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THE NONLINEAR INITIAL-BOUNDARY VALUE PROBLEM AND THE EXISTENCE OF MULTI-DIMENSIONAL SHOCK WAVE FOR QUASILINEAR HYPERBOLIC-PARABOLIC COUPLED SYSTEMS

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Abstract

For the quasilinear hyperbolic-parabolic coupled system, the nonlinear initial-boundary value problem and the shock wave free boundary problem are considered. By linear iteration, the existence and uniqueness of the local $H^m\left(m\geqslant\left[\frac{N+1}{2}\right]+4\right)$ solution are obtained under the assumption that for the fixed boundary problem, the boundary conditions are uniformly Lopatinski well-posed with respect to the hyperbolic and parabolic part, and for the free boundary problem, there exists a linear stable shock front structure. In particular, the local existence of the isothermal shock wave solution for radiative hydrodynamic eqations is proved.

§ 1. Introduction

In physical application, one often meets the following conservation law:

$$\begin{cases}\partial_t F_0(u,v)+\sum\limits_1^N\partial_{x_j}F_j(u_x,u,v)=0,\\ \partial_t G_0(v)+\sum\limits_1^N\partial_{x_j}G_j(u,v)=0,\end{cases}\tag{1.1}$$

where u, F_j are p-dimensional vectors, v, G_j are q-dimensional vectors, and F_j, G_j are smooth functions of their arguments. We write $w=(u, v)$ in the following.

Write

$$\begin{cases}\widetilde{P}_0=\partial_u F_0(u,v),\ \widetilde{P}_{ij}=\partial_{\xi_i}F_j(\xi,u,v),\ \widetilde{A}_0=\partial_v F_0(u,v),\ \widetilde{Q}_0=\partial_v G_0(v),\\ \widetilde{A}_j=\partial_v F_j(u_x,u,v),\ \widetilde{Q}_j=\partial_v G_j(u,v),\ \widetilde{B}_j=\partial_u G_j(u,v)\quad(i,j=1,\cdots\cdots,N).\end{cases}\tag{1.2}$$

Then $\widetilde{P}_0$, $\widetilde{Q}_0$ are positively definite for the considered w, and (1.1) can be written as a quasilinear hyperbolic-parabolic coupled system:

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$$\begin{cases} u_t+P_{ij}(u_x,\ u,\ v)u_{x_ix_j}+A_j(u_x,\ u,\ v)v_{x_j}+f_1(u_x,u,\ v)=0, \\ v_t+Q_j(u,\ v)v_{x_j}+B_j(u,\ v)u_{x_j}+f_2(u,\ v)=0. \end{cases} \tag{1.3}$$

Here, the operators satisfy the following condition:

$$\begin{cases} \partial_t+P_{ij}\partial_{x_ix_j} \text{ is 2-order Petrovsky parabolic operator,} \\ \partial_t+Q_j\partial_{x_j} \text{ is 1-order Kreiss hyperbolic operator,} \end{cases} \tag{1.4}$$

where $P_{ij}=\widetilde{P}_0^{-1}\widetilde{P}_{ij}$, $Q_j=\widetilde{Q}_0^{-1}\widetilde{Q}_j$, and the explicit form of A_j, B_j, f_1, f_2 are of no consequence and are omitted. And the Kreiss' hyperbolic operator is defined as in [8, 11]. In particular, strictly hyperbolic operators and symmetric hyperbolic operators often met in physics are Kreiss' hyperbolic.

For example, we consider the equations of radiative hydrodynamics [4]:

$$\begin{cases} \rho_t+\nabla\cdot(\rho\boldsymbol{u})=0, \\ (\rho u_i)_t+\nabla\cdot(\rho u_i\boldsymbol{u})+\left(R\rho T+\dfrac{4\sigma}{3c}T^4\right)_{x_i}=0 \quad (i=1,\ 2,\ 3), \\ \left(\dfrac{1}{2}\rho u^2+\dfrac{\rho RT}{\gamma-1}+\dfrac{4\sigma}{c}T^4\right)_t+\nabla\cdot\left(\dfrac{1}{2}\rho u^2+\dfrac{\gamma}{\gamma-1}\rho RT+\dfrac{16\sigma}{3c}T^4\right)\boldsymbol{u} \\ \quad -\nabla\cdot\left(\dfrac{lc}{3}\nabla\left(\dfrac{4\sigma}{c}T^4\right)\right)=0, \end{cases} \tag{1.5}$$

where R, c, σ, l, γ are positive functions or constants.

The first four equations constitute a quasilinear Kreiss hyperbolic system of $(\rho,\ \boldsymbol{u})$, which is also quasilinear of T. Using the first four equations to rewrite the fifth, we get a second order quasilinear parabolic equation of T, which is quasilinear of $(\rho,\ \boldsymbol{u})$ either[22]. It is easily verified that (1.5) is of the form (1.1) and satisfies (1.4).

Another example is the system of equations of compressible heatconductive viscous fluids:

$$\begin{cases} \rho_t+\nabla\cdot(\rho\boldsymbol{u})=0, \\ (\rho u_i)_t+(\rho u_iu_j+p\delta_{ij}-\gamma_{ij})_{x_j}=0 \quad (i=1,\ 2,\ 3), \\ \rho T(S_t+(\boldsymbol{u}\cdot\nabla)S)=\gamma_{ij}\partial_{x_j}u_i+\nabla\cdot(\varkappa\nabla T), \end{cases} \tag{1.6}$$

where the viscous tensor

$$\gamma_{ij}=\eta\left(\partial_{x_j}u_i+\partial_{x_i}u_j-\frac{2}{3}\partial_{x_h}u_h\delta_{ij}\right)+\zeta\delta_{ij}\partial_{x_h}u_h.$$

The fifth equation in (1.6) can be rewritten as

$$T_t-\frac{\varkappa}{\rho C_v}\Delta T+L_1(\rho,\ \boldsymbol{u},\ T)=0,$$

where $\dfrac{\varkappa}{\rho C_v}>0$, L_1 is a first order differential operator in space variables, and linear in $\nabla\rho$. The first equation in (1.6) is hyperbolic with ρ, the last four equations form a Petrovsky parabolic system of $(\boldsymbol{u},\ T)$. So (1.6) is of the form (1.3), satisfying (1.4).

In this paper, for the system (1, 3), we discuss the nonlinear general initial-boundary value problem and the Cauchy problem with discontinuous initial data, the latter is connected with the shock wave solution of conservation law (1.1).

First, we consider the initial-boundary value problem (1.3) which does not necessarily originate from (1.1) in a cylinder $\Omega\times(0, \infty)$, where $\Omega\subset R^N$ has smooth boundary $\partial\Omega$, noncharacteristic to (1.4). Let $\boldsymbol{n}=(n_1, \cdots, n_N)$ be the inner unit vector normal to $\partial\Omega$, and $\sum\limits_1^N Q_j n_j$ has q^- negative eigenvalues. For (1.3), we consider fhe following boundary conditions on $\partial\Omega\times R_+^1$:

$$J_1(u_x, w, x, t)=0, \ J_2(w, x, t)=0, \ J_3(w, x, t)=0, \tag{1.7}$$

where J_1, J_2, J_3 are b, $p-b$, q^- relations respectively, in which we assume

$$\partial_v J_2(0, x, 0)=0. \tag{1.8}$$

For simplicity, we consider the zero initial condition

$$w(x, 0)=0. \tag{1.9}$$

Denote

$$\begin{cases} \partial_\xi J_1(\xi, w, x, t)\,|_{w=0,\xi=0}=J_1^0(x, t), \\ \partial_u J_2(w, x, t)\,|_{w=0}=J_2^0(x, t), \\ \partial_v J_3(w, x, t)\,|_{w=}=J_3^0(x, t). \end{cases} \tag{1.10}$$

Let $k\geqslant\left[\dfrac{N+1}{2}\right]+4$. Then one of our main results is the following theorem.

Theorem 1. *For the problem* (1.3), (1.7), (1.9), *suppose*

1°) *All functions have $k+1$-th order continuous derivatives of their arguments and are zero order homogeneous of x for large $|x|$.*

2°) $f_j|_{w=0,\ u_x=0}\in H^k(\Omega\times R_+^1)$, $j=1, 2$,

$J_j|_{w=0,\ u_x=0}\in H^k(\partial\Omega\times R_+^1)$, $j=1, 2, 3$.

For bounded Ω, 2° follows immediately from 1°.

3°) *Zero compatiability condition of $k-1$-th order:*

$f_j|_{w=0,u_x=0}(j=1, 2)$ and $J_j|_{w=0,u_x=0}(j=1, 2, 3)$ have zero traces at $t=0$ up to the order $k-1$.

4°) *The following two linear problems are well-posed at every point on the boundary:*

$$\begin{cases} u_t-\sum\limits_{i,j} P_{ij}(0, 0, x, t)u_{x_i x_j}=0, & \text{in} \quad \Omega\times R_+^1, \\ J_1^0(x, t)u_x=0, \ J_2^0(x, t)u=0, & \text{on} \quad \partial\Omega\times R_+^1, \\ u(x, 0)=0, \end{cases}$$

$$\begin{cases} v_t-\sum\limits_j Q_j(0, x, t)v_{x_j}=0, \quad \text{in} \quad \Omega\times R_+^1, \\ J_3^0(x, t)v=0, \quad \text{on} \quad \partial\Omega\times R_+^1, \ v(x, 0)=0. \end{cases}$$

Then, there exists $t_0>0$, such that in $\Omega\times(0, t_0)$, (1, 3), (1.7), (1.9) has a unique solution $w\in H^k(\Omega\times(0, t_0))$, satisfying

$$\|\psi w\|^2_{k,\eta}<\infty, \quad \forall\psi\in C_0^\infty(-t_0, t_0).$$

Here

$$\|w\|^2_{k,\eta}=\sqrt{\eta}\,(\|w\|^2_{k,\eta}+\|\mathscr{E}^{-1}u_x\|^2_{k,\eta})+|w|^2_{k,\eta}+|\mathscr{E}^{-1}u_x|^2_{k,\eta}. \tag{1.11}$$

The study of the boundary value problems for quasilinear hyperbolic–parabolic coupled systems is rather limited in the case of multi–dimen–sional space (cf. [8, 13, 22]). Here we have advanced the existing results in following directions:

1°. The form of systems: $N\geqslant 1$, and there is no restriction on the number of the parabolic or hyperbolic equations. The hyperbolic part need not be symmetric, it should only have a Kreiss symmetrilizer. Besides, the coefficients of u_{xx} may depend on u_x, which is not permitted in [22].

2°. The form of boundary condition: i). The uniform Lopatinsky well–posedness of the principal part of the problem includes the Dirichlet–Neumann conditions $\lambda\partial_n u+\mu u=g$ as the special case. ii). The boundary conditions for hyperbolic and parabolic variables v, u may be coupled. iii). The boundary conditions are nonlinear in u_x, u and v.

3°. The regularity of data is considerably relaxed. For the space H^k, k is reduced from $k\geqslant 2\left[\frac{N}{2}\right]+6$ (cf. [8, 12]) to $k\geqslant\left[\frac{N+1}{2}\right]+4$, achieved mainly by employing the techniques of Beals and Reed[2].

4°. Since the non–local operator $\mathscr{E}^{-1}$ is introduced in tne energy estimate, a more efficient and simpler scheme of linear iteration is adopted.

Nevertheless, here we deal only with noncharacteristic boundary, in contrast with the case discussed in [13, 18].

Secondly, we consider for (1.1) the shock wave solution, i.e., the Cauchy problem with discontinuous initial data:

Given a smooth surface S_0 in R^N and smooth initial value $w_0^\pm(x)$ on two sides $\Omega_\pm$ of S_0, $w_0^+\neq w_0^-$ on S_0, one wants to find a surface $S(t)$ and shock wave solution $w^\pm(x, t)$ defined and satisfying (1.1) on two sides $\Omega_\pm(t)$ of $S(t)$, such that on $S(t)$ the Rankine–Hugoniot condition and p supplementary conditions hold:

$$\begin{cases} n_t(F_0(w^+)-F_0(w^-))+\sum_1^N n_j(F_j(u_x^+, w^+)-F_j(u_x^-, w^-))=0,\\ n_t(G_0(v^+)-G_0(v^-))+\sum_1^N n_j(G_j(w^+)-G_j(w^-))=0. \end{cases} \tag{1.12}$$

$$\begin{cases} \Phi(u^+, u^-)=0,\\ \Psi(u_x^+, u_x^-, w^+, w^-)=0, \end{cases} \tag{1.13}$$

where $(n_t, n_1,\cdots,n_N)$ is the normal vector of $S(t)$. And $w^\pm(x, 0)=w_0^\pm(x)$, $S(0)=S_0$.

Obviously, for given S_0, in order to have the shock wave solution, the traces of $w_0^\pm(x)$ on S_0 must satisfy certain compatibility conditions.

Let α be the parametric coordinate on S_0 which has normal $\boldsymbol{n}=(n_1,\cdots,n_N)$. Then from (1.5), (1.6), in order to have the shock wave solution, one sees that: there exists a sufficiently smooth function $\lambda(\alpha)$ defined on S_0 such that

$$\begin{cases} -\lambda(\alpha)(F_0(w_0^+)-F_0(w_0^-))+\sum_1^N n_j(\alpha)(F_j(u_{0x}^+,w_0^+)-F_j(u_{0x}^-,w_0^-))=0, \\ -\lambda(\alpha)(G_0(v_0^+)-G_0(v_0^-))+\sum_1^N n_j(\alpha)(G_j(w_0^+)-G_j(w_0^-))=0. \end{cases} \tag{1.14)'$$

and

$$\Phi(u_0^+, u_0^-)=0,\ \Psi(u_{0x}^+, u_{0x}^-, w_0^+, w_0^-)=0, \tag{1.14)''$$

$(\lambda(\alpha), n_1,\cdots,n_N)$ is uniformly noncharacteristic with respect to the parabolic and hyperbolic systems in (1.3). (1.15)

As a usual initial-boundary value problems, $w_0^\pm(x)$ should also satisfy the compatiability conditions of higher order. In fact, (1.14) is the compatiability condition of 0-order, which are $2p+q$ relations for $4p+2q+1$ variables, viz., the traces of $w_0^\pm$, $u_{0x}^\pm$ on S_0 and the function $\lambda(\alpha)$. As in [12], the compatiability conditions of higher order may be obtained by differentiating (1.12), (1.13) with respect to t. From (1.3), $w_t^\pm$ may be expressed by $u_{xx}^\pm$ and $v_x^\pm$. So the m-order compatiability conditions are $2p+q$ relations for $2p+2q+1$ variables $\partial_t^m n_t$, $\partial_n^{m+1}u_0^\pm$ and $\partial_n^m v_0^\pm$, where ∂_n is the normal differentiation to S_0, and $\partial_t^m n_t$ is to be determined by given $w_0^\pm$ and $\partial_t^0 n_t=\lambda(\alpha)$ with the help of compatiability conditions.

With $k\geqslant\left[\frac{N+1}{2}\right]+4$, we have another of our main results:

Theorem 2. *For the Cauchy problem of* (1.1) *with discontinuous initial data $w_0^\pm(x)$ having S_0 as its jump surface, assume:*

1°) *S_0 is sufficiently smooth, dividing R^N into two parts: Ω_+ and Ω_-. S_0 is a hyperplane when $|x|\gg1$.*

2°) *$u_0^\pm\in H_{ul}^{k+2}(\Omega_\pm)$, $v_0^\pm\in H_{ul}^{k+1}(\Omega_\pm)$, the space H_{ul}^k is defined as in* [5]. *$w_0^\pm$ are constants in $|x|\gg1$.*

3°) *$\exists\lambda_j(\alpha)\in H_{ul}^{k+2-j}(S_0)$, $j=0, 1,\cdots, m-1$ such that* (1.14), (1.15) *and the compatiability conditions up to the order $k-1$ are satisfied.*

4°) *At every $\alpha\in S_0$, the frozen coefficient linearized problem of* (1.1), (1.12), (1.13) *determined by $(\partial_n u_0^\pm(\alpha), w_0^\pm(\alpha), \lambda(\alpha))$ is uniformly linear stable* (*cf.* [10]).

Then there exists a $t_0>0$ such that in $[0, t_0]$ there is a hyper-surface $S(t)$ which belongs to $H_{ul}^{k+1}(S_0)$ as a function on S_0, and $S(0)=S_0$. $S(t)$ divides $R^N\times[0, t_0]$ into two parts: $\Omega_+(t)$ and $\Omega_-(t)$. There exist functions $w^\pm(x, t)=(u^\pm, v^\pm)\in H_{ul}^{k+1}(\Omega_\pm(t))\times H_{ul}^k(\Omega_\pm(t))$ satisfying (1.1), (1.12), (1.13) *and $w^\pm(x, 0)=w_0^\pm(x)$. Besides, such a shock wave solution is unique in the class of solutions with one shock front.*

Remark. The method to get the uniqueness result here applies also to the hyperbolic shock wave problem in [12], where only the existence result is mentioned.

Corollary. *For the equations of radiative hydrodynamics* (1.5), *the Cauchy* *problem with discontinuous initial data has a unique local isothermal shock wave solution if the conditions* 1°—3° *in Theorem* 2 *are satisfied and at every point on* S_0, *the Lax inequality holds:*

$$u_n^+ - a < \lambda < u_n^- - a, \tag{1.16}$$

where $u_n^\pm$ *are normal velocities on two sides of* S_0 *with* $\boldsymbol{n}$ *pointing from* Ω_- *to* Ω_+, *and* a *is the isothermal sound speed*[10].

This corollary comes directly from Theorem 2 and the stability results of shock wave solutions in [10].

In section 2, the problems (1.3), (1.7), (1.9) and (1.1), (1.12), (1.13) are transformed into equivalent forms more suitable for linear iteration. In section 3, we discuss tne dependency upon the coefficients of the energy estimate for linearized problem, which is crucial for linear iteration. In section 4, the existence and uniqueness results in Theorem 1 and Theorem 2 are proved. Section 5 is devoted to the discussion of some properties for a class of nonsmooth pseudo-differential operators, which are used in section 2.

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§2. Transformation of the Problem

We are going to solve the problems by linear iteration. As pointed out in [9], our iteration must depend on the results obtained in [9, 10, 16] for the coupled system. Since the energy estimates in [9, 10, 16] contain the nonlocal pseudo-differential operators, it is difficult to use the usual method in iteration by estimating the norms on $t=T$, or the method of Majda in [12] by adjusting η and T in the hyperbolic-weighted norms. Here we will use the technique in [7], where all the terms containing factors of w or its derivatives are iterated as the left side of the equation. The advantage of the scheme lies not only on its simplicity, but also on its applicability to the problem, where the estimate on $t=T$ is not available, or the estimates contain nonlocal operators. The method here may be used to simplify the proof in [12].

Using the Cauchy integral formula for the remainder, we may rewrite (1.3), (1.7), (1.9) into the following equivalent form which will be our start point of iteration:

$$\begin{cases} u_t - \Sigma P_{ij}(u_x, w, x, t) u_{x_i x_j} - \Sigma A_j(u_x, w, x, t) v_{x_j} + \Sigma C_j(u_x, w, x, t) u_{x_j} \\ \quad + C_{11}(w, x, t) w = F_1(x, t), \\ v_t - \Sigma Q_j(w, x, t) v_{x_j} - \Sigma B_j(w, x, t) u_{x_j} + C_{22}(w, x, t) w = F_2(x, t). \end{cases} \tag{2.1}$$

$$\begin{cases} T_0(u_x, w, x, t) u_x + T_2(w, x, t) w = g_1(x, t), \\ T_3(w, x, t) u + S_3(w, x, t) v = g_{21}(x, t), \\ S_4(w, x, t) v + T_4(w, x, t) u = g_{22}(x, t). \end{cases} \tag{2.2}$$

$$w(x, 0) = 0. \tag{2.3}$$

From conditions 2°, 3° in Theorem 1, $F_1, F_2, g_1, g_{21}, g_{22} \in H^k$, having zero traces at $t=0$ up to the order $k-1$. From (1.8) follows

$$S_3(0, x, 0) = 0.$$

As proven in [9], the stability of linear problem is equivalent to the stability of its adjoint problem when $S_3=0$, and consequently the stability implies well-posedness. But if the linear problem is well-posed for $S_3=0$, then by continuous extension, one knows that the problem remains well-posed for small $|S_3|$. Therefore we may perform linear iteration for small $|w|+|t|$.

For the shock wave problem (1.1), (1.12), (1.13), we will first reduce it to a common domain of definition by performing a transformation of variables which depends on the unknown shock front.

Let α be the parametric coordinate on S_0, and $n(\alpha)$ be the unit normal vector at $\alpha \in S_0$. Then for small $|\beta|$, $x(\alpha, \beta) = \alpha + \beta n(\alpha)$ parametrizes a tubular neighborhood of S_0. Thus for small $t < t_1 \ll 1$, the shock front $S(t)$ may be expressed as

$$S(t) = \{\alpha + \beta(\alpha, t) n(\alpha), \alpha \in S_0\}, \quad \beta(\alpha, 0) = 0. \tag{2.4}$$

Let $\rho(x) \in C^\infty(R^N)$: for some $\delta_1 > 0$, $\rho(x) = 1$, when $d(x, S_0) < \frac{\delta_1}{2}$; $\rho(x) = 0$, when $d(x, S_0) > \delta_1$. Perform the transformation

$$\tilde{x} = x - \rho(x) \beta(\alpha(x), t) n(\alpha(x)), \quad \tilde{t} = t. \tag{2.5}$$

Then, $x \mapsto \tilde{x}$ will be a sufficiently smooth diffeomorphism $R^N \to R^N$ when

$$\sup_{\alpha \in S_0, 0 \le t \le t_1} (|\beta(\alpha, t)| + |\nabla_{\tan} \beta(\alpha, t)|) < \delta_2, \tag{2.6}$$

where $\nabla_{\tan}$ denotes the gradient on Riemann manifold S_0.

After the transformation, the equation (1.1), or equivalently (1.3), becomes (we denote again the independent variables by (x, t)):

in $\Omega_\pm \times (0, t_1)$:

$$\begin{cases} u_t^\pm + P_{ij}^\pm(u_x^\pm, w^\pm, \beta_x, \beta) u_{x_i x_j}^\pm + A_j^\pm(u_x^\pm, w^\pm, \beta_x, \beta) v_{x_j}^\pm \\ \quad + f_1^\pm(u_x^\pm, w^\pm, \beta_{xx}, \beta_x, \beta_t, \beta) = 0, \\ v_t^\pm + Q_j^\pm(w^\pm, \beta_x, \beta_t, \beta) v_{x_j}^\pm + B_j^\pm(w^\pm, \beta_x, \beta) u_{x_j}^\pm + f_2^\pm(w^\pm, \beta_t, \beta_x, \beta) = 0, \end{cases} \tag{2.7}$$

where

$$P_{ij}^\pm = P_{ij}(u_x^\pm, w^\pm) - P_{ik}(u_x^\pm, w^\pm)(\rho\beta n_j)_{x_k} - P_{kj}(u_x^\pm, w^\pm)(\rho\beta n_i)_{x_k}$$
$$- P_{hk}(u_x^\pm, w^\pm) \cdot (\rho\beta n_i)_{x_h} (\rho\beta n_j)_{x_k},$$

$$Q_j^{\pm}=Q_j(w^{\pm})-\rho n_j\beta_t-Q_i(w^{\pm})(\rho\beta n_j)_{x_i},$$

while $A_j^{\pm}$, $B_j^{\pm}$, $f_1^{\pm}$, $f_2^{\pm}$ are determined similarly, the explicit forms of which are omitted because of no consequence here.

In new coordinates, the boundary conditions (1.12), (1.13) become

$$\begin{cases}\beta_t[F_0(w^+)-F_0(w^-)]-[v(\alpha)-(1+\beta\pi)^{-1}\nabla_{\tan}\beta]\cdot[F(u_x^+,w^+)-F(u_x^-,w^-)]=0,\\ \beta_t[G_0(v^+)-G_0(v^-)]-[n(\alpha)-(1+\beta\pi)^{-1}\nabla_{\tan}\beta]\cdot[G(w^+)-G(w^-)]=0,\end{cases} \tag{2.8}$$

$$\begin{cases}\Phi(u^+,u^-)=0,\\ \Psi(u_x^+,u_x^-,w^+,w^-)=0,\end{cases} \tag{2.9}$$

where $\pi(\alpha)$ is the Weingarten map: $T_\alpha(S_0)\to T_\alpha(S_0)$.

Evidently, for sufficiently smooth local solutions, the Cauchy problem (1.1) (1.12), (1.13) with discontinuous initial data is equivalent to the problem (2.7)—(2.9) with the initial condition:

$$w^{\pm}(x,0)=w_0^{\pm}(x),\ \beta(\alpha,0)=0. \tag{2.10}$$

Next, we will reduce (2.7)—(2.10) to the case of zero initial data. From the compatibility condition in Theorem 2 and the fact that for $|x|\gg 1$, S_0 is a hyperplane and $w_0^{\pm}$ are constants, we can construct $({}^{\#}u^{\pm}, {}^{\#}v^{\pm}, {}^{\#}\beta)\in H_{ul}^{k+2}(\Omega_\pm\times(0,t_1))\times H_{ul}^{k+1}(\Omega_\pm\times(0,t_1))\times H_{ul}^{k+2}(S_0\times(0,t_1))$ such that the new unknown functions $W^{\pm}=w^{\pm}-{}^{\#}w^{\pm}$ and $\phi=\beta-{}^{\#}\beta$ satisfy the following problem with homogeneous initial condition:

$$\begin{cases}\text{in } \Omega_\pm\times(0,t_1):\\ U_t^{\pm}+{}^{\#}P_{ij}^{\pm}(U_x^{\pm},W^{\pm},\phi_x,\phi)U_{x_ix_j}^{\pm}+{}^{\#}A_j^{\pm}(U_x^{\pm},W^{\pm},\phi_x,\phi)V_{x_j}^{\pm}\\ \quad+{}^{\#}C_j^{\pm}(U_x^{\pm},W^{\pm},\phi_{xx},\phi_x,\phi_t,\phi)U_{xj}^{\pm}\\ \quad+{}^{\#}C_{11}^{\pm}(W^{\pm},\phi_{xx},\phi_t,\phi_x,\phi)\cdot(W^{\pm},\rho\phi_{xx},\rho\phi_x,\rho\phi_t,\rho\phi)^t=f_1^{\pm}(x,t),\\ V_t^{\pm}+{}^{\#}Q_j^{\pm}(W^{\pm},\phi_t,\phi_x,\phi)V_{x_j}^{\pm}+{}^{\#}B_j^{\pm}(W^{\pm},\phi_t,\phi_x,\phi)U_{xj}^{\pm}\\ \quad+{}^{\#}C_{12}^{\pm}(W^{\pm},\phi_t,\phi_x,\phi)\cdot(W^{\pm},\rho\phi_t,\rho\phi_x,\rho\phi)^t=f_2^{\pm}(x,t),\end{cases} \tag{2.11}$$

$$\begin{cases}T_0^+(U_x^+,W^+,\phi_t,\phi_x,\phi)U_x^+-T_0^-(U_x^-,W^-,\phi_t,\phi_x,\phi)U_x^-\\ \quad+T_{01}(W^+,W^-,\phi_t,\phi_x,\phi)\cdot(W^+,W^-,\phi_t,\phi_x,\phi)^t=g_{11}(x,t),\\ T_{11}(U_x^+,U_x^-,W^+,W^-,\phi_x,\phi)\cdot(U_x^+,U_x^-)^t\\ \quad+T_{12}(W^+,W^-,\phi_x,\phi)(W^+,W^-,\phi_x,\phi)^t=g_{12}(x,t),\\ T_{21}(U^+,U^-)\cdot(U^+,U^-)^t=g_{21}(x,t),\\ [G_0({}^{\#}v^++v^+)-G_0({}^{\#}v^-+v^-)]\phi_t+[G({}^{\#}w^++W^+)-G({}^{\#}w^-+W^-)]\\ \quad\cdot(1+({}^{\#}\beta+\phi)\pi)^{-1}\nabla_{\tan}\phi+{}^{\#}\beta_t[G_0(V^+)V^+-{}^{\#}G_0(V^-)V^-]\\ \quad-[n(\alpha)-(1+({}^{\#}\beta+\phi)\pi(\alpha))^{-1}\nabla_{\tan}{}^{\#}\beta]\cdot[{}^{\#}G(W^+)V^+-{}^{\#}G(W^-)V^-]\\ \quad+T_{22}(U^+,U^-,\phi)\cdot(U^+,U^-,\phi)^t=g_{22}(x,t),\end{cases} \tag{2.12}$$

$$W^{\pm}(x,0)=0,\ \phi(x,0)=0. \tag{2.13}$$

Here $f_1^{\pm}, f_2^{\pm}\in H^k(\Omega_\pm\times(0,t_1))$, $g_1=(g_{11},g_{12})$, $g_2=(g_{21},g_{22})\in H^k(S_0\times(0,t_1))$, and (f,g) have zero traces at $t=0$ up to the order $k-1$. All the coefficients in (2.11), (2.12) are sufficiently smooth in $W^{\pm}$, $U_x^{\pm}$, ϕ_{xx}, ϕ_x, ϕ_t, ϕ, and belong to H_{ul}^k in (x,t).

Straightforward calculation gives their explicit forms by substituting $w^{\pm}={}^{\#}w^{\pm}+W^{\pm}$ and $\beta={}^{\#}\beta+\phi$ into (2.11), (2.12) and using the Cauchy formula for remainders.

Hence, by the above procedure, the proof of Theorem 2 is reduced to the proof of the following Theorem 2′ for the problem (2.11)—(2.13):

Theorem 2′. *If, for $W^{\pm}=0$, $\phi=0$ in the coefficients of (2.11)—(2.13), the resulted linear problem is well-posed in the sense of [10] at every point $\alpha\in S_0$,*

$$k\geqslant\left[\frac{N+1}{2}\right]+4,$$

and (f, g) are as described above, then $\exists t_0>0$, such that in $(0, t_0)$, (2.11)—(2.13) has a unique solution $(W^{\pm}, \phi)$ satisfying

$$|||(\psi W^{-}, \psi\phi)|||_{k,\eta}<\infty, \ \forall\psi\in C_0^\infty(-t_0, t_0), \tag{2.14}$$

where

$$|||(w, \phi)|||^2_{k,\eta}=\|w\|^2_{k,\eta}+|\phi|^2_{k+1,\eta}.$$

In the following, we will iterate (2.1)—(2.3) and (2.11)—(2.13) to prove Theorem 1 and Theorem 2′.

§ 3. The Dependency of the Energy Estimate upon the Coefficients

1) In order to make use of the linear results in [9, 10] to iterate the nonlinear problem (2.1)—(2.3) and (2.11)—(2.13), it is essential to know the dependency of the energy estimate for linearized problem upon its coefficients. Because of the localized deduction of the energy estimates, we need only to analyse the dependency in local coordinates. Thus. we may rewrite (2.1)—(2.3) as

in $\Omega=\{(x, y); x>0, y\in R^{N-1}\}$:

$$\begin{cases} u_t-P_0u_{xx}-P_{1j}u_{xy_j}-P_{2ij}u_{y_iy_j}-A_0v_x-A_jv_{y_j}+C_0u_x+C_ju_{y_j}+C_{11}w=F_1, \\ v_t-Q_0v_x-Q_jv_{y_j}-B_0u_x-B_ju_{y_j}+C_{22}w=F_2, \end{cases} \tag{3.1}$$

$$\begin{cases} T_0u_x+T_{1j}u_{y_j}+T_2w=g_1, \\ T_3u+S_3v=g_{21}, \qquad \text{on } x=0, \\ T_4u+S_4v=g_{22}, \end{cases} \tag{3.2}$$

$$w(x, 0)=0. \tag{3.3}$$

Our main theorem about the dependency of the energy estimate for (3.1)—(3.3) upon its coefficients is the following theorem.

Theorem 3.1. *Under the conditions of Theorem 1, fixing $W=(U, V)\in C_0^\infty(\bar\Omega\times(0, t_1))$ in the coefficients of (3.1)—(3.3), if the resulted linearized problem is well-posed, and $k\geqslant\left[\frac{N+1}{2}\right]+4$, then its solution w satisfies the following estimate*

$$\|w\|^2_{k,\eta}\leqslant C_{k,\eta}\|(F, g)\|^2_{k,\eta}, \tag{3.4}$$

where

$$|||(F, g)|||_{k,\eta}^2 = \|\mathscr{E}^{-1}F_1\|_{k,\eta}^2 + \|F_2\|_{k,\eta}^2 + |\mathscr{E}^{-1}g_1|_{k,\eta}^2 + |g_2|_{k,\eta}^2,$$

and the constant $C_{k,\eta}$ in (3.4) depends on W only by $|||W|||_k$, or consequently depends on $|||W|||_{k,\eta}$ and η.

Similarly, in local coordinate, after the reflection $x \mapsto -x$ in $x<0$, (2.11)—(2.13) can be written as the following problem for $w=(W^+, W^-)$ and ϕ:

$$\begin{cases} \text{in } x>0,\ t>0: \\ u_t - P_0 u_{xx} - P_{1j} u_{xy_j} - P_{2ij} u_{y_i y_j} - A_0 v_x - A_j v_{y_j} + C_0 u_x + C_j u_{y_j} \\ \quad + C_{11} w + E_{2ij} \phi_{y_i y_j} + E_{1j} \phi_{y_j} + E_{10} \phi_t + E_1 \phi = F_1(x, y, t), \\ v_t - Q_0 v_x - Q_j v_{y_j} - B_0 u_x - B_j u_{y_j} + C_{21} w + E_{2j} \phi_{y_j} \\ \quad + E_{20} \phi_t + E_2 \phi = F_2(x, y, t), \end{cases} \tag{3.5}$$

$$\begin{cases} \text{on } x=0,\ t>0: \\ T_0 u_x + T_{1j} u_{y_j} + T_1 w + R_{10} \phi_t + R_{1j} \phi_{y_j} + R_1 \phi = g_1(y, t), \\ T_3 u = g_{21}(y, t), \\ b_0 \phi_t + b_{1j} \phi_{y_j} + S_4 v + T_4 u + R_2 \phi = g_{22}(y, t), \end{cases} \tag{3.6}$$

$$w(x, y, 0) = 0,\quad \phi(y, 0) = 0, \tag{3.7}$$

where $P_0, P_{1j}, P_{2ij}, A_0, A_j$ depend on $(\nabla u, w, \phi_y, \phi)$; T_0, T_{1j} depend on $(\nabla u, w, \phi_t, \phi_y, \phi)$; C_0, C_j depend on $(\nabla u, w, \phi_t, \phi_{yy}, \phi_y, \phi)$; $C_{11}, E_{2ij}, E_{1j}, E_1$ depend on $(w, \phi_t, \phi_{yy}, \phi_y, \phi)$; $Q_0, Q_j, B_0, B_j, C_{21}, E_{2j}, E_{20}, E_2, T_1, R_{10}, R_{1j}, R_1$ depend on $(w, \phi_t, \phi_y, \phi)$; $T_3 = T_3(u)$; $b_0 = b_0(v)$; b_{1j}, S_4, T_4, R_2 depend on (w, ϕ). And every ϕ in (3.5) is accompanied by a smooth factor with compact support in x.

Then for the problem (3.5)—(3.7), we have the following theorem.

Theorem 3.2. *If, for fixed*

$$(w, \phi) = (w_n, \phi_n) \in C_0^\infty(\overline{\Omega} \times (0, t_1)) \times C_0^\infty(S_0 \times (0, t_1))$$

in the coefficients of (3.5)—(3.7), the resulted linear problem is well-posed,

$$k \geq \left[\frac{N+1}{2}\right] + 4,$$

and $(F, g) \in H^k(\Omega \times R_+^1) \times H^k(S_0 \times R_+^1)$, having zero traces at $t=0$ up to the order $k-1$, then the solution (w_{n+1}, ϕ_{n+1}) of the linear problem satisfies the estimate:

$$\widetilde{|||}(w_{n+1}, \phi_{n+1})\widetilde{|||}_{k,\eta}^2 \leq C_{k,\eta} |||(F, g)|||_{k,\eta}^2, \tag{3.8}$$

where the constant $C_{k,\eta}$ depends on (w_n, ϕ_n) only by the norm $\widetilde{|||}(w_n, \phi_n)\widetilde{|||}_{k,\eta}$.

2) First, let us discuss a class of pseudo-differential operators, the symbols of which have nonsmooth coeffcients and are η-weighted.

Let $z=(y, t)$, its dual variables $\xi=(\omega, \tau)$, and $s=\eta+i\tau$. Write

$$\langle \xi, \eta \rangle = (|\xi|^2 + \eta^2)^{\frac{1}{2}},$$

then we define

Definition 3.3. $a(z, \xi, \eta) \in S_\rho^{m,k}$, *if*

$$\langle \xi, \eta \rangle^{-m+\rho|\beta|} D_\xi^{\beta_1} D_\eta^{\beta_2} D_z^\alpha a(z, \xi, \eta) \in H^{k-|\alpha|},\ \forall\, \beta_1+\beta_2=\beta,\ |\alpha| \leq k. \tag{3.9}$$

If, in (3.9), $H^{k-|\alpha|}$ is replaced by $H^{k-|\alpha|}_{ul}$ (cf. Kato [4]), then we have the symbols in $S^{m,k(ul)}_{\rho}$. Corresponding to the symbol $a(z, \xi, \eta) \in S^{m,k}_{\rho}$ (or $S^{m,k(ul)}_{\rho}$), we may define a pseudo-differential operator $a(z, D, \eta)$ with parameter η:

$$a(z, D, \eta)u(z) = \int e^{iz\xi} a(z, \xi, \eta)\hat{u}(\xi, \eta) d\!\!\bar{}\xi.$$

We write then $a(z, D, \eta) \in SP^{m,k}_{\rho}$ (or $SP^{m,k(ul)}_{\rho}$).

With another weight function $\langle \xi, \eta \rangle_{\sigma} = (|\omega|^2 + |\tau| + \eta)^{\frac{1}{2}}$, we have as in [12] and in Definition 3.3

Definition 3.4. $b(z, \xi, \eta) \in S^{m,k}_{\rho,\sigma}$ *if*

$$\langle \xi, \eta \rangle_{\sigma}^{-m+\rho|\beta|} D^{\beta_1}_{\xi} D^{\beta_2}_{\eta} D^{\alpha}_{z} b(z, \xi, \eta) \in H^{k-|\alpha|}, \ \forall\ \beta_1 + \beta_2 = \beta, \ |\alpha| \leqslant k. \tag{3.10}$$

So is defined

$$b(z, D, \eta) \in SP^{m,k}_{\rho,\sigma}.$$

Similarly $S^{m,k(ul)}_{\rho,\sigma}$ and $SP^{m,k(ul)}_{\rho,\sigma}$ are defined.

In particular, $\mathscr{E} \in SP^{1,k(ul)}_{1,\sigma} \cap SP^{1,(kl)}_{\frac{1}{2}}$, $\mathscr{E}^{-1} \in SP^{-1,k(ul)}_{1,\sigma} \cap SP^{-\frac{1}{2},k(ul)}_{\frac{1}{2}}$, $\forall k \in Z^+$.

For the L^2-boundedness of the above operators and their commutators, we have the following proposition.

Proposition 3.5.

i) *If* $a(z, \xi, \eta) \in S^{0,k}_{\rho}$ (*or* $S^{0,k}_{\rho,\sigma}$), $k > \frac{N}{2}$, *then*

$$\|a(z, D, \eta)u(z)\|_{\eta} \leqslant C\|u\|_{\eta}, \ \forall u \in L^2(R^N, \eta);$$

ii) *Let* $a(z, \xi, \eta) \in S^{0,k(ul)}_{\rho}$ (*or* $S^{0,k(ul)}_{\rho,\sigma}$), $k > \frac{N}{2}$, *then*

$$\|a(z, D, \eta)u(z)\|_{\eta} \leqslant C\|u\|_{\eta}, \ \forall u \in L^2(R^N, \eta);$$

iii) *Let* $P(z, \xi, \eta) \in S^{r,k+1}_{1,\sigma}$ (*or* $S^{r,k+1(ul)}_{1,\sigma}$), $k > \frac{N}{2}$, *then*

$$[\mathscr{E}, P(z, D, \eta)] \in SP^{r,k}_{1,\sigma} (\textit{or } SP^{r,k(ul)}_{1,\sigma});$$

iv) *If* $P(z, D, \eta) \in SP^{2,k+2}_{1,\sigma}$ (*or* $SP^{2,k+2(ul)}_{1,\sigma}$), $k > \frac{N}{2}$, *then*

$$[\mathscr{E}^{-1}, P(z, D, \eta)] \in SP^{0,k}_{1,\sigma} (\textit{or } SP^{0,k(ul)}_{1,\sigma}).$$

We will need some Banach algebra properties of $H^k\left(k > \frac{N}{2}\right)$.

Proposition 3.6. *Let*

$$\min_{\substack{1 \leqslant h \leqslant m \\ j_1 < \cdots < j_h}} k_{j_1} + \cdots + k_{j_h} - (h-1)\left(\frac{N}{2} + 1\right) = r \geqslant 0.$$

Then we have

i) $\|\prod_1^m u_i\|^2_{r,\eta} \leqslant C\|u_j\|^2_{k_j,\eta} \cdot \prod_{i \neq j} \|u_i\|^2_{k_i}$;

ii) $\|\mathscr{E}^{-1} \prod_1^m u_i\|^2_{r,\eta} \leqslant C\|\mathscr{E}^{-1} u_j\|^2_{k_j,\eta} \prod_{i \neq j} \|u_i\|^2_{k_i}$;

iii) $\|\mathscr{E}^{-1}\prod_1^m u_i\|_{r,\eta}^2 \leqslant C\|\mathscr{E}^{-1}u_j\|_{k_j}^2 \cdot \|u_n\|_{k_n,\eta}^2 \cdot \prod_{i\neq j}\|u_i\|_{k_i}^2$.

The proof of Propositions 3.5 and 3.6 will be postponed till section 5.

Corollary 3.7. *Let $F(U)$ be a function sufficiently smooth. Then for*

$$k \geqslant \left[\frac{N}{2}\right]+2,$$

$$\|\mathscr{E}^{-1}(F(Uu)\|_{k,\eta} \leqslant C(\|\mathscr{E}^{-1}U\|_k)\cdot\|u\|_{k,\eta}.$$

Proof $D^k(F(U(u)$ is the finite linear combination with continuous coefficients of the terms such as $(D^{k_1}U)\cdots(D^{k_j}U)(D^{k_0}u)$. Since for $k\geqslant\left[\frac{N}{2}\right]+2$, $\forall h\in Z^+$, $k_0+\cdots+k_h\leqslant k$ implies

$$(h+1)k-(h+1)-(k_0+\cdots+k_h)+1-h\left(\left[\frac{N}{2}\right]+1\right)\geqslant 0,$$

then Proposition 3.6 gives the desired ineqvality.

3) Now we are going to prove Theorem 3.1.

Consider first the k-times tangential enlarged system of (3.1)—(3.3).

Denoting $u=u_1$, $D^k u_2=\mathscr{E}^{-1}((D^k u)_x+P_0^{-1}A_0(D^k v))$, where $D=(\partial_t, \partial_y)$, then, from (3.1), we can deduce the following equations of $D^k\widetilde{w}=(D^k u_1, D^k u_2, D^k v)$:

$$(D^k u_1)_x=\mathscr{E}(D^k u_2)-P_0^{-1}A_0(D^k v), \tag{3.11}$$

$$\begin{aligned}(D^k u_2)_x=&\mathscr{E}^{-1}P_0^{-1}(\partial_t-P_{2ij}\partial_{y_iy_j})(D^k u_1)-P_0^{-1}P_{1j}(D^k u_2)_{y_j}\\&+[\mathscr{E}^{-1}, P_0^{-1}(\partial_t-P_{2ij}\partial_{y_iy_j})](D^k u_1)+P_0^{-1}P_{1j}\partial_{y_j}\mathscr{E}^{-1}P_0^{-1}A_0(D^k v)\\&-[\mathscr{E}^{-1}, P_0^{-1}P_{1j}\mathscr{E}\partial_{y_j}]\mathscr{E}^{-1}(D^k u_1)_x-\mathscr{E}^{-1}P_0^{-1}A_j(D^k v)_{y_j}\\&-\mathscr{E}^{-1}(P_0^{-1}A_0)_x(D^k v)-\mathscr{E}^{-1}P_0^{-1}D^k(C_0u_x+C_ju_{y_j}+C_{11}w)\\&-\mathscr{E}^{-1}P_0^{-1}D^kF_1-\mathscr{E}^{-1}P_0^{-1}[D^k, P_0\partial_{xx}+P_{1j}\partial_{xy_j}+P_{2ij}\partial_{y_iy_j}]u\\&-\mathscr{E}^{-1}P_0^{-1}[D^k, A_0\partial_x+A_j\partial_{y_j}]v.\end{aligned} \tag{3.12}$$

$$\begin{aligned}(D^k v)_x=&Q_0^{-1}(D^k v)_t-Q_j(D^k v)_{y_j}-Q_0^{-1}B_0\mathscr{E}(D^k u_2)+Q_0^{-1}B_0P_0^{-1}A_0(D^k v)\\&-Q_0^{-1}B_j(D^k u_1)_{y_j}-Q_0^{-1}D^k(C_{22}w)-Q_0^{-1}[D^k, Q_0\partial_x+Q_j\partial_{y_j}]v\\&-Q_0^{-1}[D^k, B_0\partial_x+B_j\partial_{y_j}]u-Q_0^{-1}D^kF_2.\end{aligned} \tag{3.13}$$

Or briefly

$$(D^k\widetilde{w})_x=\mathscr{N}D^k\widetilde{w}+\mathscr{F}+F, \tag{3.14}$$

where $\mathscr{N}$ is a first order pseudo-differential operator with the symbol

$$\mathscr{N}(s, \omega)=\begin{bmatrix}0 & \sigma & 0\\ P_0^{-1}(s+P_2(\omega))\sigma^{-1} & -iP_0^{-1}P_1\cdot\omega & 0\\ -iQ_0^{-1}B\cdot\omega & -Q_0^{-1}B^0\sigma & Q_0^{-2}(s-iQ\cdot\omega)\end{bmatrix},$$

with

$$P_2(\omega)=\sum_{i,j}P_{2ij}\omega_i\omega_j,\ P_1\cdot\omega=\sum_j P_{1j}\omega_j,\ Q\cdot\omega=\sum_j Q_j\omega_j,\ B\cdot\omega=\sum_j B_j\omega_j.$$

$$F=(0, -\mathscr{E}^{-1}P_0^{-1}D^kF_1, -Q_0^{-1}D^kF_2)^t,$$

while $\mathscr{F}=(\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3)^t$ is defined by (3.11)—(3.14).

Let $\mathscr{R}(s, \omega)$ be the symmetrilizer constructed in [17] (There are mistakes there when treating the case of variable coefficients.). It is not difficult to check that for $\|W\|_{k,\eta}<\infty$, $\mathscr{R}(s, \omega)\in S^{0,k(u1)}_{\frac{1}{2}}$.

For the terms of lower order, we have, for $k\geqslant\left[\frac{N+1}{2}\right]+4$,

$$\|\mathscr{R}(0, \eta)\mathscr{F}\|^2_\eta\leqslant C_k(\|w\|^2_{k,\eta}+\|\mathscr{E}^{-1}u_x\|^2_{k,\eta}+\eta^{-2}\|\mathscr{E}^{-1}F_1\|^2_{k,\eta}+\eta^{-2}\|F_2\|^2_{k,\eta}). \quad (3.15)$$

Here and in the following, C_k will denote the constant depending on $\|W\|_k$, but being uniform of η.

To prove (3.15), by Proposition 3.5, iv), we need only to estimate $\|\mathscr{F}\|_\eta$.

1° By Sobolev imbedding theorem, $k\geqslant\left[\frac{N+1}{2}\right]+4$, so

$$\|\mathscr{F}_1\|^2_\eta=\|P_0^{-1}A_0(D^kv)\|^2_\eta\leqslant C_k\|D^kv\|^2_\eta;$$

2° To estimate $\mathscr{F}_3$: (see (3.13)).

From Propositions 3.5 and 3.6, we have

$$\|Q_0^{-1}B_0P_0^{-1}A_0(D^kv)\|_\eta\leqslant C_k\|D^kv\|_\eta,\ \|Q_0^{-1}D^k(C_{22}w)\|_\eta\leqslant C_k\|w\|_{k,\eta}.$$

Since Q_j, B_j are independent of ∇u, and $[D^k, Q_j\partial_{y_j}]$, $[D^k, B_j\partial_{y_j}]$ are k-order tangential operators, we have

$$\|Q_0^{-1}[D^k, Q_j\partial_{y_j}]v\|_\eta\leqslant C_k\|v\|_{k,\eta},\ \|Q_0^{-1}[D^k. B_j\partial_{y_j}]u\|_\eta\leqslant C_k\|u\|_{k,\eta}.$$

Because of

$$Q_0^{-1}[D^k, B_0\partial_x]u=Q_0^{-1}(\sum_{k_1+k_2=k}C_k(D^{k_1}B_0)(D^{k_2}\partial_xu)-B_0\partial_xD^ku),$$

its estimate is reduced to the estimation of $Q_0^-(D^{k_1}B_0)(D^{k_2}\partial_xu)$, $k_1+k_2=k$, $k_2<k$.

Since $Q_0^{-1}(D^{k_1}B_0)(D^{k_2}\partial_xu)=Q_0^{-1}(D^{k_1}B_0)(\mathscr{E}^{-1}\mathscr{E}D^{k_2}\partial_xu)$, we have

$$\|Q_0^{-1}(D^{k_1}B_0)(D^{k_2}\partial_xu)\|_\eta\leqslant C_k\|\mathscr{E}^{-1}u_x\|_{k,\eta}.$$

Similarly

$$[D^k, Q_0\partial_x]v=\sum_{k_1+k_2=k,k_2<k}(D^{k_1}Q_0)(D^{k_2}\partial_xv)-Q_0[\partial_x, D^k]v.$$

It remains to estimate $(D^{k_1}Q_0)(D^{k_2}v_x)$, $k_1+k_2=k$, $k_2<k$. The fact that Q_0 is nondegenerate implies $v_x=Q_0^{-1}(v_t-Q_jv_{y_j}-B_0u_x-B_ju_{y_j}-C_{22}w-F_2)$. Hence we get

$$\|(D^{k_1}Q_0)(D^kv_x)\|^2_\eta\leqslant C_k(\|D^kw\|^2_\eta+\|\mathscr{E}^{-1}D^ku_x\|^2_\eta+\eta^{-2}\|F_2\|^2_{k,\eta}).$$

To sum up, we get

$$\|\mathscr{F}_3\|^2_\eta\leqslant C_k(\|w\|^2_{k,\eta}+\|\mathscr{E}^{-1}u_x\|^2_{k,\eta}+\eta^{-2}\|F_2\|^2_{k,\eta}).$$

3° To estimate $\mathscr{F}_2$: (see (3.12)).

From Propositions 3.5, 3.6 and Corollary 3.7, one has

$$\|[\mathscr{E}^{-1}, P_0^{-1}(\partial_t-P_{2ij}\partial_{y_iy_j})](D^ku)\|_\eta+\|[\mathscr{E}^{-1}, P_0^{-1}P_{1j}\mathscr{E}\partial_{y_j}]\mathscr{E}^{-1}(D^ku)_x\|_\eta$$
$$\leqslant C_k(\|D^ku\|_\eta+\|\mathscr{E}^{-1}(D^ku)_x\|_\eta),$$

$$\|P_0^{-1}P_{1j}\partial_{y_j}\mathscr{E}^{-1}P_0^{-1}A_0(D^kv)\|_\eta+\|\mathscr{E}^{-1}P_0^{-1}A_j\partial_{y_j}(D^kv)\|_\eta$$
$$+\|\mathscr{E}^{-1}(P_0^{-1}A_0)_x(D^kv)\|_\eta\leqslant C_k\|D^kv\|_\eta,$$

$$\|\mathscr{E}^{-1}P_0^{-1}D^k(C_{11}w)\|_\eta\leqslant C_k\|w\|_{k,\eta},$$

$$\|\mathscr{E}^{-1}P_0^{-1}D^k(C_0u_x+C_ju_{y_j})\|_\eta\leqslant C_k(\|\mathscr{E}^{-1}u_x\|_{k,\eta}+\|u\|_{k,\eta}).$$

The estimate of $\mathscr{E}^{-1}P_0^{-1}[D^k, A_0\partial_x+A_j\partial_{y_j}]v$ can be carried out in a similar way as the corresponding terms in $\mathscr{F}_3$. The only difference is that A_0, A_j may depend on ∇U. By Proposition 3.6, it is estimated after the action of $\mathscr{E}^{-1}$ that

$$\|\mathscr{E}^{-1}P_0^{-1}[D^k, A_0\partial_x+A_j\partial_{y_j}]v\|_\eta^2\leqslant C_k(\|w\|_{k,\eta}^2+\|\mathscr{E}^{-1}u_x\|_{k,\eta}^2+\eta^{-2}\|F_2\|_{k,\eta}^2).$$

Since

$$\mathscr{E}^{-1}P_0^{-1}[D^k, P_{2ij}\partial_{y_iy_j}]u=\mathscr{E}^{-1}P_0^{-1}\sum_{k_1+k_2=k,k_2<k}(D^{k_1}P_{2ij})(D^{k_2}\partial_{y_iy_j}u)$$
$$-\mathscr{E}^{-1}P_0^{-1}P_{2ij}[\partial_{y_iy_j}, D^k]u,$$

and

$$\mathscr{E}^{-1}P_0^{-1}[D^k, P_{1j}\partial_{y_j}\partial_x]u=\mathscr{E}^{-1}P_0^{-1}\sum_{k_1+k_2=k,k_2<k}(D^{k_1}P_{1j})(D^{k_2}\partial_{xy_j}u)$$
$$-\mathscr{E}^{-1}P_0^{-1}P_{1j}[\partial_{xy_j}, D^k]u,$$

Proposition 3.6 again implies

$$\|\mathscr{E}^{-1}P_0^{-1}[D^k, P_{1j}\partial_{xy_j}+P_{2ij}\partial_{x_iy_j}]u\|_\eta\leqslant C_k(\|u\|_{k,\eta}+\|\mathscr{E}^{-1}u_x\|_{k,\eta}).$$

Finally, since

$$\mathscr{E}^{-1}P_0^{-1}[D^k, P_0\partial_{xx}]u=\mathscr{E}^{-1}P_0^{-1}\sum_{k_1+k_2=k,k_2<k}(D^{k_1}P_0)(D^{k_2}u_{xx})$$
$$-\mathscr{E}^{-1}[\partial_{xx}, D^k]u,$$

noticing that $[\partial_{xx}, D^k]u$ is the linear combination of terms of the form $D^{k_2}u_{xx}$, and using the parabolic equation to express u_{xx}, its estimation is reduced to the cases treated before. Hence

$$\|\mathscr{E}^{-1}P_0^{-1}[D^k, P_0\partial_{xx}]u\|_\eta^2\leqslant C_k(\|w\|_{k,\eta}^2+\|\mathscr{E}^{-1}u_x\|_{k,\eta}^2+\eta^{-2}\|\mathscr{E}^{-1}F_1\|_{k,\eta}^2).$$

To sum up, we have

$$\|\mathscr{F}_2\|_\eta^2\leqslant C_k(\|w\|_{k,\eta}^2+\|\mathscr{E}^{-1}u_x\|_{k,\eta}^2+\eta^{-2}\|\mathscr{E}^{-1}F_1\|_{k,\eta}^2).$$

Combining the estimates for $\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{F}_3$, we get (3.15).

4) Next, consider the k-times tangential enlarged boundary condition of (3.2). For $D^k\tilde{w}=(D^ku_1, D^ku_2, D^kv)$, we have:

$$\begin{cases} T_0D^ku_2+T_{1j}\mathscr{E}^{-1}\partial_{y_j}(D^ku)=\mathscr{E}^{-1}D^kg_1-\mathscr{E}^{-1}[D^k, T_0\partial_x+T_{1j}\partial_{y_j}]u \\ \quad -\mathscr{E}^{-1}D^k(T_2w)-[\mathscr{E}^{-1}, T_{1j}\partial_{y_j}\mathscr{E}]\mathscr{E}^{-1}D^ku_1 \\ \quad -[\mathscr{E}^{-1}, T_0\mathscr{E}^2]\mathscr{E}^{-2}D^ku_x+T_0\mathscr{E}^{-1}P_0^{-1}A_0D^kv. \\ T_3(D^ku)+S_3(D^kv)=D^kg_{21}-[D^k, T_3]u-[D^k, S_3]v, \\ T_4(D^ku)+S_4(D^kv)=D^kg_{22}-[D^k, T_4]u-[D^k, S_4]v. \end{cases} \tag{3.16}$$

Or briefly

$$\mathscr{T}(D^k\tilde{w})=G+\mathscr{G}. \tag{3.17}$$

Here

$$\mathscr{T}(D^k\tilde{w})=\begin{bmatrix} T_0D^ku_2+T_{1j}\mathscr{E}^{-1}\partial_{y_j}D^ku_1 \\ T_3D^ku_1+S_3D^kv \\ T_4D^ku_1+S_4D^kv \end{bmatrix},$$

$G=(\mathscr{E}^{-1}D^kg_1, D^kg_{21}, D^kg_{22})^t$, and $\mathscr{G}$ is defined by (3.16), (3.17).

For $\mathscr{G}=(\mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3)^t$, one have evidently

$$|\mathscr{G}_2|_\eta^2+|\mathscr{G}_3|_\eta^2=|[D^k, T_3]u+[D^k, S_3]v|_\eta^2+|[D^k, T_4]u+[D^k, S_4]v|_\eta^2$$
$$\leqslant C_k\eta^{-2}|w|_{k,\eta}^2.$$

Since $|\mathscr{E}^{-1}D^k(T_2w)+T_0\mathscr{E}^{-1}P_0^{-1}A_0D^kv|_\eta^2\leqslant C_k\eta^{-1}|w|_{k,\eta}^2$, we have

$$|[\mathscr{E}^{-1}, T_{1j}\partial_{y_j}\mathscr{E}]\mathscr{E}^{-1}D^ku_1|_\eta^2+|[\mathscr{E}^{-1}, T_0\mathscr{E}^2]\mathscr{E}^{-2}D^ku_x|_\eta^2$$
$$\leqslant C^k\eta^{-1}(|w|_{k,\eta}^2+|\mathscr{E}^{-1}u_x|_{k,\eta}^2).$$

$[D^k, T_{ij}\partial_{y_j}]$ is a k-order tangential operator, therefore

$$|\mathscr{E}^{-1}[D^k, T_{1j}\partial_{y_j}]u|_\eta^2\leqslant C_k\eta^{-1}|u|_{k,\eta}^2.$$

Since

$$\mathscr{E}^{-1}[D^k, T_0\partial_x]u=\sum_{k_1+k_2=k-1}C_{k_1,k_2}\mathscr{E}^{-1}(D^{k_1+1}T_0)(D^{k_2}\partial_xu)-\mathscr{E}^{-1}T_0[\partial_x, D^k]u,$$

and $[\partial_x, D^k]u$ can also be expressed in the form $D^{k-1}u_x$, by Proposition 3.6 and Corollary 3.7 we have

$$|\mathscr{E}^{-1}[D^k, T_0\partial_x]u|_\eta^2\leqslant C_k\eta^{-1}|\mathscr{E}^{-1}u_x|_{k,\eta}^2.$$

Consequently we get $|\mathscr{G}_1|_\eta^2\leqslant C_k\eta^{-1}(|w|_{k,\eta}^2+|\mathscr{E}^{-1}u_x|_{k,\eta}^2)$.

To sum up, we have

$$|\mathscr{G}|_\eta^2\leqslant C_k\eta^{-1}(|w|_{k,\eta}^2+|\mathscr{E}^{-1}u_x|_{k,\eta}^2). \tag{3.18}$$

5) To prove Theorem 3.1, we still need the following sharp Garding inequality, the proof of which we postpone till section 5.

Proposition 3.8. *Let $\mathscr{R}_\lambda(z, \xi, \eta)$ be the symmetrilizer constructed in* [17]:

$$\mathscr{R}_\lambda(z, \xi, \eta)=\mathscr{R}_\lambda^*(z, \xi, \eta)=\begin{bmatrix}\lambda\mathscr{R}_{11} & \mathscr{R}_{21}^*\\ \mathscr{R}_{21} & \mathscr{R}_{22}\end{bmatrix},$$

where $\mathscr{R}_{11}\in S_{1,\sigma}^{0,k(u1)}$, $\mathscr{R}_{21}$, $\mathscr{R}_{22}\in S_1^{0,k(u1)}$.

Define the Hermite symcol $H(z, \xi, \eta)$:

$$H(z, \xi, \eta)=\begin{bmatrix}H_{11} & H_{12}\\ H_{12}^* & H_{22}\end{bmatrix}=\mathrm{Re}\ \mathscr{R}_\lambda(z, \xi, \eta)\mathscr{N}(z, \xi, \eta),$$

where $H_{11}\in S_{1,\sigma}^{1,k(u1)}\cap S_{\frac{1}{2}}^{1,k(u1)}$, H_{12}, $H_{22}\in S_1^{1,k(u1)}$.

If

$$H(z, \xi, \eta)\geqslant C_0^1\begin{bmatrix}\sigma_0I_{2p} & \\ & \eta I_q\end{bmatrix}, \text{ with } \sigma_0=\mathrm{Re}\ \sigma,$$

then $\exists\lambda$, η_0 (η_0 may depend on λ), such that $\forall\eta\geqslant\eta_0$, the following inequality holds:

$$\mathrm{Re}\ (\widetilde{w}, \mathscr{R}_\lambda(z, D, \eta)\mathscr{N}(z, D, \eta)\widetilde{w})_\eta\geqslant C_0\ \mathrm{Re}(\widetilde{u}, \mathscr{E}\widetilde{u})_\eta+C_1\eta\|v\|_\eta^2, \tag{3.19}$$

where $\widetilde{w}=(\widetilde{u}, v)$, $\widetilde{u}$, v are C_0^∞ vector functions of $2p$ and q dimensions.

Applying the preceding results, we can prove the following tangential version of dependency theorem.

Lemma 3.9. *For $k\geqslant\left[\dfrac{N+1}{2}\right]+4$, the solutions of the well-posed linearized problem (3.1)—(3.3) satisfy the following istimate*

$$\mathrm{Re}\ (D^ku, \mathscr{E}D^ku)_\eta+\mathrm{Re}(D^ku_x, \mathscr{E}^{-1}D^ku_x)_\eta+\eta\|D^kv\|_\eta^2+|w|_{k,\eta}^2+|\mathscr{E}^{-1}u_x|_{k,\eta}^2$$
$$\leqslant C_k(\|(F, g)\|_{k,\eta}^2+\|w\|_{k,\eta}^2+\|\mathscr{E}^{-1}u_x\|_{k,\eta}^2). \tag{3.20}$$

Proof For simplicity, we drop λ in $\mathscr{R}_\lambda$. From the construction of $\mathscr{R}$ (z, ξ, η)

[17], we know

i) $\operatorname{Re} \mathscr{R}(x, y, t, s, \omega)\mathscr{N}(x, y, t, s, \omega) \geqslant C_0 \operatorname{diag}(\sigma_0 I_{2p}, \eta I_q)$;

ii) $\hat{\widetilde{w}}^t \mathscr{R}(x, y, t, s, \omega)\hat{\widetilde{w}} \geqslant C_1^{\prime}(K|\hat{\widetilde{w}}^+(s, \omega)|^2 - |\hat{\widetilde{w}}^-(s, \omega)|^2)$.

Here, $\hat{\widetilde{w}} = \hat{\widetilde{w}}^+ + \hat{\widetilde{w}}^-$ is the decomposition of $\hat{\widetilde{w}}$ with regard to the generalized eigenspaces of matrix $\mathscr{N}(0, y, t, s, \omega)$ ($\operatorname{Re} s > 0$), corresponding to the eigenvalues with positive and negative real parts respectively. As pointed out in [8], taking the localized neighborhood sufficiently small, the decomposition may be practised with $\mathscr{N}(0, y_0, t_0, s, \omega)$, here (y_0, t_0) is a fixed point in the considered neighborhood.

From (3.14) we have

$$\operatorname{Re}(D^k\widetilde{w}, \mathscr{R}(z, D, \eta)D^k\widetilde{w}_x)_\eta = \operatorname{Re}(D^k\widetilde{w}, \mathscr{R}(z, D, \eta)\mathscr{N}(z, D, \eta)D^k\widetilde{w})_\eta + \operatorname{Re}(D^k\widetilde{w}, \mathscr{R}(z, D, \eta)\mathscr{F})_\eta + \operatorname{Re}(D^k\widetilde{w}, \mathscr{R}(z, D, \eta)F)_\eta.$$

Integrating by parts the left side, since $\mathscr{R}(z, \xi, \eta)$ is Hermitian, we have

$$(D^k\widetilde{w}, \mathscr{R}(z, D, \eta)(C^k\widetilde{w})_x) = -\frac{1}{2}\langle C^k\widetilde{w}, \mathscr{R}(z, D, \eta)D^k\widetilde{w}\rangle_\eta + O(\|D^k\widetilde{w}\|_\eta^2).$$

By Proposition 3.5, (3.15) and Proposition 3.8, we have

$$\begin{aligned} &\frac{1}{2}\langle D^k\widetilde{w}, \mathscr{R}(z, D, \eta)D^k\widetilde{w}\rangle_\eta + C_0 \operatorname{Re}(D^k\widetilde{w}, \mathscr{E} D^k w)_\eta + C_1\eta\|D^k v\|_\eta^2 \\ &\leqslant C_k(\|D^k\widetilde{w}\|_\eta^2 + \|\mathscr{R}\mathscr{F}\|_\eta^2 + \|\mathscr{R}F\|_\eta^2) \\ &\leqslant C_k(\|D^k\widetilde{w}\|_\eta^2 + \|\mathscr{E}^{-1}F_1\|_{k,\eta}^2 + \|F_2\|_{k,\eta}^2 + \|w\|_{k,\eta}^2 + \|\mathscr{E}^{-1}u_x\|_{k,\eta}^2). \end{aligned} \tag{3.21}$$

Let $\mathscr{T}_0 = \mathscr{T}(y_0, t_0)$. Fourier–Laplace transform gives

$$\mathscr{T}_0(\widehat{D^k\widetilde{w}}) = \mathscr{T}_0^+(\widehat{D^k\widetilde{w}^+}) + \mathscr{T}_0^-(\widehat{D^k\widetilde{w}})^- = \hat{G} + \hat{\mathscr{G}}. \tag{3.22}$$

From the well-posedness, $\det \mathscr{T}_0^- \neq 0$ uniformly, therefore

$$|(\widehat{D^k\widetilde{w}})^-| \leqslant C(|\widehat{G+\mathscr{G}}| + |(\widehat{D^k\widetilde{w}})^+|).$$

From the condition ii) satisfied by $\mathscr{R}(z, \xi, \eta)$, we get for $k \gg 1$,

$$\begin{aligned} \langle D^k\widetilde{w}, \mathscr{R}(z, D, \eta)D^k\widetilde{w}\rangle_\eta &= \langle \widehat{D^k\widetilde{w}}, \mathscr{R}(z_0, \xi, \eta)\widehat{D^k\widetilde{w}}\rangle + \varepsilon|D^k\widetilde{w}|^2 \\ &\geqslant C_i^{\prime}\int (k|(\widehat{D^k\widetilde{w}})^+|^2 - |(\widehat{D^k\widetilde{w}})^-|^2)d\tau\, d\omega - \varepsilon|D^k\widetilde{w}|_\eta^2 \\ &\geqslant C_i^{\prime}\int |\widehat{D^k\widetilde{w}}|^2 d\tau\, d\omega - C_2^{\prime}\int |\widehat{G+\mathscr{G}}|^2 d\tau\, d\omega - \varepsilon|D^k\widetilde{w}|_\eta^2 \\ &\geqslant C_1^{\prime\prime}|D^k\widetilde{w}|_\eta^2 - C_2^{\prime\prime}|G|^2 - C_2^{\prime\prime}|\mathscr{G}|_\eta^2. \end{aligned} \tag{3.23}$$

Since $\mathscr{T}$ is an operator of 0-order, the operator norm of $\mathscr{T} - \mathscr{T}_0$ may be taken sufficiently small, depending only on $\|W\|_k$, by shrinking the localized neighborhood. From (3.17)—(3.18), we have

$$\begin{aligned} \langle D^k\widetilde{w}, \mathscr{R}(z, D, \eta)D^k\widetilde{w}\rangle_\eta \geqslant C_1^{\prime\prime}|\widetilde{w}|_{k,\eta}^2 - C_k(|\mathscr{E}^{-1}g_1|_{k,\eta}^2 + |g_2|_{k,\eta}^2 \\ + \eta^{-1}|D^k\widetilde{w}|_\eta^2) - \delta|D^k\widetilde{w}|_\eta^2, \end{aligned} \tag{3.24}$$

where δ may be taken sufficiently small.

Letting $\eta \gg 1$ in (3.21) and (3.24), one gets (3.20), since $D^k\widetilde{w} \sim (D^k u, D^k v, \mathscr{E}^{-1}D^k u_x)$.

Now it is easy to prove Theorem 3.1.

By Lemma 3.9, it remains to estimate the normal derivatives. We can proceed as in [8]. Differentiating the parabolic and hyperbolic equations in the normal direction $k-1$ times, we get the estimates of normal derivatives by tangential derivatives. Evidently this does not make stronger the dependency upon the coefficients. From (3.20) it follows that

$$\sqrt{\eta}\,(\|w\|_{k,\eta}^2+\|\mathscr{E}^{-1}u_x\|_{k,\eta}^2)+|w|_{k,\eta}^2+|\mathscr{E}^{-1}u_x|_{k,\eta}^2$$
$$\leqslant C_k(\|\!|(F,\ g)|\!\|_{k,\eta}^2+\|w\|_{k,\eta}^2+\|\mathscr{E}^{-1}u_x\|_{k,\eta}^2+\eta^{-1}|w|_{k,\eta}^2+\eta^{-1}|\mathscr{E}^{-1}u_x|_{k,\eta}^2). \quad (3.25)$$

Keeping in mind that C_k is independent of η, taking $\eta\gg 1$, we have

$$\|\!|w|\!\|_{k,\eta}^2\leqslant C_k\|\!|(F,\ g)|\!\|_{k,\eta}^2, \text{ where } C_k=C_k(\|\!|W|\!\|_k). \quad (3.26)$$

Since the support of W in t is contained in the compct set $[0,\ t_1]$, the norms $\|\!|W|\!\|_{k,\eta}$ and $\|\!|W|\!\|_k$ are equivalent for fixed η, and the equivalent constant depends on η. Thus

$$C_k(\|\!|W|\!\|_k)=C_{k,\eta}(\|\!|W|\!\|_{k,\eta})=C_{k,\eta}.$$

Substituting them into (3.26) gives (3.4).

6) Now we turn to the proof of Theorem 3.2.

Let $e(y,\ t,\ s,\ \omega)=sb_0+ib_{1j}\omega_j$. For a fixed point $(y_0,\ t_0)$ in the localized neighborhood, construct a pair of projection operators $\Pi_1(s,\ \omega)$ and $\Pi_2(s,\ \omega)$ as follows:

$$\Pi_1(s,\ \omega)\varphi=\frac{\langle\varphi,\ e(y_0,\ t_0,\ s,\ \omega)\rangle}{|e(y_0,\ t_0,\ s,\ \omega)|^2}\,e(y_0,\ t_0,\ s,\ \omega),\ \Pi_2=I-\Pi_1.$$

Let $\Pi_2(D_t+\eta,\ D_y)$ act on the last boundary condition in (3.6). We get

$$\begin{aligned}&\Pi_2(D_t+\eta,\ D_y)e(y_0,\ t_0,\ D_t+\eta,\ D_y)+\Pi_2S_4v+\Pi_2(R_2+e(y,\ t,\ D_t+\eta,\ D_y)\\&\quad-e(y_0,\ t_0,\ D_t+\eta,\ D_y))\phi+\Pi_2T_4u\\&\quad=\Pi_2(S_4v+R_2\phi+T_4u+e(y,\ t,\ D_t+\eta,\ D_y)\phi-e(y_0,\ t_0,\ D_t+\eta,\ D_y)\phi)\\&\quad=\Pi_2g_{22}.\end{aligned} \quad (3.27)$$

Substituting (3.27) for the last equation in (3.6), we will denote by (3.6′) the resulted boundary condition.

Fixing $(w,\ \phi)=(w_n,\ \phi_n)\in C_0^\infty(\bar{\Omega}\times(0,\ t_1))\times C_0^\infty(S_0\times(0,\ t_1))$, we may treat (3.5), (3.6′), (3.7) as an initial-boundary value problem for hyperbolic-parabolic coupled system we have discussed in Theorem 3.1, and the dependency of its energy estimate upon its coefficients may be analysed in the same way as above, except that two new features need careful examination.

First, the coefficients of w_{n+1} and its derivatives in (3.5), (3.6′) depend upon ϕ_n, $\partial_t\phi_n$ and $\partial_y^2\phi_n$, the terms $\partial_y^2\phi_n$ appearing only in the coefficients of the parabolic part of (3.5) and only in the coefficients of the terms which is of lower order with regard to $\partial_t u_{n+1}$, $\nabla^2 u_{n+1}$ and ∇v_{n+1}. From the proof of Theorem 3.1, we see that the constant in the k-order energy estimate depends on $\partial_x u_n$ in the coefficients of parabolic system only in the form $|\mathscr{E}^{-1}\partial_x u_n|_k$. Consequently, our dependency on $\partial_y^2\phi_n$ here is of the form $|\mathscr{E}^{-1}\partial_y^2\phi_n|_k\leqslant C|\phi_n|_{k+1}$. The dependency upon ϕ_n, $\partial_t\phi_n$ and $\partial_y\phi_n$ in the coefficients of hyperbolic or parabolic part is evidently of the order $|\phi_n|_{k+1}$. Therefore, in the energy estimate of k-order for the problem (3.5), (3.6′), (3.7), the constant depends only on $\tilde{|||}(w_n, \phi_n)\tilde{|||}_k$.

Secondly, as terms of lower order, (3.5), (3.6′) contains the terms ϕ_{n+1}, $\partial_t\phi_{n+1}$, $\partial_y\phi_{n+1}$ and $\partial_y^2\phi_{n+1}$, with $\partial_y^2\phi_{n+2}$ appearing only in the parabolic part of (3.5). Noticing the proof of Theorem 3.1, and $|\mathscr{E}^{-1}\partial_y^2\phi_{n+1}|_{k,\eta}\leqslant C|\phi_{n+1}|_{k+1,\eta}$, we get

$$\begin{aligned}|||w_{n+1}|||_{k,\eta}^2\leqslant C_k(|||(F, g)|||_{k,\eta}^2+\eta^{-\frac{1}{2}}|\phi_{n+1}|_{k+1,\eta}^2\\ +|M_2\phi_{n+1}|_{k,\eta}^2+\mathscr{E}|\phi_{n+1}|_{k+1,\eta}^2),\end{aligned}\tag{3.28}$$

where the last terms on the right side come from the two terms in (3.27), containing ϕ. Taking the localized neighborhood small enough, we may make ε as small as we wish, while the operator norm of M_2 may increase correspondingly. The constant C_k in (3.28) depends on $\tilde{|||}(w_n, \phi_n)\tilde{|||}_k$, and is independent of η.

For well-posed problem, $e(t, y, s, \omega)\neq 0$, so from (3.6) one may solve $\partial_t\phi_{n+1}$ and $\partial_y\phi_{n+1}$ in term of w_{n+1}, ϕ_{n+1} and g_{22}. Thus, the Proposition 3.6 implies

$$|\phi_{n+1}|_{k+1,\eta}^2\leqslant C_k^1(|w_{n+1}|_{k,\eta}^2+|g_{22}|_{k,\eta}^2),\tag{3.29}$$

where the constnnt C_k^1 depends on $\tilde{|||}(w_n, \phi_n)\tilde{|||}_k$, and is independent of η.

Adding (3.29), multiplied by small positive, upon (3.28) and taking the localized neighborhood sufficiently small to make $\varepsilon\ll 1$, then making $\eta\gg 1$, we have

$$\tilde{|||}(w_{n+1}, \phi_{n+1})\tilde{|||}_{k,\eta}^2\leqslant C_k|||(F, g)|||_{k,\eta}^2,\tag{3.30}$$

where the constant C_k depends on $\tilde{|||}(w_n, \phi_n)\tilde{|||}_k$, and is independent of η. As in the proof of Theorem 3.1, $C_k(\tilde{|||}(w_n, \phi_n)\tilde{|||}_k)\leqslant C_{k,\eta}(\tilde{|||}(w_n, \phi_n)\tilde{|||}_{k,\eta})$, therefore we get (3.8).

§ 4. The Existence and Uniqueness of Local Solution

Here, we will use the results of Theorems 3.1 and 3.2 to prove Theorems 1 and 2′ by linear iteration. We only give the proof of Theorem 1. The proof of Theorem 2′ can be done similarly, and is omitted here.

1° Existence:

Denote (2.1)—(2.3) briefly by

$$\begin{cases}u_t-Pu-Av=F_1,\\ v_t-Qv-Bu=F_2,\end{cases}\quad \text{in } \Omega\times R_+^1,\tag{2.1′}$$

$$T_0u+T_{1j}u_{y_j}+T_1w=g_1,\ T_2w=g_2,\ \text{on } \partial\Omega\times R_+^1,\tag{2.2′}$$

$$w(x, 0)=0.\tag{3.3′}$$

For $r>0$, let $\varphi_r(t)\in C^\infty(R^1)$: $0\leqslant\varphi_r\leqslant 1$; $\varphi_r(t)=1$ when $t<\frac{r}{2}$; $\varphi_r(t)=0$ when $t>r$.

We formulate the following new initial-boundary value problem from (2.1′)—(2.3′);

$$\begin{cases} u_t-\widetilde{P}u-\widetilde{A}v=\widetilde{F}_1, \\ v_t-\widetilde{Q}v-\widetilde{B}u=\widetilde{F}_2, \end{cases} \text{ in } \Omega\times R^1_+, \tag{4.1}$$

$$\widetilde{T}_0u_x+\widetilde{T}_{1j}u_{y_j}+\widetilde{T}_1w=\widetilde{g}_1,\ \widetilde{T}_2w=\widetilde{g}_2, \quad \text{on } \partial\Omega\times R^1_+, \tag{4.2}$$

$$w(x,\ 0)=0, \tag{4.3}$$

where $\widetilde{P}$ is obtained by multiplying all arguments in the coefficients of P by $\varphi_{t_1}(t)$; $\widetilde{A}$, $\widetilde{B}$, $\widetilde{Q}$, $\widetilde{T}_0$, $\widetilde{T}_{1j}$, $\widetilde{T}_1$, $\widetilde{T}_2$ are defined in the same way; and $(\widetilde{F},\ \widetilde{g})=\varphi_{t_2}(t)(F,\ g)$ with $t_2\leqslant t_1$.

Write

$\mathscr{B}(\varepsilon,\ \tilde{t}_1)=\{w;\ \text{supp}\ w\subset\overline{\Omega}\times[0,\ \tilde{t}_1),\ \|w\|_{k,\eta}\leqslant\varepsilon,\ D^j_tw|_{t=0}=0,\ j=0,\ 1,\cdots,k-1.\}$.

For small ε and t_1, after substituting any $W\in\mathscr{B}(\varepsilon,\ \infty)$ into the coefficients of (4.1)—(4.3), the resulted linear problem is well-posed everywhere on $\partial\Omega\times R^1_+$. On account of Theorem 4.1 in [9] and Theorem 3.1 of section 3 in this paper, the linearized problem has a solution w, satisfying

$$\|w\|_{k,\eta}\leqslant C_{k,\eta}\|(\widetilde{F},\ \widetilde{g})\|_{k,\ \eta}. \tag{4.4}$$

Since $(F,\ g)$ has zero traces up to the order $k-1$ at $t=0$, we have $\|(\widetilde{F},\ \widetilde{g})\|_{k,\eta}\to 0$ as $t_2\to 0$. Fixing η, and taking $t_2\ll 1$, we may make the right side of (4.4) smaller than ε. Denote the linearized iteration operator of (4.1)—(4.3) by $\mathscr{A}$. Then $\mathscr{A}$: $W\mapsto w$ maps $\mathscr{B}(\varepsilon,\ \infty)$ into $\mathscr{B}(\varepsilon,\ \infty)$.

Taking any $w_0\in\mathscr{B}(\varepsilon,\ \infty)$, we construct $w_{j+1}=\mathscr{A}w_j(j=0,\ 1,\cdots\cdots)$. Define $W_{j+1}=w_{j+1}-w_j$. Then W_{j+1} satisfies

$$\begin{cases} \partial_tU_{j+1}-\widetilde{P}(w_j)U_{j+1}-\widetilde{A}(w_j)V_{j+1}=(\widetilde{P}(w_j)-\widetilde{P}(w_{j-1}))u_j+(\widetilde{A}(w_j)-\widetilde{A}(w_{j-1}))v_j, \\ \partial_tV_{j+1}-\widetilde{Q}(w_j)V_{j+1}-\widetilde{B}(w_j)U_{j+1}=(\widetilde{Q}(w_j)-\widetilde{Q}(w_{j-1}))v_j+(\widetilde{B}(w_j)-\widetilde{B}(w_{j-1}))u_j, \end{cases} \tag{4.5}$$

$$\begin{cases} \widetilde{T}_0(w_j)\partial_xU_{j+1}+\widetilde{T}_{1h}(w_j)\partial_{y_h}U_{j+1}+\widetilde{T}_1(w_j)W_{j+1}=(\widetilde{T}_0(w_{j-1})-\widetilde{T}_0(w_j))\partial_xu_j \\ \quad+(\widetilde{T}_{1h}(w_{j-1})-\widetilde{T}_{1h}(w_j))\partial_{y_h}u_j+(\widetilde{T}_1(w_{j-1})-\widetilde{T}_1(w_j))w_j, \\ \widetilde{T}_2(w_j)W_{j+1}=(\widetilde{T}_2(w_{j-1})-\widetilde{T}_2(w_j))w_j, \end{cases} \tag{4.6}$$

$$W_{j+1}(x,\ 0)=0. \tag{4.7}$$

From Proposition 3.6, the right sides of (4.5) and (4.6) can be estimated as follows

$$\|\mathscr{E}^{-1}(\widetilde{P}(w_j)-\widetilde{P}(w_{j-1}))u_j\|_{k-2,\eta}\leqslant C_{k,\eta}\cdot\|u_j\|_{k,\eta}\cdot(\|W_j\|_{k-2,\eta}+\|\mathscr{E}^{-1}(U_j)_x\|_{k-2,\eta}),$$

$$\|\mathscr{E}^{-1}(\widetilde{A}(w_j)-\widetilde{A}(w_{j-1}))v_j\|_{k-2,\eta}\leqslant C_{k,\eta}\cdot\|v_j\|_{k,\eta}\cdot(\|W_j\|_{k-2,\eta}+\|\mathscr{E}^{-1}(U_j)_x\|_{k-2,\eta}),$$

$$\|(\widetilde{Q}(w_j)-\widetilde{Q}(w_{j-1}))v_j+(\widetilde{B}(w_j)-\widetilde{B}(w_{j-1}))u_j\|_{k-2,\eta}\leqslant C_{k,\eta}\cdot\|w_j\|_{k,\eta}\cdot\|W_j\|_{k-1,\eta},$$

$$\begin{aligned} &|\mathscr{E}^{-1}(\widetilde{T}_0(w_{j-1})-\widetilde{T}_0(w_j))\partial_xu_j+\mathscr{E}^{-1}(\widetilde{T}_{1h}(w_{j-1})-\widetilde{T}_{1h}(w_j))\partial_{y_h}u_j \\ &\quad+\mathscr{E}^{-1}(\widetilde{T}_1(w_{j-1})-\widetilde{T}_1(w_j))w_j|_{k-2,\eta}\leqslant C_{k,\eta}(|w_j|_{k,\eta} \\ &\quad+|\mathscr{E}^{-1}\partial_xu_j|_{k,\eta})(|W_j|_{k-2,\eta}+|\mathscr{E}^{-1}(U_j)_x|_{k-2,\eta}),\end{aligned}$$

$$|(\widetilde{T}_2(w_{j-1})-\widetilde{T}_2(w_j))w_j|_{k-2,\eta}\leqslant C_{k,\eta}|w_j|_{k,\eta}\cdot|W|_{k-2,\eta}.$$

Applying the $(k-2)$-order energy inequality for well-posed problem (4.5)–(4.7), we have

$$\|W_{j+1}\|_{k-2,\eta} \leqslant C_\eta \|W_j\|_{k-2,\eta} \cdot \|w_j\|_{k,\eta}, \tag{4.8}$$

where the constant C_η depends on η, but is independent of $t_2 \ll 1$. Take $t_2 \ll 1$ such that $\varepsilon \ll 1$ and hence $C_\eta \|w_j\|_{k,\eta} < \frac{1}{2}$. Thus (3.8) implies that $\{w_j\}$ is a Cauchy sequence in the norm $\|\cdot\|_{k-2}$. Since $w_j \in \mathscr{B}(\varepsilon, \infty)$, we see that $w_j \to w \in \mathscr{B}(\varepsilon, \infty)$ by Banach-Saks Theorem, and w is the fixed point of $\mathscr{A}$, i.e., the solution of (4.1)—(4.3). But (4.1)—(4.3) is identical with (2.1′)—(2.3′) in $\left(0, \frac{1}{2} t_2\right)$. Taking $t_0 = \frac{1}{2} t_2$, we have the solution w of (2.1′)—(2.3′) in $(0, t_0)$.

2° Uniqueness:

Let w, w' are two solutions of (2.1′)—(2.3′) in $(0, t_0)$ and $(0, t_0^1)$ respectively:

$$w \in H^k(\Omega \times (0, t_0)): \|\psi w\|_{k,\eta} < \infty, \ \forall \psi \in C_0^\infty(-t_0, t_0);$$
$$w' \in H^k(\Omega \times (0, t_0^1)): \|\psi' w'\|_{k,\eta} < \infty, \ \forall \psi' \in C_0^\infty(-t_0', t_0').$$

Let $t_0'' = \min(t_0, t_0')$. We want to prove $w = w'$ in $(0, t_0'')$.

Because the set of all $t \in [0, t_0'']$ for which $w = w'$ is closed, it remains to prove it to be open, i.e., $\forall r_0 < t_0''$, if $w = w'$ in $[0, r_0]$, then $\exists \delta > 0$, $r_0 + \delta \leqslant t_0''$, such that $w = w'$ in $[0, r_0 + \delta]$. Without loss of generality, we may assume $r_0 = 0$.

Since w, $w' = 0$ for $t < 0$, we see that $\forall \varepsilon > 0$, $\exists r_1 \leqslant t_0''$, such that $\|w\|_{k,\eta,r_1} \leqslant \varepsilon$, $\|w'\|_{k,\eta,r_1} \leqslant \varepsilon$. Let $(\tilde{F}, \tilde{g}) = \varphi_{r_2}(t)(F, g)$, where φ_r is defined as in 1°. Similarly choosing r_1 and r_2, we consider (4.1)—(4.3).

It is evident that when $r_1 \ll 1$, $\varepsilon \ll 1$, (4.1)—(4.3) is well-posed for any $w \in \mathscr{B}(\varepsilon, \infty)$. As in 1°, for $r_2 \leqslant r_1$, $r_2 \ll 1$, the linearized iteration operator $\mathscr{A}$ of (4.1)—(4.3) maps $\mathscr{B}(\varepsilon, \infty)$ to $\mathscr{B}(\varepsilon, \infty)$. And for $\varepsilon \ll 1$, $\mathscr{A}$ is contracted in the norm $\|\cdot\|_{k-2,\eta}$. Since $\mathscr{B}(\varepsilon, \infty)$ is convex and closed in the Hilbert space with norm $\|\cdot\|_{k,\eta}$, by Banach-Saks Theorem, it is closed in the topology with the norm $\|\cdot\|_{k-2,\eta}$. From the contraction of $\mathscr{A}$, we know that the fixed point of $\mathscr{A}$ is unique in $\mathscr{B}(\varepsilon, \infty)$.

Substituting w, w' into the coefficients of (4.1)—(4.3), we have the well-posed linearized problem, so there are solutions w_1, w_1' in $\mathscr{B}(\varepsilon, \infty)$, therefore the iteration may proceed:

$$w_{j+1} = \mathscr{A} w_j, \ w_{j+1}' = \mathscr{A} w_j', \quad j = 1, 2, \cdots.$$

As in 1°, for $r_2 \ll 1$, the sequences $\{w_j\}$ and $\{w_j'\}$ converge to w^* and $w'^* \in \mathscr{B}(\varepsilon, \infty)$, respectively, w^*, w'^* being the fixed point of $\mathscr{A}$. But the fixed point of $\mathscr{A}$ is unique, so $w^* = w'^*$.

Because of (4.1)—(4.3) is identical with (2.1′)—(2.3′) in $\left(0, \frac{1}{2} r_2\right)$, and w, w' are solutions of (2.1′)—(2.3′) in $\left(0, \frac{1}{2} r_2\right)$, from the uniqueness theorem for linear problem in [9], it follows that $w=w_1$, $w'=w'_1$ in $\left(0, \frac{1}{2}r_2\right)$, i.e., w_1, w'_1 are solutions of (2.1′)—(2.3′) in $\left(0, \frac{1}{2}r_2\right)$. In the same way, we can prove $w_1=w_2=\cdots=w^*$, $w'_1=w'_2=\cdots=w'^*$ in $\left(0, \frac{1}{2}r_2\right)$. But $w^*=w'^*$, thus $w=w'$ in $\left(0, \frac{1}{2}r_2\right)$, and $\frac{1}{2}r_2$ can be taken as the desired δ.

§5. The Relevant Properties of η-Weighted Pseudo-differential Operators

In this section, we discuss the properties of a class of pseudo-differential operators whose symbols have only H^k regularity and are η-weighted. We will give the proof for Propositions 3.5, 3.6, and 3.8. Our approach of proof follows Beals–Reed[2] and changes the corresponding proof for smooth coefficients in [17]. Compared with the result of Majda in [11, 12], the requirement on the regularity of the coefficient is considerably relaxed.

I) The Proof of Proposition 3.5.

First we cite a useful lemma (cf. Lemma 1.4, [2]):

Lemma 5.1. *Suppose*

$$C_g^2=\sup_\xi\int|g(\xi', \xi)|^2d\xi'<\infty,$$

$$C_G^2=\sup_\xi\int|G(\xi', \xi)|^2d\xi<\infty.$$

Then

$$\forall h\in L^2(R^N_\xi),\ (Th)(\xi')=\int G(\xi', \xi)g(\xi'-\xi, \xi)h(\xi)d\xi\in L^2(R^N_{\xi'}),$$

and $\|Th\|_{L^2}\leqslant C_gC_G\|h\|_{L^2}$.

Using this lemma, we now prove Proposition 2.4:

i) Since

$$\widehat{a(z,D,\eta)u}\,(\xi', \eta)=\int\hat{a}(\xi'-\xi, \xi, \eta)\hat{u}(\xi, \eta)d\xi$$
$$=\int\langle\xi'-\xi\rangle^{-k}\langle\xi'-\xi\rangle^{k}\hat{a}(\xi'-\xi, \xi, \eta)\hat{u}(\xi,\eta)d\xi,$$

where $\hat{a}$ denotes the Fourier transform of a with the argument z, all the rest of "$\wedge$" denote Fourier–Laplace transform.

Let $G(\xi', \xi)=\langle\xi'-\xi\rangle^{-k}$, $g(\xi', \xi)=\langle\xi'\rangle^{k}\hat{a}(\xi', \xi, \eta)$. Since $k>\frac{1}{2}N$, from Lemma 5.1 it follows that

$$\|a(z, D, \eta)u(z)\|_\eta=\|\widehat{a(z,D,\eta)u}(\xi', \eta)\|\leqslant C\|\hat{u}(\xi', \eta)\|=C\|u\|_\eta.$$

ii) We need only to show that $\forall u,\ v\in L^2(R^N, \eta)$,

$$|(a(z, D, \eta)u, v)_\eta| \leq C\|u|_\eta\|v\|_\eta.$$

We can choose $\{\varphi_j\}$ such that $1=\Sigma\varphi_j(z)$, where $\varphi_j(z)=\varphi(z-z_j)$ with some z_j, and $\varphi\in C_0^\infty$. Let $\psi_j(z)=\psi(z-z_j)$, $\psi\in C_0^\infty$, $\psi=1$ on supp φ. Hence

$$\begin{aligned}
&|(a(z, D, \eta)u(z), v(z))_\eta| \\
&=\left|\int a(z, \xi, \eta)\hat{u}(\xi, \eta)v(z)e^{iz\xi-\eta t}d\xi\, dz\right| \\
&=\leq\sum_j\left|\int(\psi_j a)(z, \xi, \eta)e^{iz\xi}\hat{u}(\xi, \eta)d\xi e^{-\eta t}(\varphi_j v)(z)dz\right| \\
&\leq\sum_j\|\varphi_j v\|_\eta\cdot\left\|\int e^{iz}(\psi_j a)(z, \xi, \eta)\hat{u}(\xi, \eta)d\xi\right\|_\eta.
\end{aligned}$$

But $(\psi_j a)\in S_\rho^{0,k}$, so from i), $(\psi_j a)(z, D, \eta)$ is a uniformly bounded operator in $L^2(R^N, \eta)$ and its norm is independent of j, by the make-up of ψ_j and the definition of H_{ul}^k. Therefore

$$|(a(z, D, \eta)u, v)_\eta|\leq C\sum_j\|\varphi_j v\|_\eta\|u\|_\eta\leq C\|u\|_\eta\|v\|_\eta.$$

iii) Let $r(z, D, \eta)=[\mathscr{E}, P(z, D, \eta)]$. Then

$$\widehat{r(z,D,\eta)u}(\xi, \eta)=\int\hat{r}(\xi-\zeta, \zeta, \eta)\hat{u}(\zeta, \eta)d\zeta,$$

where

$$\hat{r}(\xi, \zeta, \eta)=\sum_{|\alpha|=1}C_\alpha\int_0^1\partial_\zeta^\alpha\sigma(\zeta+t\xi, \eta)dt\,\widehat{D_z^\alpha P}(\xi, \zeta, \eta).$$

Since $\partial_\zeta^\alpha\sigma(\zeta+t\xi, \eta)$ is uniformly bounded for $|\alpha|=1$, and $\langle\zeta, \eta\rangle_\sigma^{-m}\cdot\langle\xi\rangle^k\widehat{D_z^\alpha P}(\xi, \zeta, \eta)$ is a uniformly bounded set in $L^2(R_\xi^N)$ for all ζ, η, we have $r(z, \zeta, \eta)\in S_{1,\sigma}^{m,k}$.

iv) From iii),

$$\begin{aligned}
\mathscr{E}^{-1}P(z, D, \eta)&=P(z, D, \eta)\mathscr{E}^{-1}+\mathscr{E}^{-1}[P(z, D, \eta), \mathscr{E}]\mathscr{E}^{-1}\\
&=P(z, D, \eta)\mathscr{E}^{-1}+\mathscr{E}^{-1}r_1(z, D, \eta),
\end{aligned}$$

where $r_1(z, D, \eta)\in SP_{1,\sigma}^{1,k+1}$.

Similarly, $\mathscr{E}^{-1}r_1(z, D, \eta)=r_1(z, D, \eta)\mathscr{E}^{-1}+\mathscr{E}^{-1}r_2(z, D, \eta)$, where $r_2(z, D, \eta)\in SP_{1,\sigma}^{0,k}$. Evidently, $r_1(z, D, \eta)\mathscr{E}^{-1}\in SP_{1,\sigma}^{0,k+1}$, so $[\mathscr{E}^{-1}, P(z, D, \eta)]\in SP_{1,\sigma}^{0,k}$.

II) The Proof of Proposition 3.6.

i) The proof may proceed in the same way as in [15], one needs only to substitute the η-weighted norms for the usual ones.

ii) We have only to show that

$$\|\mathscr{E}^{-1}(uv)\|_{k,\eta}\leq C\|\mathscr{E}^{-1}u\|_{k,\eta}\|v\|_k,\quad k>\frac{N}{2}.$$

Following [15], we have

$$\langle\xi, \eta\rangle^k\sigma^{-1}\widehat{uv}(\xi, \eta)=\langle\xi, \eta\rangle^k\sigma^{-1}(\xi, \eta)\int\langle\xi', \eta\rangle^{-k}\langle\xi'-\xi\rangle^{-k}\cdot f(\xi', \eta)g(\xi'-\xi)d\xi',$$

where $f(\xi, \eta)=\langle\xi, \eta\rangle^k\hat{u}(\xi, \eta)$, $g(\xi)=\langle\xi\rangle^k\hat{v}(\xi)$. The notation "$\wedge$" means the Fourier-Laplace transform or Fourier transform according to whether the expression containing η or not. Thus

$$\langle\xi,\ \eta\rangle^k\sigma^{-1}(\xi,\ \eta)\widehat{uv}(\xi,\ \eta)=\Big(\int_{|\xi'|<\frac{1}{2}|\xi|}+\int_{|\xi'|\geq\frac{1}{2}|\xi|}\Big)g(\xi'-\xi)\sigma^{-1}(\xi,\ \eta)\sigma(\xi',\ \eta)$$
$$\cdot\langle\xi,\ \eta\rangle^k\langle\xi',\ \eta\rangle^{-k}\langle\xi'-\xi\rangle^{-k}f(\xi',\ \eta)\sigma(\xi',\ \eta)^{-1}d\xi'. \tag{5.1}$$

When $|\xi'|\leq\frac{1}{2}|\xi|$, we have $|\xi'-\xi|\sim|\xi|$. Expanding the terms $\langle\xi,\ \eta\rangle^k$ in the above relation, we can estimate each term in turn:

$$\langle\xi,\ \eta\rangle^k\langle\xi',\ \eta\rangle^{-k}\leq\sum_{k'\leq k}C_{k'}\langle\xi\rangle^{k'}\langle\xi'\rangle^{-k'},$$
$$\sigma^{-1}(\xi,\ \eta)\sigma(\xi',\ \eta)\langle\xi,\ \eta\rangle^k\langle\xi',\ \eta\rangle^{-k}\langle\xi'-\xi\rangle^{-k}\leq\sum_{k'\leq k}C_{k'}\langle\xi\rangle^{k'},$$
$$\langle\xi'\rangle^{-k'}\langle\xi\rangle^{-k}\leq C\langle\xi'\rangle^{-k}.$$

When $|\xi'|\geq\frac{1}{2}|\xi|$,

$$\sigma(\xi',\ \eta)\langle\xi',\ \eta\rangle^{-k}\sigma(\xi,\eta)^{-1}\langle\xi,\ \eta\rangle^k$$

is uniformly bounded, because of the monotonous decrease of $\sigma(\xi,\ \eta)\langle\xi,\ \eta\rangle^{-k}$ about $|\xi|$. Since $\langle\xi'-\xi\rangle^{-k}\in L^2(R^N_{\xi'})$, $\sigma^{-1}(\xi,\ \eta)\sigma(\xi',\ \eta)\langle\xi,\ \eta\rangle^k\langle\xi',\ \eta\rangle^{-k}\langle\xi'-\xi\rangle^{-k}$ is a uniformly bounded set in $L^2(R^N_{\xi'})$ for all ξ and η.

Combining the estimates for $|\xi'|\leq\frac{1}{2}|\xi|$ and $|\xi'|\geq\frac{1}{2}|\xi|$, and applying Lemma 5.1 to (5.1), we get the desired estimate.

iii) We proceed as in ii). For $|\xi'|\leq\frac{1}{2}|\xi|$, $\frac{1}{2}|\xi|\leq|\xi'|\leq2|\xi|$ and $|\xi'|\geq2|\xi|$, we estimate separately the integration in (5.1) and arrive at the desired inequality, details omitted.

III) The Proof of Proposition 3.8 (Sharp Garding Inequality).

The proof consists of the following steps:

1° For the composition of the operators $\mathscr{R}_\lambda(z,\ D,\ \eta)$ and $\mathscr{N}(z,\ D,\ \eta)$;

Lemma 5.2. *Under the condition of Proposition* 3.8, $(\mathscr{R}_\lambda\mathscr{N})(z,\ D,\ \eta)-\mathscr{R}_\lambda(z,\ D,\ \eta)\circ\mathscr{N}(z,\ D,\ \eta)$ *is a bounded operator in* $L^2(R^N,\ \eta)$.

Proof Because of the block structure of $\mathscr{R}_\lambda$, $\mathscr{N}$, we need only to consider the composition of the corresponding blocks.

$$\mathscr{R}_\lambda\mathscr{N}=\begin{bmatrix}\lambda\mathscr{R}_{11}\mathscr{N}_{11}+\mathscr{R}^*_{21}\mathscr{N}_{21} & \mathscr{R}^*_{21}\mathscr{N}_{22}\\ \mathscr{R}_{21}\mathscr{N}_{11}+\mathscr{R}_{22}\mathscr{N}_{21} & \mathscr{R}_{22}\mathscr{N}_{22}\end{bmatrix}.$$

Denote by $\langle D,\ \eta\rangle$ the operator with symbol $(|\xi|^2+\eta^2)^{\frac{1}{2}}$. As in [2], making use of the Cauchy integral formula for remainders of 1-order Taylor expansion, we have $r(z,\ D,\ \eta)=[\mathscr{N}_{22}(z,\ D,\ \eta),\ \langle D,\ \eta\rangle]\in SP^{1,k-1}_1$, and

$$\mathscr{R}_{22}(z,\ D,\ \eta)\langle D,\ \eta\rangle\circ\mathscr{N}_{22}(z,\ D)\langle D,\ \eta\rangle^{-1}=(\mathscr{R}_{22}\mathscr{N}_{22})(z,\ D,\ \eta)+r_1(z,D,\ \eta),$$

where $r_1(z,\ D,\ \eta)\in SP^{0,k-1}_1$. Thus

$$\begin{aligned}\mathscr{R}_{22}\circ\mathscr{N}_{22}&=\mathscr{R}_{22}\circ\mathscr{N}_{22}\circ\langle D,\ \eta\rangle\langle D,\ \eta\rangle^{-1}\\&=\mathscr{R}_{22}\langle D,\ \eta\rangle\circ\mathscr{N}_{22}\langle D,\ \eta\rangle^{-1}+\mathscr{R}_{22}\circ r(z,\ D,\ \eta)\langle D,\ \eta\rangle^{-1}\\&=(\mathscr{R}_{22}\mathscr{N}_{22})(z,\ D,\ \eta)+r_1(z,\ D,\ \eta)+\mathscr{R}_{22}\circ r(z,\ D,\ \eta)\langle D,\ \eta\rangle^{-1},\end{aligned}$$

where $r_1(z, D, \eta)+\mathscr{R}_{22}\circ r(z, D, \eta)\langle D, \eta\rangle^{-1}$ is bounded in $L^2(R^N, \eta)$.

$\mathscr{R}^*_{21}\mathscr{N}_{22}$ may be treated similarly.

The rest four terms in $\mathscr{R}_\lambda\mathscr{N}$ may be discussed in the same way, if we change $\langle D, \eta\rangle$ by $\mathscr{E}$.

2° For the conjugate of the operator, we have the following lemma.

Lemma 5.3. *Assume the conditions in Proposition 3.8. Then* $(\mathscr{R}_\lambda\mathscr{N})^*(z, D, \eta) - (\overline{\mathscr{R}_\lambda\mathscr{N}})'(z, D, \eta)$ *is a bounded operator in* $L^2(R^N, \eta)$.

Proof Since

$$(\mathscr{R}_\lambda\mathscr{N})^*(z, D, \eta)u(z)=e^{\eta t}\int e^{i\langle z-z', \xi\rangle}e^{-\eta t}(\overline{\mathscr{R}_\lambda\mathscr{N}})'(z, \xi, \eta)u(z')dz'd\xi,$$

we have

$$\widehat{(\mathscr{R}_\lambda\mathscr{N})^*u}(\xi, \eta)=\int(\widehat{\overline{\mathscr{R}_\lambda\mathscr{N}}})'(\xi-\zeta, \xi, \eta)\hat{u}(\zeta, \eta)d\zeta.$$

But

$$\begin{aligned}(\widehat{\overline{\mathscr{R}_\lambda\mathscr{N}}})'(\xi-\zeta, \xi, \eta)&=(\widehat{\overline{\mathscr{R}_\lambda\mathscr{N}}})'(\xi-\zeta, \zeta, \eta)\\&+\sum_{|\alpha|=1}C_\alpha\int_0^1\partial^\alpha_{\zeta_1}(\widehat{\overline{\mathscr{R}_\lambda\mathscr{N}}})'(\xi-\zeta, \zeta_1+t(\xi-\zeta), \eta)_{\zeta_1=\zeta}dt(\xi-\zeta)^\alpha,\\ \widehat{(\mathscr{R}_\lambda\mathscr{N})^*u}(\xi, \eta)&=(\widehat{\overline{\mathscr{R}_\lambda\mathscr{N}})'u}(\xi, \eta)+\int\hat{r}(\xi-\zeta, \zeta, \eta)\hat{u}(\zeta, \eta)d\zeta,\end{aligned}$$

where

$$\hat{r}(\xi, \zeta, \eta)=\sum_{|\alpha|=1}C_\alpha\int_0^1\partial^\alpha_\zeta(\widehat{\overline{\mathscr{R}_\lambda\mathscr{N}}})'(\xi, \zeta+t\xi, \eta)dt\,\xi^\alpha.$$

Evidently, $\langle\xi\rangle^{k-1}\hat{r}(\xi, \zeta, \eta)$ is a uniformly bounded set in $L^2(R^N_\xi)$ for all ζ, η Hence Proposition 3.5 implies that $r(z, D, \eta)$ is bounded.

3° By Lemmas 5.2 and 5.3, the proof of Proposition 3.8 is reduced to the proof of the following inequality

$$\operatorname{Re}(\tilde{w}, H(z, D, \eta)\tilde{w})_\eta\geqslant C_0\operatorname{Re}(\tilde{u}, \mathscr{E}\tilde{u})_\eta+C_1\eta\|v\|^2_\eta. \tag{5.2}$$

Let

$$\begin{aligned}G(z, \xi, \eta)&=\operatorname{Re}(\mathscr{R}_\lambda\mathscr{N})-c_0\begin{bmatrix}\sigma_0I_{2p} & \\ & \eta I_q\end{bmatrix}-(\lambda-1)\begin{bmatrix}\operatorname{Re}(\mathscr{R}_{11}\mathscr{N}_{11}) & \\ & 0\end{bmatrix}\\&=\operatorname{Re}(\mathscr{R}_1\mathscr{N})-c_0\begin{bmatrix}\sigma_0I_{20} & \\ & I_q\end{bmatrix}.\end{aligned} \tag{5.3}$$

We shall prove the following Lemmas.

Lemma 5.4. $\operatorname{Re}(\tilde{u}, \mathscr{R}_{11}\mathscr{N}_{11}(z, D, \eta)\tilde{u})_\eta\geqslant C\operatorname{Re}(\tilde{u}, \mathscr{E}\tilde{u})_\eta$.

Lemma 5.5. $\operatorname{Re}(\tilde{w}, \mathring{G}(z, D, \eta)\tilde{w})_\eta\geqslant -c_1\|\mathscr{E}^{\frac{1}{2}}u\|^2-c_2\|v\|^2_\eta$.

Suppose that these two lemmas are proved. Then we have

$$\begin{aligned}\operatorname{Re}(\tilde{w}, \mathscr{R}_\lambda\mathscr{N}\tilde{w})_\eta&\geqslant\operatorname{Re}(\tilde{w}, G(z, D, \eta)\tilde{w})_\eta+c_0\operatorname{Re}(\tilde{u}, \mathscr{E}\tilde{u})_\eta\\&\quad+c_0\eta\|v\|^2_\eta+(\lambda-1)c\operatorname{Re}(\tilde{u}, \mathscr{E}\tilde{u})_\eta\\&\geqslant((\lambda-1)c+c_0-c_1)\operatorname{Re}(\tilde{u}, \mathscr{E}\tilde{u})_\eta+(c_0\eta-c_1)\|v\|^2_\eta.\end{aligned}$$

Taking $\lambda\gg 1$, $\eta\gg 1$ leads to (5.2).

4° The proof of Lemma 5.4.

As in [2], denote $H_1(z,\xi,\eta)=\operatorname{Re}\mathscr{R}_{11}\mathscr{N}_{11}(z,\xi,\eta)$. From the structure of $\mathscr{R}_{11}$ in [17], we know $H_1(z,\xi,\eta)\geqslant c\sigma_0 I_{2p}$.

Let $Q(z,\xi,\eta)=H_1(z,\xi,\eta)-c_0\sigma_0 I_{2p}$, $Q(z,\xi,\eta)\in S^{1,k(u1)}_{1,\sigma}$, and $Q\geqslant 0$.

As in [20], let $q(D,z,D,\eta)$ be the Friedrichs symmetrilization of Q:

$$q(\xi',z,\xi,\eta)=\int F_\sigma(\xi',\zeta,\eta)Q(z,\zeta,\eta)F_\sigma(\xi,\zeta,\eta)d\zeta, \tag{5.4}$$

where

$$F_\sigma(\xi,\zeta,\eta)=\langle\xi,\eta\rangle_\sigma^{-\frac{1}{4}N}\varphi(\langle\xi,\eta\rangle^{-\frac{1}{2}}(\zeta-\xi)), \tag{5.5}$$

with φ being a smooth even function of compact support, supp $\varphi\subset\{\xi\in R^N,\ |\xi|\leqslant 1\}$, and $\int\varphi^2(\xi)d\xi=1$.

Let $r(z,D,\eta)=q(D,z,D,\eta)-Q(z,D,\eta)$. Since $q(D,z,D,\eta)$ is nonnegative, $(\tilde{u},q(D,z,D,\eta)\tilde{u})_\eta\geqslant 0$, it remains to show $r(z,\xi,\eta)\in S^{0,k-2(u1)}_{1,\sigma}$.

Since

$$\widehat{r(z,D,\eta)u}(\xi',\eta)=\int\hat{r}(\xi'-\xi,\xi,\eta)\hat{u}(\xi,\eta)d\xi, \tag{5.6}$$

$$\hat{r}(\xi',\xi,\eta)=\hat{r}_{\rm I}+\hat{r}_{\rm II}=\int F_\sigma(\xi'+\xi,\zeta,\eta)[\hat{Q}(\xi',\zeta,\eta)-\hat{Q}(\xi',\xi,\eta)]F_\sigma(\xi,\zeta,\eta)d\zeta$$
$$+\int[F_\sigma(\xi'+\xi,\zeta,\eta)-F_\sigma(\xi,\zeta,\eta)]\hat{Q}(\xi',\xi,\eta)F_\sigma(\xi,\zeta,\eta)d\zeta. \tag{5.7}$$

For brevity, we will denote by $g(\xi',\xi,\eta)$ or $g(\xi',\xi,\zeta,\eta)$ the functions in $L^2(R^N_{\xi'})$, uniformly bounded for the parameters ξ,ζ,η. We should prove

$$\langle\xi'\rangle^{k-2}\hat{r}(\xi',\xi,\eta)\in L^2(R^N_{\xi'}),\ \text{uniformly in}\ \xi,\eta. \tag{5.8}$$

i) $\langle\xi'\rangle\geqslant\frac{1}{2}\langle\xi,\eta\rangle_\sigma^{\frac{1}{2}}$:

On supp F_σ, $|\xi-\zeta|\leqslant\langle\xi,\eta\rangle_\sigma^{\frac{1}{2}}$. And

$$\begin{aligned}&\tilde{Q}(\xi',\zeta,\eta)-\tilde{Q}(\xi',\xi,\eta)\\&=\sum_{|\alpha|=1}C_\alpha\int_0^1 D^\alpha_{\zeta_1}\hat{Q}(\xi',\zeta_1+t(\xi-\zeta),\eta)_{\zeta_1=\zeta}\,dt\cdot(\xi-\zeta)^\alpha\\&=\langle\xi'\rangle^{-k}\langle\xi,\eta\rangle_\sigma^{\frac{1}{2}}g(\xi',\xi,\zeta,\eta),\end{aligned}$$

so

$$\begin{aligned}\hat{r}_{\rm I}&=\langle\xi,\eta\rangle_\sigma^{\frac{1}{2}}\langle\xi'\rangle^{-k}\int F_\sigma(\xi'+\xi,\zeta,\eta)F_\sigma(\xi,\zeta,\eta)g(\xi',\xi,\zeta,\eta)d\zeta\\&=\langle\xi,\eta\rangle_\sigma^{\frac{1}{2}}\langle\xi'\rangle^{-k}g(\xi',\xi,\eta);\\\hat{r}_{\rm II}&=\hat{Q}(\xi',\xi,\eta)\int[F_\sigma(\xi'+\xi,\zeta,\eta)-F_\sigma(\xi,\zeta,\eta)]F_\sigma(\xi,\zeta,\eta)d\zeta\\&=\langle\xi,\eta\rangle_\sigma\langle\xi'\rangle^{-k}g(\xi',\xi,\eta).\end{aligned}$$

Therefore, (5.8) is valid on $\langle\xi'\rangle\geqslant\frac{1}{2}\langle\xi,\eta\rangle_\sigma^{\frac{1}{2}}$.

ii). $\langle\xi'\rangle\geqslant\frac{1}{2}\langle\xi,\eta\rangle_\sigma^{\frac{1}{2}}$.

Now we have $\langle\xi+k\xi',\eta\rangle_\sigma\sim\langle\xi,\eta\rangle_\sigma$, $(0\leqslant k\leqslant 1)$. From [17] it follows that

$$D_\xi^\alpha F_\sigma(\xi,\zeta,\eta)=\langle\xi,\eta\rangle^{-\frac{1}{4}N}\sum_{\gamma\leqslant\beta\leqslant|\alpha|}\psi_{\alpha,\beta,\gamma}(\xi,\eta)\mu^\gamma D_\mu^\beta\varphi(\mu), \tag{5.9}$$

where

$$\mu=(\xi-\zeta)\langle\xi,\eta\rangle_\sigma^{-\frac{1}{2}}\quad \psi_{\alpha,\beta,\gamma}\in S_{1,\sigma}^{\frac{1}{2}|\beta-\gamma|-|\alpha|,\infty}.$$

Therefore

$$\hat{Q}(\xi',\zeta,\eta)-\hat{Q}(\xi',\xi,\eta)=\sum_{|\alpha|=1}\partial_\xi^\alpha\hat{Q}(\xi',\xi,\eta)(\zeta-\xi)^\alpha$$

$$+\sum_{|\alpha|=2}\int_0^1 C_\alpha\partial_{\zeta_1}^\alpha\hat{Q}(\xi',\zeta_1+t(\zeta-\xi),\eta)_{\zeta_1=\xi}(1-t)dt(\zeta-\xi)^\alpha$$

$$=\sum_{|\alpha|=1}\partial_\xi^\alpha\hat{Q}(\xi',\xi,\eta)(\zeta-\xi)^\alpha+|\zeta-\xi|^2\langle\xi'\rangle^{-k}\langle\xi+k(\zeta-\xi),\eta\rangle_\sigma^{-1}g(\xi',\xi,\zeta,\eta). \tag{5.10}$$

$$F_\sigma(\xi'+\xi,\zeta,\eta)-F_\sigma(\xi,\zeta,\eta)=\sum_{|\alpha|=1}C_\alpha\partial_\xi^\alpha F_\sigma(\xi+k\xi',\zeta,\eta)\xi'^\alpha$$

$$=\sum_{|\alpha|=1}C_\alpha\xi'^\alpha\langle\xi+k\xi',\eta\rangle^{-\frac{1}{4}N}\sum_{\gamma\leqslant\beta\leqslant1}\psi_{1,\beta,\gamma}(\xi+k\xi',\eta)\bar{\mu}^\gamma\partial_\mu^\beta\varphi(\bar{\mu}), \tag{5.11}$$

where

$$\bar{\mu}=(\xi+k\xi'-\zeta)\langle\xi+k\xi',\eta\rangle_\sigma^{-\frac{1}{2}}.F_\sigma(\xi'+\xi,\zeta,\eta)-F_\sigma(\xi,\zeta,\eta)$$

$$=\sum_{|\alpha|=1}C_\alpha\xi'^\alpha\langle\xi,\eta\rangle_\sigma^{-\frac{1}{4}N}\sum_{\gamma\leqslant\beta\leqslant1}\psi_{1,\beta,\gamma}(\xi,\eta)\mu^\gamma D_\mu^\beta\varphi(\mu)$$

$$+\sum_{|\alpha|=2}C_\alpha\xi'^\alpha\langle\xi+k\xi',\eta\rangle_\sigma^{-\frac{1}{4}N}\sum_{\gamma\leqslant\beta\leqslant2}\psi_{2,\beta,\gamma}(\xi,k\xi',\eta)\bar{\mu}^\gamma\partial_\mu^\beta\varphi(\bar{\mu}). \tag{5.12}$$

Noticing that $F_\sigma^2(\xi,\zeta,\eta)(\zeta-\xi)^\alpha$ is odd in $(\zeta-\xi)$ when $|\alpha|=1$, from (5.10) (5.11), we have

$$\hat{r}_{\rm I}=\sum_{|\alpha|=1}\int[F_\sigma(\xi'+\xi,\zeta,\eta)-F_\sigma(\xi,\zeta,\eta)]\partial_\xi^\alpha\hat{Q}(\xi',\xi,\eta)(\zeta-\xi)^\alpha F_\sigma(\xi,\zeta,\eta)d\zeta$$

$$+\sum_{|\alpha|=2}\int C_\alpha F_\sigma(\xi'+\xi,\zeta,\eta)\int_0^1\partial_\xi^\alpha\hat{Q}(\xi',\xi+t(\zeta-\xi),\eta)(1-t)dt(\zeta-\xi)^\alpha F_\sigma(\xi,\zeta,\eta)d\zeta$$

$$=\langle\xi'\rangle^{1-k}|\zeta-\xi|\sum_{\gamma\leqslant\beta\leqslant1}\psi_{1,\beta,\gamma}(\xi+k\xi',\eta)g(\xi',\xi,\eta)$$

$$+|\zeta-\xi|^2\langle\xi'\rangle^{-k}\langle\xi+t(\zeta-\xi),\eta\rangle^{-1}g(\xi',\xi,\eta)\quad(\text{for }\zeta\in\operatorname{supp}F_\sigma)$$

$$=(\langle\xi,\eta\rangle_\sigma^{\frac{1}{2}}\langle\xi'\rangle^{1-k}\sum_{\gamma\leqslant\beta\leqslant1}\langle\xi+k\xi',\eta\rangle_\sigma^{\frac{1}{2}(\beta-\gamma)-1}+\langle\xi'\rangle^{-k})g(\xi',\xi,\eta)$$

$$=\langle\xi'\rangle^{1-k}g(\xi',\xi,\eta). \tag{5.13}$$

$$\hat{r}_{\rm II}=\sum_{|\alpha|=1}C_\alpha\xi'^\alpha\int\partial_\xi^\alpha F_\sigma(\xi,\zeta,\eta)\hat{Q}(\xi',\xi,\eta)F_\sigma(\xi,\zeta,\eta)d\zeta$$

$$+\int O(|\xi'|^2\partial_\xi^2F_\sigma(\xi+k\xi',\zeta,\eta)\hat{Q}(\xi',\xi,\eta)F_\sigma(\xi,\zeta,\eta)d\zeta$$

$$=\sum_{\substack{|\alpha|=1\\ \gamma\leq\beta\leq1}} C_\alpha \xi'^\alpha \int \langle \xi,\ \eta\rangle_\sigma^{-\frac{1}{4}N} \psi_{1,\beta,\gamma}(\xi,\ \eta)\mu^\gamma \partial_\mu^\beta \varphi(\mu)\hat{Q}(\xi',\ \xi,\ \eta)F_\sigma(\xi,\ \zeta,\ \eta)d\zeta$$
$$+\int O(|\xi'|^2 \langle \xi,\ \eta\rangle_\sigma^{-\frac{1}{4}N} \psi_{2,\beta,\gamma}(\xi+k\xi',\ \eta)\bar{\mu}^\gamma \partial_\mu^\beta \varphi(\bar{\mu}))\tilde{Q}(\xi',\ \xi,\ \eta)F_\sigma(\xi,\ \zeta,\ \eta)d\zeta. \tag{5.14}$$

For $\gamma<\beta$, $\partial_\mu \varphi(\mu)$ is odd in μ, so the first integral in the above expression is zero. For $\gamma=\beta$, $\psi_{1,\beta,\gamma}\in S_{1,\sigma}^{-1,\infty}$, then

$$\begin{aligned}\hat{r}_{II}&=\langle\xi'\rangle^{1-k} g(\xi',\ \xi,\ \eta)+\langle\xi'\rangle^{2-k}\langle \xi,\ \eta\rangle_\sigma \langle \xi+k\xi',\ \eta\rangle_\sigma^{-} g(\xi',\ \xi,\ \eta)\\ &=\langle\xi'\rangle^{2-k} g(\xi',\ \xi,\ \eta).\end{aligned} \tag{5.15}$$

Combining (5.14) and (5.15), we know (5.8) holds for $\langle\xi'\rangle\leq\frac{1}{2}\langle\xi,\ \eta\rangle_\sigma^{\frac{1}{2}}$.

5° The Proof of Lemma 5.5.

Let $b(D,\ z,\ D,\ \eta)$ be the usual Friedrichs symmetrilization of $G(z,\ D,\ \eta)$ with parameter η. i.e., in (5.4) and (5.5), F_σ is substituted by

$$F=\langle \xi,\ \eta\rangle^{-\frac{1}{4}N} \varphi(\langle \xi,\ \eta\rangle^{-\frac{1}{2}}(\zeta-\xi)).$$

As in 4°, let $r(z,\ D,\ \eta)=b(D,\ z,\ D,\ \eta)-G(z,\ D,\ \eta)$. Since $G_{21},\ G_{22}\in SP_1^{1,k}$, we may show $r_{21},\ r_{22}\in SP_1^{0,k-2}$ similarly as in 4°. Thus, the proof of Lemma 5.5 is reduced to proving the following inequality;

$$\operatorname{Re}(\tilde{u},\ r_{11}(z,\ D,\ \eta)\tilde{u})_\eta\leq C\|\mathscr{E}^{\frac{1}{2}}\tilde{u}\|_\eta^2. \tag{5.16}$$

In the following, we will briefly write $r=r_{11}$ and $G=G_{11}$. As in 4° ((5.7)), we have

$$\hat{r}(\xi',\ \xi,\ \eta)=\hat{r}_I+\hat{r}_{II}=\int F(\xi'+\xi,\ \zeta,\ \eta)[\hat{G}(\xi',\ \zeta,\ \eta)-\hat{G}(\xi',\ \xi,\ \eta)]F(\xi,\ \zeta,\ \eta)d\zeta$$
$$+\int [F(\xi'+\xi,\ \zeta,\ \eta)-F(\xi,\ \zeta,\ \eta)]\hat{G}(\xi',\ \xi,\ \eta)F(\xi,\ \zeta,\ \eta)d\zeta. \tag{5.17}$$

When $\langle\xi'\rangle\geq\frac{1}{2}\langle\xi,\ \eta\rangle^{\frac{1}{2}}$, noting that in the proof of Lemma 5.4 we have at most once differentiated $\hat{Q}$, we proceed here in the same way to have $\langle\xi'\rangle^{k-2}\hat{r}(\xi',\ \xi,\ \eta)$ uniformly bounded in $L^2(R_{\xi'}^N)$, for all $\xi,\ \eta$.

When $\langle\xi'\rangle\leq\frac{1}{2}\langle\xi,\ \eta\rangle^{\frac{1}{2}}$, the estimate of $\hat{r}_{II}$ is similar to (5.14). Hence we have similar estimate as (5.15). In the expression for $\hat{r}_I$ (similar to (5.13)), the first integrand contains only the first derivatives of $\hat{Q}$ about ξ, therefore we can estimate it as in 4°. Nevertheless, the second integral must be considered anew. Now we have

$$\begin{aligned}&\int F(\xi'+\xi,\ \zeta,\ \eta)\int_0^1 \sum_{|\alpha|=2} C_\alpha \partial_{\xi_1}^\alpha \hat{G}(\xi',\ \xi_1+t(\zeta-\xi),\ \eta)_{\xi_1=\xi}(1-t)dt(\zeta-\xi)^\alpha F(\xi,\ \zeta,\ \eta)d\zeta\\ &=|\zeta_1-\xi|^2\cdot\langle\xi'\rangle^{-k}\langle \xi+k(\zeta_1-\xi),\ \eta\rangle^{-1} g(\xi',\ \xi,\ \eta)\quad (\text{for } \zeta_1\in\operatorname{supp} F)\\ &=\langle\xi'\rangle^{-k}\langle \xi,\ \eta\rangle^{\frac{1}{2}} g(\xi',\ \xi,\ \eta).\end{aligned} \tag{5.18}$$

Combining it with the estimate in $\langle\xi'\rangle \geqslant \frac{1}{2}\langle\xi, \eta\rangle^{\frac{1}{2}}$ gives

$$\langle\xi'\rangle^{k-2}\langle\xi, \eta\rangle^{-\frac{1}{2}}\hat{r}(\xi', \xi, \eta) \tag{5.19}$$

is uniformly bounded in $L^2(R^N_{\xi'})$ for all ξ, η.

Since

$$\begin{aligned}(\tilde{u}, r_{11}(z, D, \eta)\tilde{u})_\eta &= (\tilde{u}, r_{11}(z, D, \eta)\mathscr{E}^{\frac{1}{2}}\mathscr{E}^{-\frac{1}{2}}\tilde{u})_\eta \\ &= (\mathscr{E}^{\frac{1}{2}}\tilde{u}, r_{11}(z, D, \eta)\mathscr{E}^{-1}\mathscr{E}^{\frac{1}{2}}\tilde{u})_\eta + (\tilde{u}, [r_{11}, \mathscr{E}^{\frac{1}{2}}]\mathscr{E}^{-\frac{1}{2}}\tilde{u})_\eta,\end{aligned}$$

and

$$[r_{11}, \mathscr{E}^{\frac{1}{2}}] \in SP^{\frac{1}{2}, k-3}_{\frac{1}{2}}, \text{ we have } |(\tilde{u}, r_{11}(z\, D, \eta)\tilde{u})_\eta| \leqslant C\|\mathscr{E}^{\frac{1}{2}}\tilde{u}\|^2_\eta.$$

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