta\in R^1;\ \text{or}\ \gamma=0,\ \delta>1;\ \text{or}\ \gamma=1,\ \delta\leqslant 0, \tag{23}$$

then

$$\int_{B_h(x)}|x-y|^{\gamma-n}(1+|\ln|x-y||)^{-\delta}dy\leqslant Ch^{\tilde{\gamma}}(1-\ln h)^{-\tilde{\delta}}, \tag{24}$$

$$\int_{B_1(x)\setminus B_h(x)}|x-y|^{\gamma-n-1}(1+|\ln|x-y||)^{-\delta}dy\leqslant Ch^{\tilde{\gamma}-1}(1-\ln h)^{-\tilde{\delta}}, \tag{25}$$

where $(\tilde{\gamma},\tilde{\delta})$ is determined by (2).

Proof Remove the origin to x and use spherical coordinates, the left hand sides of (24), (25) can be reduced to

$$\omega_{n-1}\int_0^h t^{\gamma-1}(1-\ln t)^{-\delta}\,dt,\ \omega_{n-1}\int_h^1 t^{\gamma-2}(1-\ln t)^{-\delta}\,dt.$$

i) For the first integral, if $\gamma>0$, $\delta\leqslant 0$, take $h_1=h^2$, then

$$\int_0^h t^{\gamma-1}(1-\ln t)^{-\delta}\,dt\leqslant\int_0^{h_1} t^{\frac{\gamma}{2}-1}\,t^{\frac{\gamma}{2}}(1-\ln t)^{-\delta}dt+\int_{h_1}^h t^{\gamma-1}(1-\ln h_1)^{-\delta}\,dt$$
$$\leqslant C\,(h_1^{\frac{\gamma_1}{2}}+(1-\ln h)^{-\delta}h^{\gamma})\leqslant Ch^{\gamma}(1-\ln h)^{-\delta};$$

if $\gamma>0$, $\delta>0$, then

$$\int_0^h t^{\gamma-1}(1-\ln t)^{-\delta}dt\leqslant(1-\ln h)^{-\delta}\int_0^h t^{\gamma-1}dt\leqslant Ch^{\gamma}(1-\ln h)^{-\delta};$$

if $\gamma=0$, $\delta>1$, then

$$\int_0^h t^{-1}(1-\ln t)^{-\delta}dt=\frac{1}{\delta-1}(1-\ln h)^{-(\delta-1)}.$$

ii) For the second integral, if $\gamma<1$, $\delta\leqslant 0$, then

$$\int_h^1 t^{\gamma-2}(1-\ln t)^{-\delta}dt\leqslant\int_h^1 t^{\gamma-2}dt(1-\ln h)^{-\delta}\leqslant Ch^{\gamma-1}(1-\ln h)^{-\delta};$$

if $\gamma<1$, $\delta>0$, then

$$\begin{aligned}\int_h^1 t^{\gamma-2}(1-\ln t)^{-\delta}dt&=\int_h^{\sqrt{h}} t^{\gamma-2}(1-\ln t)^{-\delta}dt+\int_{\sqrt{h}}^1 t^{\gamma-2}(1-\ln t)^{-\delta}dt\\&\leqslant C(1-\ln h)^{-\delta}t^{\gamma-1}\big|_h^{\sqrt{h}}+C\int_{\sqrt{h}}^1 t^{\gamma-2}dt\\&\leqslant C(1-\ln h)^{-\delta}h^{\gamma-1}+Ch^{(\gamma-1)/2}\leqslant C(1-\ln h)^{-\delta}h^{\gamma-1};\end{aligned}$$

if $\gamma=1$, $\delta\leqslant 0$, then

$$\int_h^1 t^{-1}(1-\ln t)^{-\delta}dt=\frac{1}{1-\delta}(1-\ln t)^{1-\delta}\big|_h^1\leqslant C(1-\ln h)^{-(\delta-1)}.$$

Thus the proof is complete.

Lemma 7. *Let $B_h^+=B_h(0)\cap\{x_n>0\}$, $f\in C^{\gamma,\delta}(B_2^+)$. Then*

$$w=\int_{B_2^+}G(x-y)f(y)dy\in C^{m+\bar{\gamma},\bar{\delta}}(B_1^+),$$

where G is the Green's function given in Lemma 5.

Proof Since $\partial^\alpha G(x-y)$ has only weak singularity $|x-y|^{m-n-|\alpha|}$, we have

$$\partial_x^\alpha\omega(x)=\int_{B_2^+}\partial_x^\alpha G(x-y)f(y)dy,\quad |\alpha|\leqslant m-1. \tag{26}$$

Its derivatives of m^{th} order can be written as

$$\begin{aligned}\partial_{x_j}\partial_x^\alpha\omega(x)&=\int_{B_2^+}\partial_{x_j}\partial_x^\alpha G(x-y)(f(y)-f(x))dy\\&\quad-f(x)\int_{\partial B_2^+}\partial_x^\alpha G(x-y)\cos(\nu,y_j)dy,\ |\alpha|=m-1.\end{aligned} \tag{27}$$

Here we have assumed $j<n$, hence there is no integral on $y_n=0$ in the second term on the right hand side. Recall $f\in C^{\gamma,\delta}$ and (γ,δ) satisfies (23), we affirm that the first integral on the right hand side is convergent.

For x, $\bar{x}\in B_1^+$, writing $\Delta=|x-\bar{x}|$, $\xi=\frac{1}{2}(x+\bar{x})$, we obtain

$$\begin{aligned}\partial_{x_j}\partial_x^\alpha w(\bar{x})-\partial_{x_j}\partial_x^\alpha w(x)&=f(x)\int_{\partial B_2^+}(\partial_x^\alpha G(x-y)-\partial_x^\alpha G(\bar{x}-y))\cos(\nu,y_j)dy\\&+(f(x)-f(\bar{x}))\int_{\partial B_2^+}\partial_x^\alpha G(\bar{x}-y)\cos(\nu,y_j)dy+\int_{B_2^+\cap B_\Delta(\xi)}\partial_{x_j}\partial_x^\alpha G(x-y)(f(x)-f(y))dy\end{aligned}$$

$$+\int_{B_{\frac{1}{2}}^{+}\cap B_{\Delta}(\xi)} \partial_{x_j}\partial_x^{\alpha} G(\bar{x}-y)(f(y)-f(\bar{x}))dy+\int_{B_{\frac{1}{2}}^{+}\setminus B_{\Delta}(\xi)} \partial_{x_j}\partial_x^{\alpha} G(x-y)dy(f(x)-f(\bar{x}))$$

$$+\int_{B_{\frac{1}{2}}^{+}\setminus B_{\Delta}(\xi)} (\partial_{x_j}\partial_x^{\alpha} G(x-y)-\partial_{x_j}\partial_x^{\alpha} G(\bar{x}-y))(f(\bar{x})-f(y))dy$$

$$=I_1+I_2+I_3+I_4+I_5+I_6.$$

Obviously, $|I_1|\leqslant C|x-\bar{x}|$, $|I_2|\leqslant C|x-\bar{x}|^{\gamma}(1+|\ln|x-\bar{x}||)^{-\delta}$. Using (24) in Lemma 6, we have

$$|I_3|\leqslant C\int_{B_{\frac{3}{2}\Delta}(x)} |x-y|^{-n}|x-y|^{\gamma}(1+|\ln|x-\bar{x}||)^{-\delta}dy\leqslant C\Delta^{\tilde{r}}(1+|\ln \Delta|)^{-\delta}.$$

The estimate for I_4 is the same. Since ihe integrand in I_5 is bounded, we have

$$|I_5|\leqslant C|x-\bar{x}|^{\gamma}(1+|\ln|x-\bar{x}||)^{-\delta}.$$

Finally, by the mean value theorem, we have

$$|I_6|\leqslant C\Delta\int_{|y-\xi|>\Delta}|\tilde{x}-y|^{-n-1}|\bar{x}-y|^{\gamma}(1+|\ln|\bar{x}-y||)^{-\delta}dy,$$

where x is a point on $\overline{x\bar{x}}$. By virture of the equivalence of $\tilde{x}-y$ and $\bar{x}-y$ in $|y-\xi|>\Delta$, $|I_6|\leqslant C\Delta^{\tilde{\gamma}}(1+|\ln\Delta|)^{-\tilde{\delta}}$ can be obtained from (25) immediately.

The remainder is to estimate $\partial_{x_n}^m w$. Noting $P(D)w=f$, we have

$$\partial_{x_n}^m w=\sum_{|\alpha|=m,\alpha_n<m} b_{\alpha}\partial^{\alpha}w+f,$$

but each term on the right hand side is of $C^{\tilde{\gamma},\tilde{\delta}}$, then so is $\partial_{x_n}^m w$, hence $w\in C^{m+\tilde{\gamma},\tilde{\delta}}$ and the lemma is proved.

Up to now we have completed the main procedure to prove Theorem 1 with homogeneous Dirichlet conditions. The remainder work is to "glue" these local conclusions and then to pass from the case of constant coefficients to the case of variable coefficients by pertubation. This process can go in the standard way, and we omit it here.

§ 3. The Case of Inhomogeneous Boundary Conditions

Let us go back to the case of flattened boundary. We have the following lemma.

Lemma 8. *If u satisfies*

$$\begin{cases} P_+(D)u=0, \\ \left.\dfrac{\partial^{j_0}u}{\partial t^{j_0}}\right|_{t=0}=\delta(x), \\ \left.\dfrac{\partial^{j}u}{\partial t^{j}}\right|_{t=0}=0, \qquad 0\leqslant j\leqslant m^+-1,\ j\neq j_0, \end{cases} \tag{28}$$

where $P_+(D)$ is an operator as in Lemma 4, then

$$|u(t, x)|\leqslant C(t^2+|x|^2)^{-\frac{1}{2}(n-j_0)}, \quad t>0. \tag{29}$$

Proof By Fourier transform with respect to x, we have

$$\begin{cases} \prod_{j=1}^{m_+}(D_t-\lambda_j(\xi))\hat{u}(t,\xi)=0, \\ \left.\dfrac{d^{j_0}\hat{u}}{dt^{j_0}}\right|_{t=0}=1, \\ \left.\dfrac{d^j\hat{u}}{dt^j}\right|_{t=0}=0, \quad 0\leqslant j\leqslant m^+-1, \quad j\neq j_0. \end{cases} \tag{30}$$

Assume that $\lambda_1, \cdots, \lambda_{m_+}$ are different from each other first, then u may be expressed as

$$\hat{u}(t,\xi)=\sum_{k=1}^{m_+} c_k(\xi)e^{it\lambda_k(\xi)} \tag{31}$$

with c_k satisfying

$$\begin{aligned} &\Sigma c_k(\xi)(\lambda_k(\xi))^j=0, \quad 0\leqslant j\leqslant m^+-1, \quad j\neq j_0, \\ &\Sigma c_k(\xi)(\lambda_k(\xi))^{j_0}=1. \end{aligned} \tag{32}$$

Since $\det|(\lambda_k(\xi))^j|_{\substack{k=1,\cdots,m_+\\ j=0,\cdots,m_+-1}}\neq 0$, $c_1(\xi), \cdots, c_{m_+}(\xi)$ can be determined uniquely. Furthermore, we have

all $c_k(\xi)$ are homogeneous of $(-j_0)^{\text{th}}$ degree.

By Fourier inverse transform we obtain

$$u(t,x)=\sum_{k=1}^{m_+}\int c_k(\xi)e^{it\lambda_k(\xi)}e^{ix\xi}d\xi. \tag{33}$$

When $t+|x|\neq 0$, the integral is continuous with respect to t and x. Moreover, $u(t,x)$ is homogeneous of degree $-n+j$ with respect to t, x, thus it implies (29).

Now if $P_+(x,\xi)$ has multiple roots, then the solution to (30) is of form

$$\hat{u}(t,\xi)=\sum_{k=1}^{l} q_k(t,\xi)e^{it\lambda_k(\xi)} \tag{34}$$

where $\lambda_1, \cdots, \lambda_l$ are different from each other, λ_k is a root of $P_+(\tau,\xi)$ with multiplicity m_k, $\sum_{k=1}^{l} m_k=m_+$, $q_k(t,\xi)=\sum_{s=0}^{m_k-1}q_{k,s}(\xi)t^s$. Noticing that the coefficients determinant of the linear algebraic equation satisfied by $q_{k,s}(\xi)$ is nothing but a derivative of Vandermonde determinant, we may also determine $q_{k,s}(\xi)$ uniquely as a homogeneous function of degree $(-j_0+s)$. Substituting them into (34), we come to the same conclusion (29) readily.

Remark. As in the remark after Lemma 4, we may replace $P_+(D)$ by $P(D)$, and additionally ask u to be bounded, then the estimate (31) is still true.

Now let us complete the proof of Theorem 1.

By virture of the linearity of the problem and the conclusion in § 2, we only need to discuss

$$\begin{cases} P(D)w=0, \\ \left.\dfrac{\partial^{j_0}w}{\partial x_n^{j_0}}\right|_{x_n=0}=g_{j_0}(x'), \\ \left.\dfrac{\partial^j w}{\partial x_n^j}\right|_{x_n=0}=0 \quad (j\neq j_0). \end{cases} \tag{35}$$

Here we have come back to the space $(x_1, \cdots, x_n)$. Therefore, when we apply Lemma 8, n must be replaced by $n-1$. Denoting by $H(x', x_n)$ the solution to (35) with $\delta(x')$ replacing $g_{j_0}(x')$, we obtain

$$w(x', x_n) = \int H(x'-y', x_n) g_{j_0}(y') dy', \tag{36}$$

which is just the integral studied in Lemma 7. In view of

$$|\partial^\alpha H| \leqslant C[|x'-y'|^2 + x_n^2]^{-\frac{1}{2}(n-1-j_0-|\alpha|)},$$

$g_{j_0} \in C^{m-j_0+\gamma,\delta}$, it is easy to obtain $w \in C^{m-j_0+j_0+\tilde{\gamma},\tilde{\delta}} = C^{m+\tilde{\gamma},\tilde{\delta}}$ by the same way as that in Lemma 7. Furthermore, noticing the remark in the end of last section, we can transfer the result to elliptic equations with variable coefficients in a genaral domain.

Remark. The above proof is proceeded under the assumption that n is large. Otherwise we may consider $\widetilde{P} = P + \sum_{j=n+1}^{n+k} D_j^m$ and replace problem (3) by

$$\begin{cases} \widetilde{P}(x, D)u = f, & \text{in } \Omega \times D \\ D_\nu^j u = g_j, \quad 0 \leqslant j \leqslant \dfrac{m}{2} - 1, & \text{on } \partial\Omega \times D, \end{cases} \tag{37}$$

where D is a periodic domain in $(x_{n+1}, \cdots, x_{n+k})$, and u is a periodic function in these variables. Since the theory on elliptic boundary problems is applicable to (37) and the conditions on $\partial\Omega \times D$ are still of Dirichlet's type, we may confirm that any solution to corresponding homogeneous problem for $\widetilde{P}$ in $\Omega \times D$ is in C^∞. Noting that if we regard any function $f(x_1, \cdots, x_n)$ in $C^{\gamma,\delta}$ as a function of $x_1, \cdots, x_n, x_{n+1}, \cdots, x_{n+k}$, it is still in the class with the same indices. Therefore, the desired property of $u(x_1, \cdots, x_n)$ can be deduced from the property of solutions to (37), and the later has been established.

§ 4. The Case $\gamma=0$, $\delta<1$

As a complement of Theorem 1, we discuss the case $\gamma=0$, $\delta<1$. The corresponding proposition is:

Theorem 2. *Under the same assumptions in Theorem 1, but replacing by $\{\gamma = 0, 0 \leqslant \delta < 1\}$ the other restrictions on (γ, δ), we have*

$$u \in C^{(m-1)+1, \delta-1}(\Omega). \tag{38}$$

Proof The technique to prove Theorem 2 is similar to that in Theorem 1. What we have to do is to find the analogue of Lemma 7, which should be read as follows.

Lemma 9. *Let $B^+ = B_h(0) \cap \{x_n > 0\}$, $f \in C^{0,\delta}(B_{\frac{1}{2}}^+)$, $(0 \leqslant \delta < 1)$. Then*

$$w = \int_{B_{\frac{1}{2}}^+} G(x-y) f(y) dy \in C^{(m-1)+1, \delta-1}(B_{\frac{1}{4}}^+),$$

where G is the Green's function.

Certainly, if Lemma 9 is true, then the whole procedure in proving Theorem 1 can be translated to the case $\gamma=0$, $\delta<1$. Hence (38) holds.

The proof of Lemma 9. Noting that $\partial^\alpha G(x-y)$ with $|\alpha|=m-1$ has weak singularity $O(|x-y|^{-n+1})$ at $x=y$, the proposition is then reduced to verifyng

$$\int g(x-y)f(y)dy\in C^{0+1,\delta-1}$$

if $f\in C^{0,\delta}$ and $g(x-y)=O(|x-y|^{-n+1})$. As in Lemma 7, taking x, $\bar{x}\in B_{\frac{1}{2}}^{+}$ and writing $\Delta=|x-\bar{x}|$, $\xi=\frac{1}{2}(x+\bar{x})$, we have

$$\begin{aligned}&\int_{B_{\frac{1}{2}}^{+}} g(x-y)f(y)dy-\int_{B_{\frac{1}{2}}^{+}} g(\bar{x}-y)f(y)dy\\ &\leqslant\left|\int_{B_{\frac{1}{2}}^{+}\backslash B_\Delta(\xi)} (g(x-y)-(g(\bar{x}-y))(f(y)-f(x))dy\right|\\ &\quad+\left|\int_{B_{\frac{1}{2}}^{+}} g(x-y)-g(\bar{x}-y))dy\cdot f(x)\right|\\ &\quad+\int_{B_\Delta(\xi)} (|g(x-y)|+|g(\bar{x}-y)|)(|f(x)|+|f(\bar{x})|)dy\\ &=I_1+I_2+I_3.\end{aligned}$$

Since I_2 amounts to the corresponding integral for the solution to $P(D)u=\text{const.}$, which is smooth, we have $|I_2|\leqslant C|x-\bar{x}|$. The estimate $|I_3|\leqslant C|x-\bar{x}|$ is also obvious. As for I_1, by the mean value theorem

$$I_1\leqslant C|x-\bar{x}|\int\frac{1}{|x-y|^n}(1+|\ln|x-y||)^{-\delta}dy.$$

Using spherical coordinates the integral on the right hand side can be estimated by

$$C\int_\Delta^2\frac{t^{n-1}}{t^n}(1+|\ln t|)^{-\delta}dt\leqslant C(1+|\ln\Delta|)^{-(\delta-1)}.$$

Combining these estimates together, we have

$$I_1+I_2+I_3\leqslant C|x-\bar{x}|\cdot(1+|\ln|x-\bar{x}||)^{-(\delta-1)},$$

which implies Lemma 9 immediately.

Remark. The special case $m=2$, $\delta=0$ is considered also by T. Kato in [8].

Summerizing our conclusion we may define a smoothness gain map on the plane (γ,δ). Let R_+^2 be the right half plane $\gamma>0$, Γ be the set $\{(k, 1)\}$ with integer k. Then every point on $R_+^2\backslash\Gamma$ corresponds to a class of function with given multi-Hölder index under the convention:

$$\begin{aligned}&(\gamma,\delta)\to C^{[\gamma]+(\gamma-[\gamma]),\delta}, &&\text{if } \gamma\neq[\gamma] \text{ or } \gamma=[\gamma],\ \delta\geqslant 0,\\ &(\gamma,\delta)\to C^{(\gamma-1)+1,\delta}, &&\text{if } \gamma=[\gamma],\ \delta<0.\end{aligned}$$

The smoothness gain map M on $R_+^2\backslash\Gamma$ is difined as

$$(\gamma,\delta)\to(m+\tilde{\gamma},\tilde{\delta}),$$

where

$$(\tilde{\gamma}, \tilde{\delta}) = \begin{cases} (\gamma, \delta), & \text{if } \gamma \neq [\gamma], \\ (\gamma, \delta - 1), & \text{if } \gamma = [\gamma]. \end{cases}$$

Obviously, according to Theorems 1 and 2, the map M corresponds to the smoothness gain of solutions to the boundary problem under above convention. This map has jump at all vertical line $\gamma = [\gamma]$, and the domain of M has holes Γ on R^2_+. Comparing with the case of classical Hölder estimate, the smoothness gain map M_1 is defined on $R^1_+ \backslash N$, where N represents the set of all natural numbers, and M_1 corresponds to the classical result $Pu \in C^\gamma \Rightarrow u \in C^{m+\gamma}$. Obviously, the map M is a refinement of the map M_1, and the appearance of the holes offer us the room to refine the result further.

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