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ISOPARAMETRIC FINITE ELEMENT METHODS FOR NONLINEAR DIRICHLET PROBLEM

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Abstract

In this paper, the author solves the nonlinear Dirichlet problem with nonhomogenous boundary condition by use of the isoparametric finite element method, and obtains the optimal error estimate.

§1. Introduction

We consider the nonlinear Dirichlet problem

$$\begin{cases} -\nabla\cdot(a(x,\ u)\nabla u)=f(x), & x\in\Omega, \\ u=g(x), & x\in\Gamma, \end{cases} \tag{1.1}$$

where $x=(x_1,\ x_2)$, and Ω is a bounded open subset of R^2, its boundary Γ is sufficiently smooth. In [1] and [2] Douglas and Dupont obtained an approximate solution of (1.1) by F. E. M. with penalty. In § 4 of [2] Douglas also obtained an approximate solution of (1.1) by F. E. M. without penalty, but the finite element space is a finite dimensional subspace of $W_2^2(\Omega)\cap C^1(\Omega)$. In this paper we obtain an approximate solution of (1.1) by the isoparametric finite element method. Moreover, the finite element space is a finite dimensional subspace of $C^0(\Omega)$. The method in this paper is a straightforward generalization of a method in [3] to the nonlinear case. In § 2 we shall give triangulation and some inequalities. In § 3 we shall prove existence of an approximate solution. In § 4 we shall evaluate the error $\|u^*-\tilde{u}_h^*\|_{1,\Omega}$ and $\|u^*-\tilde{u}_h^*\|_{0,\Omega}$, where u^* denotes the solution of (1.1) and $\tilde{u}_h^*$ denotes the approximate solution with numerical integration being taken into account. For simplicity, we shall not evaluate the error $\|u^*-u_h^*\|_{1,\Omega}$ and the error $\|u^*-u_h^*\|_{0,\Omega}$, where u_h^* denotes the approximate solution without taking into account numerical integration.

In this paper C denotes a generic constant with possibly different values in different contexts.

For $1\leqslant p\leqslant\infty$ and n, a nonnegative integer, let $W^{n,p}(K)$ be a Sobolev space.

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Let $\|\cdot\|_{n,p,K}$ and $|\cdot|_{n,p,K}$ be the norm and the seminorm on $W^{n,p}(K)$ respectively. When $p=2$, we write $H^n(K)=W^{n,2}(K)$, $\|\cdot\|_{n,K}=\|\cdot\|_{n,2,K}$, $|\cdot|_{n,K}=|\cdot|_{n,2,K}$.

§2. Triangulation and Some Inequalities

For simplicity, we only discuss the isoparametric 2-simplex of type 2. Such a finite element is discussed in [3]. Here we briefly describe them. Let $\hat{K}$ be a reference finite element of an isoparametric family. We take $\hat{K}$ as the triangle $\triangle\hat{O}\hat{A}\hat{B}$, which is described in Fig. 1 in [3]. Let K_{1Bi} $(i=1, \cdots, m)$ be all boundary elements and K_{1Ii} $(i=m+1, \cdots, m+m_0)$ be all interior elements, where K_{1Bi} consists of OB and AB and arc OA on Γ_h as shown in Fig. 1 in [3]. For simplicity we denote $K_{1i}=K_{1Bi}$ $(i=1, \cdots, m)$ and $K_{1i}=K_{1Ii}(i=m+1, \cdots, m+m_0)$, and we have $\Omega_h=\bigcup_{i=1}^{m+m_0}K_{1i}$. In general, Ω_h is not equal to Ω. In this paper we take K_{2Bi} $(i=1, \cdots, m)$ as boundary elements, where K_{2Bi} consists of segments OB and AB and arc OEA on Γ, as shown in Fig. 1 in [3]. We denote by K_{2Ii} $(i=m+1, \cdots, m+m_0)$ the interior elements which equal to K_{1Ii}. We also denote $K_{2i}=K_{2Bi}$ $(i=1, \cdots, m)$ and $K_{2i}=K_{2Ii}$ $(i=m+1, \cdots, m+m_0)$. We have $\bar{\Omega}=\bigcup_{i=1}^{m+m_0}K_{2i}$. Let $h_{K_{1i}}=\mathrm{diam}(K_{1i})$, $h=\max_{1\leqslant i\leqslant m+m_0}h_{K_{1i}}$. We assume that triangulations satisfy the regular condition in [4]. We know[4] that there exists an isoparametric mapping F_{K_i}: $\hat{K}\to K_{1i}$ $(i=1, \cdots, m+m_0)$. Let $\hat{K}^0$ be a bounded open convex subset containing $\hat{K}$. We also know[3] that the inverse mapping of F_{K_i} on $\hat{K}^0$ exists for $h<h_0$, where h_0 is sufficiently small. Let $K_i=F_{K_i}(\hat{K}^0)$. We have $K_i\supset K_{2i}$. Let $G_{1j}=K_{1j}\backslash K_{2j}$, $G_{2j}=K_{2j}\backslash K_{1j}$, $G_j=G_{1j}\cup G_{2j}$ $(j=1, \cdots, m)$, $\tilde{G}_1=\bigcup_{j=1}^{m}G_{1j}$, $\tilde{G}_2=\bigcup_{j=1}^{m}G_{2j}$, $\tilde{G}=\tilde{G}_1\cup\tilde{G}_2$. Let B_j $(j=1, \cdots, m_1)$ and I_j $(j=m_1+1, \cdots, m_1+m_2)$ represent boundary nodes and interior nodes respectively. We denote by $\hat{P}_2(\hat{K}^0)$ the space of all polynomials of degree $\leqslant 2$ on $\hat{K}^0$. We define

$$V_h=\{w_h(x_1, x_2)\,|\,w_h(x_1, x_2)\in C^0(\bar{\Omega}),\ w_h(x_1, x_2)$$
$$=\hat{w}_{hi}(\hat{x}_1, \hat{x}_2)\cdot F_{K_i}^{-1} \text{ on } K_{2i}, \text{ where } \hat{w}_{hi}(\hat{x}_1, \hat{x}_2)\in\hat{P}_2(\hat{K}^0)\}.$$

Let g be a continuous function defined on Γ. We define

$$V_h^g=\{w_h(x_1, x_2)\,|\,w_h(x_1, x_2)\in V_h,\ w_h(B_j)=g(B_j)\ (j=1, \cdots, m_1)\}.$$

When $g=0$, we denote $V_h^0=V_h^g$.

Theorem 2.1. *Let $w_h(x_1, x_2)\in V_h^0$. Then*

$$\int_{\gamma_i}|w_h(x_1, x_2)|^2ds\leqslant Ch^5\int_{K_{2i}}|w_h(x_1, x_2)|^2dx_1dx_2, \quad i=1, \cdots, m,$$

where $\gamma_i=K_{2i}\cap\Gamma$.

Theorem 2.2. *Suppose that $p(x_1, x_2)\in V_h$. Then*

$$\int_{G_j} |p(x_1, x_2)|^2 dx_1 dx_2 \leqslant Ch^2 \int_{K_{lj}} |p(x_1, x_2)|^2 dx_1 dx_2, \quad l=1, 2; \ j=1, \cdots, m.$$

Theorem 2.3. *Suppose that* $p(x_1, x_2) \in V_h$. *Then*

$$|p(x_1, x_2)|^2_{1,G_j} \leqslant Ch^2 |p(x_1, x_2)|^2_{1,K_{lj}}, \quad l=1, 2; \ j=1, \cdots, m.$$

Proof of Theorem 2.1—Theorem 2.3 could be found in [3].

Let $u \in H^2(\Omega)$, and Π_h be an interpolation operator, which is defined by

(i) $\Pi_h u \in V_h$,

(ii) $\Pi_h u(B_j) = u(B_j), \quad j=1, \cdots, m_1$,

(iii) $\Pi_h u(I_j) = u(I_j), \quad j=m_1+1, \cdots, m_1+m_2$.

Arguing as in the case of § 4 in [5], we could obtain

$$\begin{cases} \|u-\Pi_h u\|_{n,\Omega} \leqslant Ch^{3-n} \|u\|_{3,\Omega}, \quad n=0, 1, \\ \left[\sum_{i=1}^{m+m_0} \|u-\Pi_h u\|^2_{n, K_{2i}}\right]^{\frac{1}{2}} \leqslant Ch^{3-n} \|u\|_{3,\Omega}, \quad n=2, 3. \end{cases} \tag{2.1}$$

§ 3. Existence of Approximate Solution

Problem P: Find $u \in H^1(\Omega)$ such that

$$\begin{cases} u=g(x), \quad x \in \Gamma, \\ (a(x, u)\nabla u, \nabla v) = (f, v), \quad \forall v \in H_0^1(\Omega). \end{cases} \tag{3.1}$$

In general, we do not know if the weak solutions of (1.1) are unique. However, under some conditions[1] the solutions of Problem P are sufficiently smooth. Douglas-Dupont-Serrin[6] have proved that the solution of (1.1) is unique under certain conditions. Thus, under certain conditions the week solution of (1.1), i.e., the solution of (3.1), is in fact a classical solution of (1.1). Under these conditions we discuss the approximate solution. These conditions may be found in [9], we omit them. But, we notice that the assumption about $a(x, u)$, f, g and Ω in next paragraph will guarantee that the weak solution of (1.1) is a classical solution, which exists and is unique.

In this paper we assume that $a(x, u)$ is a sufficiently smooth mapping of $\tilde{\Omega} \times R$ into $[\alpha_0, \alpha_1]$, where α_0, α_1 are constants satisfying $0 < \alpha_0 \leqslant \alpha_1 < \infty$, $\tilde{\Omega}$ is a bounded open subset of R^2 and satisfies $\bar{\Omega} \subset \tilde{\Omega}$. Moreover, since $h \to 0$, we will make the following assumpation

$$\Omega_h \subset \tilde{\Omega} \text{ for all } h.$$

We also assume that the derivatives of $a(x, u)$ through certain order are bounded on $\tilde{\Omega} \times R$, g is a trace of $G \in C^3(\bar{\Omega})$, f is sufficiently smooth defined over $\tilde{\Omega}$ and $u \in H^3(\Omega)$. Because Γ is sufficiently smooth, there exists an extension operator E: $H^3(\Omega) \to H^3(R^2)$, i.e. for all $u \in H^3(\Omega)$, the function $Eu \in H^3(R^2)$ satisfies $Eu|_\Omega = u$ and, besides, the operator E is continuous, i. e., there exists a constant $C(\Omega)$

such that

$$\|Eu\|_{3,\Omega}\leqslant C(\Omega)\|u\|_{3,\Omega},\quad \forall u\in H^3(\Omega).$$

In what follows, we will assume that solution u^* of Problem P is defined on R^2 and satisfies above inequality. But we omit E.

Problem P_h: Find $u_h\in V_h^g$ such that

$$(a(x, u_h)\nabla u_h, \nabla v_h)=(f, v_h),\quad \forall v_h\in V_h^0. \tag{3.2}$$

u_h are the approximate solutions of Problem P.

Now we prove the existence of Problem P_h.

Let u_{h1} be a certain element in V_h^g.

Problem P'_h: Find $u_{h0}\in V_h^0$ such that

$$(a(x, u_{h0}+u_{h1})\nabla u_{h0}, \nabla v_h)=(f, v_h)-((a(x, u_{h0}+u_{h1})\nabla u_{h1}, \nabla v_h),\quad \forall v_h\in V_h^0. \tag{3.3}$$

Theorem 3.1. *The mapping S: $V_h^0\to V_h^0$, defined by*

$$(a(x, u_{h0}+u_{h1})\nabla Su_{h0}, \nabla v_h)=(f, v_h)-(a(x, u_{h0}+u_{h1})\nabla u_{h1}, \nabla v_h),\quad \forall v_h\in V_h^0, \tag{3.4}$$

exists and is unique.

Proof Let v_h be an arbitrary element in the space V_h^0. Using Theorem 2.2 and Theorem 2.3 and Poincaré inequality, we have

$$\begin{aligned}(a(x, u_{h0}+u_{h1})\nabla v_h, \nabla v_h)&\geqslant\alpha_0|v_h|^2_{1,\Omega}=\alpha_0[|v_h|^2_{1,\Omega_h}+|v_h|^2_{1,\tilde G_1}-|v_h|^2_{1,\tilde G_2}]\\ &\geqslant C[\|v_h\|^2_{1,\Omega_h}-\|v_h\|^2_{1,\tilde G}]=C[\|v_h\|^2_{1,\Omega}-\|v_h\|^2_{1,\tilde G_1}+\|v_h\|^2_{1,\tilde G_2}-\|v_h\|^2_{1,\tilde G}]\\ &\geqslant C[\|v_h\|^2_{1,\Omega}-2\|v_h\|^2_{1,\tilde G}]\geqslant C\|v_h\|^2_{1,\Omega},\end{aligned} \tag{3.5}$$

the last inequality holds for h sufficiently small.

From (3.4) and (3.5) we easily know that solution Su_{h0} of (3.4) exists and is unique, i.e. the mapping S exists and is unique.

Theorem 3.2. *The range of S is contained in a ball.*

Proof Let u_{h0} be an arbitrary element. The particular choice $v_h=Su_{h0}$ in (3.4) gives

$$C\|\nabla Su_{h0}\|^2_{0,\Omega}\leqslant\|f\|_{0,\Omega}\|Su_{h0}\|_{0,\Omega}+\alpha_1\|u_{h1}\|_{1,\Omega}\|Su_{h0}\|_{1,\Omega}.$$

The conclusion of the Theorem follows from above inequality.

Theorem 3.3. *The mapping S: $V_h^0\to V_h^0$, defined by* (3.4) *is continuous.*

Proof Let sequence $\{u_n\}$ in V_h^0 be convergence and u_0 be an element in V_h^0 such that $u_n\to u_0$ in V_h^0. Since S is bounded, sequence $\{Su_n\}$ is also bounded. Since V_h^0 is finite dimensional space, we can extract from $[Su_n\}$ a subsequence $\{Su_{n_i}\}$ such that $\{Su_{n_i}\}$ converges in the V_h^0. Let $\bar u$ be an element in V_h^0 such that $Su_{n_i}\to\bar u$ in V_h^0. Now we prove $\bar u=Su_0$. We have

$$(a(x, u_{n_i}+u_{h1})\nabla Su_{n_i}, \nabla v_h)=(f, v_h)-(a(x, u_{n_i}+u_{h1})\nabla u_{h1}, \nabla v_h),\quad \forall v_h\in V_h^0.$$

Let us fix v_h in V_0^h, above equalities converge to

$$(a(x, u_0+u_{h1})\nabla\bar u, \nabla v_h)=(f, v_h)-(a(x, u_0+u_{h1})\nabla u_{h1}, \nabla v_h),\quad \forall v_h\in V_h^0. \tag{3.6}$$

From (3.4) and (3.6), we obtain $\bar u=Su_0$. Since $\bar u$ is unique, sequence $\{Su_n\}$ is convergence, we thus have proved our Theorem.

Theorem 3.4. *Problem P_h has a solution.*

Proof Problem P_h and Problem P'_h are equivalent. Using Theorem 3.2 and Theorem 3.3 and Brouwer fixed point theorem, we see that operator equation $u_h=Su_h$ has a solution. We denote by u_{h0} the solution of $u_h=Su_h$. Obviously, u_{h0} is a solution of P'_h. Therefore $u_{h0}+u_{h1}$ is a solution of P_h.

§ 4. Error Estimate

For simplicity, we only consider error estimate with the numerical integration being taken into account. Using the same method, we could obtain error estimate without taking into account numerical integration.

Let $f(x_1, x_2)$ be a continuous function defined on $\widetilde{\Omega}$. We have

$$\int_\Omega f(x_1, x_2)\,dx_1dx_2=\sum_{i=1}^{m}\int_{K_{hi}} f(x_1, x_2)\,dx_1dx_2+\sum_{i=m+1}^{m+m_0}\int_{K_{hi}} f(x_1, x_2)dx_1dx_2. \tag{4.1}$$

Let K be an arbitrary interior element and F_K be an isoparametric mapping such that $K=F_K(\hat{K})$. If $f(x_1, x_2)=\hat{f}(\hat{x}_1, \hat{x}_2)\cdot F_K^{-1}$, then we have

$$\int_K f(x_1, x_2)dx_1\,dx_2=\int_{\hat{K}}\hat{f}(\hat{x}_1, \hat{x}_2)J_{F_K}(\hat{x}_1, \hat{x}_2)\,d\hat{x}_1d\hat{x}_2, \tag{4.2}$$

where $J_{F_K}(\hat{x}_1, \hat{x}_2)$ denotes the Jacobian of mapping F_K. If we apply the quadrature formula over $\hat{K}$

$$\int_{\hat{K}}\hat{f}(\hat{x}_1, \hat{x}_2)d\hat{x}_1d\hat{x}_2\sim\sum_{l=1}^{L}\hat{\omega}_l\hat{f}(\hat{b}_{1l}, \hat{b}_{2l}), \tag{4.3}$$

where $\hat{\omega}_l>0(l=1, \cdots, L)$, then we have the quadrature formula over K

$$\int_K f(x_1, x_2)dx_1dx_2\sim\sum_{l=1}^{L}\omega_{l,K}f(b_{1lK}, b_{2lK}), \tag{4.4}$$

where

$$\begin{cases}\omega_{l,K}=\hat{\omega}_lJ_{F_K}(\hat{b}_{1l}, \hat{b}_{2l}),\\ b_{l_K}=F_K(\hat{b}_l),\ \hat{b}_l=(\hat{b}_{1l}, \hat{b}_{2l}),\\ b_{lK}=(b_{1lK}, b_{2lK}).\end{cases} \tag{4.5}$$

We denote

$$\hat{E}(\hat{f})=\int_{\hat{K}}\hat{f}(\hat{x}_1, \hat{x}_2)d\hat{x}_1d\hat{x}_2-\sum_{l=1}^{L}\hat{\omega}_l\hat{f}(\hat{b}_{1l}, \hat{b}_{2l}). \tag{4.6}$$

In what follows, we assume that $\hat{\omega}_l$, $\hat{b}_l$ $(l=1, \cdots, L)$ satisfy the following conditions:

$$\begin{aligned}&\text{(i)}\ \hat{E}(\hat{v}J_{F_K})=0,\quad \forall\hat{v}\in P_2(\hat{K}),\\ &\text{(ii)}\ \hat{E}(\hat{\omega})=0,\quad \forall\hat{w}\in P_2(\hat{K}),\\ &\text{(iii)}\ \hat{E}(\hat{v}\hat{w})=0,\ \forall\hat{v}\in \hat{P}_1(\hat{K}),\quad \hat{w}\in P_2(\hat{K}).\end{aligned} \tag{4.7}$$

Let $\widetilde{K}$ be an arbitrary boundary element and $F_{\widetilde{K}}$ be an isoparametric mapping. We shall apply the same quadrature formula (4.4) for boundary element $\hat{K}$, i. e.

$$\int_{\hat{K}} f(x_1, x_2)dx_1dx_2\sim\sum_{l=1}^{L}\omega_{l,\widetilde{K}}f(b_{1l\widetilde{K}}, b_{2l\widetilde{K}}),$$

where $\omega_{l\tilde{K}}$ and $b_{l\tilde{K}}$ are defined by (4.5).

Define

$$A(y;\, u,\, v) = (a(x,\, y)\nabla u,\, \nabla v), \tag{4.8}$$

$$A_h(u_h;\, u_h,\, v_h) = \sum_{i=1}^{m+m_0} \sum_{l=1}^{L} \omega_{lK_{2i}} a(b_{lK_{2i}},\, u_h(b_{lK_{2i}})) \nabla u_h(b_{lK_{2i}}) \nabla v_h(b_{lK_{2i}}), \tag{4.9}$$

$$f(v) = (f,\, v), \tag{4.10}$$

$$f_h(v_h) = \sum_{i=1}^{m+m_0} \sum_{l=1}^{L} \omega_{lK_{2i}} f(b_{lK_{2i}}) v_h(b_{lK_{2i}}). \tag{4.11}$$

Problem $\widetilde{P}_h$: Find $\tilde{u}_h \in V_h^g$ such that

$$A_h(u_h;\, u_h,\, v_h) = f_h(v_h), \quad \forall v_h \in V_h^0. \tag{4.12}$$

Theorem 4.1. *The mapping $\tilde{S}$: $V_h^0 \to V_h^0$, defined by*

$$A_h(\tilde{u}_{h0} + \tilde{u}_{h1};\, \tilde{S}\tilde{u}_{h0},\, v_h) = f_h(v_h) - A_h(\tilde{u}_{h0} + \tilde{u}_{h1};\, \tilde{u}_{h1},\, v_h), \quad \forall v_h \in V_h^0, \tag{4.13}$$

exists and is unique, where $\tilde{u}_{h1}$ is a fixed element in V_h^g.

Proof Using the property of $a(x, u)$ and the Theorem 4.4.2 in [4], we obtain

$$A_h(\tilde{u}_{h0} + \tilde{u}_{h1};\, u_h,\, u_h) \geqslant C \|u_h\|_{1,\Omega_h}^2.$$

By Theorem 2.2 and Theorem 2.3, the following inequality holds:

$$A_h(\tilde{u}_{h0} + \tilde{u}_{h1};\, u_h,\, u_h) \geqslant C \|u_h\|_{1,\Omega}^2. \tag{4.14}$$

The conclusion of Theorem follows from above inequality.

Theorem 4.2. *The range of $\tilde{S}$ is constrained in a ball.*

Using the method of § 4.4 in [4] and arguing as in the Theorem 3.2, we could prove our Theorem.

Theorem 4.3. *The mapping $\tilde{S}$ is continuous.*

Theorem 4.4. *Problem $\widetilde{P}_h$ has a solution.*

Using the same methods in the Theorem 3.3 and Theorem 3.4, we can prove that Theorem 4.3 and Theorem 4.4 are exact.

Let us next turn to the error estimate.

Let $\tilde{u}_h^*$ and u^* be solutions of Problem $\widetilde{P}_h$ and Problem P respectively. Let v_h be an arbitrary element in V_h^g. We can write

$$\begin{aligned} A_h(\tilde{u}_h^*;\, \tilde{u}_h^* - v_h,\, \tilde{u}_h^* - v_h) &= A(u^*;\, u^* - v_h,\, \tilde{u}_h^* - v_h) + A(u^*;\, v_h,\, \tilde{u}_h^* - v_h) \\ &\quad - A_h(\tilde{u}_h^*;\, v_h,\, \tilde{u}_h^* - v_h) + [f_h(\tilde{u}_h^* - v_h) - f(\tilde{u}_h^* - v_h)] \\ &\quad - [A(u^*;\, u^*,\, \tilde{u}_h^* - v_h) - f(\tilde{u}_h^* - v_h)] \\ &= A(u^*;\, u^* - v_h,\, \tilde{u}_h^* - v_h) + [A(u^*;\, v_h,\, \tilde{u}_h^* - v_h) - A(\tilde{u}_h^*;\, v_h,\, \tilde{u}_h^* - v_h)] \\ &\quad + [A(\tilde{u}_h^*;\, v_h,\, \tilde{u}_h^* - v_h) - A_h(\tilde{u}_h^*;\, v_h,\, \tilde{u}_h^* - v_h)] \\ &\quad + [f_h(\tilde{u}_h^* - v_h) - f(\tilde{u}_h^* - v_h)] - [A(u^*;\, u^*,\, \tilde{u}_h^* - v_h) - f(\tilde{u}_h^* - v_h)]. \end{aligned} \tag{4.15}$$

Now we estimate each term on the right hand side of (4.15). We denote each term on the right hand side of (4.15) by I, II, III, IV, V respectively. We have

$$|\mathrm{I}| \leqslant \alpha_1 \|u^* - v_h\|_{1,\Omega} \|\tilde{u}_h^* - v_h\|_{1,\Omega}. \tag{4.16}$$

$$|\mathrm{II}| \leqslant C \|a(x,\, u^*) - a(x,\, \tilde{u}_h^*)\|_{0,3,\Omega} \|v_h\|_{1,6,\Omega} \|\tilde{u}_h^* - v_h\|_{1,\Omega}. \tag{4.17}$$

Using the method of proving (4.19) in [3], we could obtain

$$|\mathrm{III}|\leqslant Ch^2\Big(\sum_{i=1}^{m+m_0}\|v_h\|_{2,K_{3i}}^2\Big)^{\frac{1}{2}}|\tilde{u}_h^*-v_h|_{1,\Omega}. \tag{4.18}$$

Using the method of proving (4.21) and (4.24) in [3], we could obtain

$$|\mathrm{IV}|=|f_h(\tilde{u}_h^*-v_h)-f(\tilde{u}_h^*-v_h)|\leqslant Ch^2\|f\|_{2,q,\tilde{\Omega}}\|\tilde{u}_h^*-v_h\|_{1,\Omega}, \tag{4.19}$$

where $q>2$. We may write

$$|\mathrm{V}|=\left|\int_\Gamma a(x,u^*)\frac{\partial u^*}{\partial n}(\tilde{u}_h^*-v_h)ds\right|.$$

By Theorem 2.1, there exists constant C such that

$$|\mathrm{V}|\leqslant C\|u^*\|_{2,\Omega}h^{\frac{5}{2}}\|\tilde{u}_h^*-v_h\|_{1,\Omega}. \tag{4.20}$$

Taking $v_h=\Pi_hu^*$ and using inequalities (4.14) and (4.16)—(4.20), we obtain

$$\|\tilde{u}_h^*-\Pi_hu^*\|_{1,\Omega}\leqslant C[h^2+\|a(x,u^*)-a(x,\tilde{u}_h^*)\|_{0,3,\Omega}].$$

Since

$$\|u^*-\tilde{u}_h^*\|_{0,3,\Omega}\leqslant C\|u^*-\tilde{u}_h^*\|_{0,\Omega}^{\frac{1}{2}}\|u^*-\tilde{u}_h^*\|_{1,\Omega}^{\frac{1}{2}}$$

and

$$\|a(x,u^*)-a(x,\tilde{u}_h^*)\|_{0,3,\Omega}\leqslant C\|u^*-\tilde{u}_h^*\|_{0,3,\Omega},$$

we obtain

$$\|u^*-\tilde{u}_h^*\|_{1,\Omega}\leqslant C[h^2+\|u^*-\tilde{u}_h^*\|_{0,\Omega}]. \tag{4.21}$$

Let L^* be an operator defined by

$$L^*v=-\nabla\cdot(a(x,u^*)\nabla v)+a_u(x,u^*)\nabla u^*\cdot\nabla v.$$

By assumption in § 2, we know that operator L^* is well defined and that the solution χ of the following problem

$$\begin{cases}L^*\chi=u^*-\tilde{u}_h^*, & x\in\Omega,\\ \chi=0, & x\in\Gamma\end{cases}$$

belongs to $H^2(\Omega)\cap H_0^1(\Omega)$. Moreover

$$\|\chi\|_{2,\Omega}\leqslant C\|u^*-\tilde{u}_h^*\|_{0,\Omega}. \tag{4.22}$$

Therefore

$$\begin{aligned}\|u^*-\tilde{u}_h^*\|_{0,\Omega}^2&=(u^*-\tilde{u}_h^*,L^*\chi)\\
&=(u^*-\tilde{u}_h^*,-\nabla\cdot(a(x,u^*)\nabla\chi))+(u^*-\tilde{u}_h^*,a_u(x,u^*)\nabla u^*\cdot\nabla\chi)\\
&=(\nabla(u^*-\tilde{u}_h^*),a(x,u^*)\nabla\chi)-\int_\Gamma a(x,u^*)(u^*-\tilde{u}_h^*)\frac{\partial\chi}{\partial n}ds\\
&\quad+(u^*-\tilde{u}_h^*,a_u(x,u^*)\nabla u^*\cdot\nabla\chi)\\
&=A(u^*;u^*,\chi)-A(\tilde{u}_h^*;\tilde{u}_h^*,\chi)+([a(x,\tilde{u}_h^*)-a(x,u^*)]\nabla\tilde{u}_h^*,\nabla\chi)\\
&\quad+(a_u(x,u^*)(u^*-\tilde{u}_h^*)\nabla u^*,\nabla\chi)-\int_\Gamma a(x,u^*)(u^*-\tilde{u}_h^*)\frac{\partial\chi}{\partial n}ds\\
&=A(u^*;u^*,\chi)-A(\tilde{u}_h^*;\tilde{u}_h^*,\chi)+((u^*-\tilde{u}_h^*)\bar{a}_u(x)\nabla(u^*-\tilde{u}_h^*)\\
&\quad+(u^*-\tilde{u}_h^*)\bar{a}_{uu}(x)\nabla u^*,\nabla\chi)-\int_\Gamma a(x,u^*)(u^*-\tilde{u}_h^*)\frac{\partial\chi}{\partial n}ds\\
&=[A(u^*;u^*,\chi-\Pi_h\chi)-A(\tilde{u}_h^*;\tilde{u}_h^*,\chi-\Pi_h\chi)]\end{aligned}$$

$$+((u^*-\tilde{u}_h^*)\bar{a}_u(x)\nabla(u^*-\tilde{u}_h^*)+(u^*-\tilde{u}_h^*)^2\bar{a}_{uu}(x)\nabla u^*, \nabla\chi)$$
$$+[A(u^*; u^*, \Pi_h\chi)-A(\tilde{u}_h^*; \tilde{u}_h^*, \Pi_h\chi)]-\int_\Gamma a(x, u^*)(u^*-\tilde{u}_h^*)\frac{\partial\chi}{\partial n}ds, \tag{4.23}$$

where

$$\bar{a}_u(x)=\int_0^1 a_u(x, u^*-t(u^*-\tilde{u}_h^*))dt, \tag{4.24}$$

$$\bar{a}_{uu}(x)=\int_0^1 (1-t)a_{uu}(x, u^*-t(u^*-\tilde{u}_h^*))dt. \tag{4.25}$$

We may write

$$A(u^*; u^*, \Pi_h\chi)-A(\tilde{u}_h^*; \tilde{u}_h^*, \Pi_h\chi)$$
$$=-[A(\tilde{u}_h^*; \tilde{u}_h^*, \Pi_h\chi)-A_h(\tilde{u}_h^*; \tilde{u}_h^*, \Pi_h\chi)]+[f(\Pi_h\chi)-f_h(\Pi_h\chi)]$$
$$+\int_\Gamma a(x, u^*)\frac{\partial u^*}{\partial n}\Pi_h\chi\, ds. \tag{4.26}$$

Define

$$\tilde{A}_h(u; v, w)=\int_{\Omega_h} a(x, u)\nabla v\cdot\nabla w\, dx. \tag{4.27}$$

We may write

$$A(\tilde{u}_h^*; \tilde{u}_h^*, \Pi_h\chi)-A_h(\tilde{u}_h^*; \tilde{u}_h^*, \Pi_h\chi)$$
$$=[\tilde{A}_h(\tilde{u}_h^*; \tilde{u}_h^*-\Pi_h u^*, \Pi_h\chi)-A_h(\tilde{u}_h^*; \tilde{u}_h^*-\Pi_h u^*, \Pi_h\chi)]$$
$$+[\tilde{A}_h(\tilde{u}_h^*; \Pi_h u^*, \Pi_h\chi)-A_h(\tilde{u}_h^*; \Pi_h u^*, \Pi_h\chi)]$$
$$+[A(\tilde{u}_h^*; \tilde{u}_h^*, \Pi_h\chi)-\tilde{A}_h(\tilde{u}_h^*; \tilde{u}_h^*, \Pi_h\chi)].$$

Arguing as (5.18) in [7] and (5.19) in [7], we obtain

$$|A_h(\tilde{u}_h^*; \Pi_h u^*, \Pi_h\chi)-A_h(\tilde{u}_h^*; \Pi_h u^*, \Pi_h\chi)|\leqslant Ch^3\|u^*\|_{3,\Omega}\|u^*-\tilde{u}_h^*\|_{0,\Omega},$$
$$|\tilde{A}_h(\tilde{u}_h^*; \tilde{u}_h^*-\Pi_h u^*, \Pi_h\chi)-A_h(\tilde{u}_h^*; \tilde{u}_h^*-\Pi_h u^*, \Pi_h\chi)|$$
$$\leqslant Ch\|\tilde{u}_h^*-\Pi_h u^*\|_{1,\Omega}\|u^*-\tilde{u}_h^*\|_{0,\Omega}$$
$$\leqslant C[h^3+h\|u^*-\tilde{u}_h^*\|_{1,\Omega}]\|u^*-\tilde{u}_h^*\|_{0,\Omega}.$$

By (4.27) we may write

$$A(\tilde{u}_h^*; \tilde{u}_h^*, \Pi_h\chi)-\tilde{A}_h(\tilde{u}_h^*; \tilde{u}_h^*, \Pi_h\chi)$$
$$=\int_{\Omega\backslash\Omega_h} a(x, \tilde{u}_h^*)\nabla\tilde{u}_h^*\cdot\nabla\Pi_h\chi\, dx-\int_{\Omega_h\backslash\Omega} a(x, \tilde{u}_h^*)\nabla\tilde{u}_h^*\cdot\nabla\Pi_h\chi dx.$$

Arguing as (3.27) in [8], we obtain

$$|A(\tilde{u}_h^*; \tilde{u}_h^*, \Pi_h\chi)-\tilde{A}_h(\tilde{u}_h^*; \tilde{u}_h^*, \Pi_h\chi)|\leqslant Ch^3\|u^*-\tilde{u}_h^*\|_{0,\Omega}.$$

Therefore

$$|A(\tilde{u}_h^*; \tilde{u}_h^*, \Pi_h\chi)-A_h(\tilde{u}_h^*; \tilde{u}_h^*, \Pi_h\chi)|$$
$$\leqslant C[h^3+h\|u^*-\tilde{u}_h^*\|_{1,\Omega}]\|u^*-\tilde{u}_h^*\|_{0,\Omega}. \tag{4.28}$$

We may write

$$f(\Pi_h\chi)-f_h(\Pi_h\chi)=\int_{\Omega_h} f\Pi_h\chi\, dx-f_h(\Pi_h\chi)+\left[\int_{\Omega\backslash\Omega_h} f\Pi_h\chi\, dx-\int_{\Omega_h\backslash\Omega} f\Pi_h\chi\, dx\right]. \tag{4.29}$$

Using (4.24) in [3] and (3.36) in [8], we obtain

$$\begin{cases} \left|\int_{\Omega\backslash\Omega_h} f\Pi_h\chi\,dx - \int_{\Omega_h\backslash\Omega} f\Pi_h\chi\,dx\right| \leqslant Ch^3\|f\|_{2,q,\tilde{\Omega}}\|\chi\|_{2,\Omega} \\ \leqslant Ch^3\|f\|_{2,q,\tilde{\Omega}}\|u^*-\tilde{u}_h^*\|_{0,\Omega}, \\ \left|\int_{\Omega_h} f\Pi_h\chi\,dx - f_h(\Pi_h\chi)\right| \leqslant Ch^3\|f\|_{3,\Omega_h}\|u^*-\tilde{u}_h^*\|_{0,\Omega}. \end{cases} \tag{4.30}$$

By Cauchy inequality and Theorem 2.1, there exists C such that

$$\left|\int_\Gamma a(x,u^*)\frac{\partial u^*}{\partial n}\Pi_h\chi\,ds\right| \leqslant Ch^{\frac{5}{2}}\|u^*\|_{2,\Omega}\left[\sum_{i=1}^m \|\Pi_h\chi\|_{1,K_{2i}}^2\right]^{\frac{1}{2}}$$

$$\leqslant Ch^{\frac{5}{2}}\|u^*\|_{2,\Omega}\left[\sum_{i=1}^m \|\chi-\Pi_h\chi\|_{1,K_{2i}}^2+\sum_{i=1}^m\|\chi\|_{1,K_{2i}}^2\right]^{\frac{1}{2}}$$

$$\leqslant Ch^{\frac{5}{2}}\|u^*\|_{2,\Omega}[h^2\|\chi\|_{2,\Omega}^2+h\|\chi\|_{2,\Omega}^2]^{\frac{1}{2}} \leqslant Ch^3\|u^*\|_{2,\Omega}\|u^*-\tilde{u}_h^*\|_{0,\Omega}. \tag{4.31}$$

Combining (4.26)—(4.31), we obtain

$$|A(u^*;u^*,\Pi_h\chi)-A(\tilde{u}_h^*;\tilde{u}_h^*,\Pi_h\chi)| \leqslant C[h^3+h\|u^*-\tilde{u}_h^*\|_{1,\Omega}]\,\|u^*-\tilde{u}_h^*\|_{0,\Omega}. \tag{4.32}$$

Let us next turn to the error estimate of another terms in (4.23).

We may write

$$A(u^*;u^*,\chi-\Pi_h\chi)-A(\tilde{u}_h^*;\tilde{u}_h^*,\chi-\Pi_h\chi)$$
$$=A(\tilde{u}_h^*;u^*-\tilde{u}_h^*,\chi-\Pi_h\chi)+(\bar{a}_u(x)(u^*-\tilde{u}_h^*)\nabla u^*,\nabla(\chi-\Pi_h(\chi))).$$

Therefore

$$|A(u^*;u^*,\chi-\Pi_h\chi)-A(\tilde{u}_h^*;\tilde{u}_h^*,\chi-\Pi_h\chi)| \leqslant C\|u^*-\tilde{u}_h^*\|_{1,\Omega}\|\chi-\Pi_h\chi\|_{1,\Omega}$$
$$\leqslant Ch\|u^*-\tilde{u}_h^*\|_{1,\Omega}\|u^*-\tilde{u}_h^*\|_{0,\Omega}. \tag{4.33}$$

Using Cauchy inequality, we have

$$|((u^*-\tilde{u}_h^*)\bar{a}_u(x)\nabla(u^*-\tilde{u}_h^*)+(u^*-\tilde{u}_h^*)^2\bar{a}_{uu}(x)\nabla u^*;\nabla\chi)|$$
$$\leqslant C\|u^*-\tilde{u}_h^*\|_{0,3,\Omega}\|u^*-\tilde{u}_h^*\|_{1,\Omega}\|\chi\|_{1,6,\Omega} \leqslant C\|u^*-\tilde{u}_h^*\|_{0,3,\Omega}\|u^*-\tilde{u}_h^*\|_{1,\Omega}\|\chi\|_{2,\Omega}$$
$$\leqslant C\|u^*-\tilde{u}_h^*\|_{1,\Omega}^{\frac{3}{2}}\|u^*-\tilde{u}_h^*\|_{0,\Omega}^{\frac{3}{2}}. \tag{4.34}$$

Obviously, the following equality holds:

$$\int_\Gamma a(x,u^*)(u^*-\tilde{u}_h^*)\frac{\partial\chi}{\partial n}ds=\int_\Gamma a(x,u^*)(u^*-\Pi_h u^*)\frac{\partial\chi}{\partial s}ds$$
$$+\int_\Gamma a(x,u^*)(\Pi_h u^*-\tilde{u}_h^*)\frac{\partial\chi}{\partial n}ds. \tag{4.35}$$

Using Cauchy inequality and Theorem 2.1 and inequality (1.2.3) in [4], we obtain

$$\left|\int_\Gamma a(x,u^*)(\Pi_h u^*-\tilde{u}_h^*)\frac{\partial\chi}{\partial n}ds\right| \leqslant C\|\Pi_h u^*-\tilde{u}_h^*\|_{0,\Gamma}\left\|\frac{\partial\chi}{\partial n}\right\|_{0,\Gamma}$$
$$\leqslant Ch^{\frac{5}{2}}\|\Pi_h u^*-\tilde{u}_h^*\|_{1,\Omega}\|\chi\|_{2,\Omega} \leqslant C[h^{\frac{9}{2}}+h^{\frac{5}{2}}\|u^*-\tilde{u}_h^*\|_{1,\Omega}]\,\|u^*-\tilde{u}_h^*\|_{0,\Omega}, \tag{4.36}$$
$$\left|\int_\Gamma a(x,u^*)(u^*-\Pi_h u^*)\frac{\partial\chi}{\partial n}ds\right| \leqslant C\|u^*-\Pi_h u^*\|_{0,\Gamma}\left\|\frac{\partial\chi}{\partial n}\right\|_{0,\Gamma}$$
$$\leqslant C\|u^*-\Pi_h u^*\|_{0,\Gamma}\|u-\tilde{u}_h^*\|_{0,\Omega}.$$

Let us turn to the error estimate of $\|u^*-\Pi_h u^*\|_{0,\Gamma}$. Since $u^*=G$ over Γ and $G\in C^3(\bar{\Omega})$, we have

$$\|u^*-\Pi_h u^*\|_{0,\Gamma}\leqslant\|u^*-G-\Pi_h(u^*-G)\|_{0,\Gamma}+\|G-\Pi_h S\|_{0,\Gamma}$$
$$=\|\Pi_h(u^*-G)\|_{0,\Gamma}+\|G-\Pi_h G\|_{0,\Gamma}$$
$$\leqslant C_1 h^{\frac{5}{2}}\Big[\sum_{i=1}^{m}\|\Pi_h(u^*-G)\|^2_{1,K_{2i}}\Big]^{\frac{1}{2}}+C_2h^3$$
$$\leqslant C_1 h^{\frac{5}{2}}\Big[\sum_{i=1}^{m}\|u^*-\Pi_h u^*\|^2_{1,K_{2i}}+\sum_{i=1}^{m}\|u^*-G\|^2_{1,K_{2i}}$$
$$+\sum_{i=1}^{m}\|G-\Pi_h G\|^2_{1,K_{2i}}\Big]^{\frac{1}{2}}+C_2h^3.$$

Since $u^*-G\in H^1_0(\Omega)\cap H^2(\Omega)$, there exists C such that

$$\sum_{i=1}^{u}\|u^*-G\|^2_{1,K_{2i}}\leqslant Ch\|u^*-G\|^2_{2,\Omega}.$$

Therefore, we have

$$\|u^*-\Pi_h u^*\|_{0,\Gamma}\leqslant Ch^3$$

and

$$\Big|\int_\Gamma a(x,u^*)(u^*-\Pi_h u^*)\frac{\partial\chi}{\partial n}ds\Big|\leqslant Ch^3\|u^*-\tilde{u}_h^*\|_{0,\Omega}. \tag{4.37}$$

Combining (4.35)—(4.37), we obtain

$$\Big|\int_\Gamma a(x,u^*)(u^*-\tilde{u}_h^*)\frac{\partial\chi}{\partial n}ds\Big|\leqslant C[h^3+h\|u^*-\tilde{u}_h^*\|_{1,\Omega}]\|u^*-\tilde{u}_h^*\|_{0,\Omega}. \tag{4.38}$$

We could obtain the following inequality by combining (4.23), (4.32)—(4.35) and (4.38)

$$\|u^*-\tilde{u}_h^*\|_{0,\Omega}\leqslant C[h^3+h\|u^*-\tilde{u}_h^*\|_{1,\Omega}+\|u^*-\tilde{u}_h^*\|^{\frac{3}{2}}_{1,\Omega}\|u^*-\tilde{u}_h^*\|^{\frac{1}{2}}_{0,\Omega}].$$

From above inequality, we obtain

$$\|u^*-\tilde{u}_h^*\|_{0,\Omega}\leqslant C[h^3+h\|u^*-\tilde{u}_h^*\|_{1,\Omega}+\|u^*-u_h^*\|^3_{1,\Omega}]. \tag{4.39}$$

Using (4.21) in (4.39), we see that for h sufficiently small

$$\|u^*-\tilde{u}_h^*\|_{0,\Omega}+h\|u^*-\tilde{u}_h^*\|_{1,\Omega}\leqslant C[h^3+\|u^*-\tilde{u}_h^*\|^3_{0,\Omega}]. \tag{4.40}$$

Arguing as § 4 in [9], we may prove $\tilde{u}_h^*$ weakly converges to u^* in $H^1(\Omega)$ and strongly to u^* in $L^2(\Omega)$. Thus, we obtain the following Theorem 4.1 from (4.40).

Theorem 4.1. *Let u^* and $\tilde{u}_h^*$ be solutions of Problem P and Problem $\tilde{P}_h$ respectively. Let the coefficients in (1.1) satisfy the hypotheses in § 3. Then we have*

$$\|u^*-\tilde{u}_h^*\|_{0,\Omega}+h\|u^*-\tilde{u}_h^*\|_{1,\Omega}\leqslant Ch^3.$$

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