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A NON-LINEAR FILTERING PROBLEM AND ITS APPLICATIONS*

SITU RONG (司徒荣)**

Abstract

For partially observed process in n-dimensional space

$$\begin{cases}\beta_t=\beta_0+\int_0^t A_1(s,\ \beta_s)ds+\int_0^t B_1(s,\ \beta_s)dw_s^{(1)},\\ \xi_t=\int_0^t B(s,\ \xi_s)B^*(s,\ \xi_s)\varphi(s,\ \beta_s)ds+\int_0^t B(s,\ \xi_s)dw_s,\end{cases} \tag{0}$$

under non-Lipschitz (even discontinuous) condition, a Bayes formula different from [1] is derived (Theorem 1). By means of this formula the innovation problem for the above process under rather weak condition is solved (Theorem 2). Then the existence of an optimal pathwise Bang-Bang control for a partially observed process with bounded controls is obtained (Theorem 4).

§1. Introduction

In the non-linear filtering problem Bayes formula plays an important role. Some abstract version of this formula can be found in [1] (non-Lipschitz diffusion coefficient is assumed for observable process in n-dimensional space) and [6] (Lipschitz diffusion coefficient is assumed for observable process in 1-dimensional space). Here for one more concrete partially observed process under rather weak condition (non-Lipschitz diffusion coefficient even discontinuous drift are assumed only) we derive a Bayes formula different from [1]. Then by it the so-called "innovation problem" is solved. That is to say, a new Brownian motion process (B. M.) $\bar{w}_s,\ s\leqslant t$, is found, which carries the same "information" as $\xi_s,\ s\leqslant t$, does. This problem is crucial, difficult, important, and useful, especially for optimal filtering and optimal control problem ([6]). So it has been discussed by many papers such as [2—6], [8] and [9]. However, the proofs given by some papers such as [5], [8] and [6] appear to be incorrect (See [3]). And all results except [2] and [6] are got under the assumption that the diffusion coefficient $B(t,\ x)=I$. (In [24, 25] it considers the non-Markovian case but still under some Lipschitz condition and

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** Department of Mathematics, Zhongshan, University, Guangzhou, China.

still assume that $B(t, x)=I$ and still in 1-dimensional space. In [6] it considers the conditional Guassian processes in 1-dimensional case). Up to now only in [2] it considers the case $B(t, y)\neq I$. And one of the best results probably is got in [2]. But it assumes that the diffusion coefficient $B(t, y)$ of the observable process satisfies a jointly Lipschitz condition and the drift coefficient $A_1(t, x, y)$ of the signal process has two continuous derivatives etc. It is obviously some restriction. Here we consider (0) in n-dimensional space, and solve this problem. Then by using this innovation result the existence of a Bang-Bang pathwise optimal stochastic control for one kind of partially observed deffusion process with bounded controls is obtained based on the idea of separation principle due to Wohnam[15]. However, it is a result stronger than that in [15] in some sense, since the admissible set considered here is not Hölder continuous in t under the sup norm in $\mathscr{C}$. Moreover, it is also different from [11—14], since there the optimal control only exists in some wider class-randomized set or under some rather restricted assumption for the admissible set, e.g. assume that it is sequentially compact, which is not easy to check, because the observation ξ also depends on u. Anyway, since up to now there are not many existence theorems of the strict admissible optimal control for the partially observed process, the existence of the pathwise Bang-Bang optimal control (in the strict sense) for one kind of partially observed stochastic process with bounded controls in n-dimensional space is at least one more concrete result for it. Moreover, our result here is for pathwise.

§ 2. Bayes Formula and "Innovation Problem" for Non-Linear Filtering

Consider a signal diffusion process β_t and an observable diffusion process ξ_t, which satisfy the following stochastic differential equation (S. D. E.)*

$$\begin{cases} d\beta_t=A_1(t,\ \beta_t)dt+B_1(t,\ \beta_t)dw_t^{(1)}, \ \beta_0=\beta_0\in R^n, \\ d\xi_t=B(t,\ \xi_t)B^*(t,\ \xi_t)\varphi(t,\ \beta_t)dt+B(t,\ \xi_t)dw_t,\ \xi_0=0, \quad t\in[0,\ T], \end{cases} \tag{1}$$

where β_t, ξ_t are n-dimensional random processes, $w_t^{(1)}$ and w_t are n-dimensional standard Brownian Motion processes (B. M.), A_1, φ are n-dimensional vectors, B_1 and B are matrixes. Under a given probability space $(\Omega,\ \mathscr{F},\ \mathscr{F}_t,\ P)$ we have the followsng theorem.

Theorem 1 (Bayes formula). *Assume that (all functions are jointly measurable)*

(i) $|A_1(t, x)|\leqslant k_0(1+|x|)$, $|B(t, x)|\leqslant k_0$,

$|\varphi(t, x)|\leqslant k_0(1+|x|)$, $|B(t, y)|\leqslant k_0$;

(ii) *there exists a* $\delta>0$ *such that*

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* B^* means the transposition of B.

$$\langle B(t, x)\lambda, \lambda\rangle \geqslant \delta|\lambda|^2,$$
$$\langle B_1(t, x)\lambda, \lambda\rangle \geqslant \delta|\lambda|^2,$$

for all $\lambda, x, y \in R^n$, *where* $\langle\cdot, \cdot\rangle$ *is the inner product in* R^n;

(iii) a) $|B_1(t, x)-B_1(t, y)|^2+2\langle x-y, A_1(t, x)-A_1(t, y)\rangle \leqslant c_1(t)\rho_1(|x-y|^2)$,

b) $|B(t, x)-B(t, y)|^2 \leqslant c_2(t)\rho_2(|x-y|^2)$,

where $\rho_i(t)$, $i=1, 2$, *are increasing, concave, positive, defined on* $u\in(0, \infty)$ *such that*

$$\int_{0+} du/\rho_i(u) = +\infty,$$

and $c_i(t)=c_i(t, \omega)$ *is such that P-a.s.*

$$\int_0^T |c_i(t)|^2 dt < +\infty, \quad c_i(t)>0, \; i=1, 2;$$

(iv) $w^{(1)}$ *is independent of* w.

Then for any $g_t(\beta, \xi)$, *an n-dimensional vector,* $\mathscr{F}_t^{\beta,\xi}$*-measurable with*

$$E|g_t(\beta, \xi)| < +\infty, \text{ for all } t\in[0, T],$$

we have

$$E(g_t(\beta, \xi)|\mathscr{F}_t^{\xi}) = \frac{\left\{\begin{array}{l}\int_\Omega g_t(\beta(\tilde{\omega}), \xi\omega))\exp\Big(\int_0^t \langle\varphi(s, \beta_s(\tilde{\omega})), d\xi_s(\omega)\rangle \\ -\frac{1}{2}\int_0^t |B^*(s, \xi_s(\omega))\varphi(s, \beta_s(\tilde{\omega}))|^2 ds\Big)\, dP(\tilde{\omega})\end{array}\right\}}{\left\{\begin{array}{l}\int_\Omega \exp\Big(\int_0^t \langle\varphi(s, \beta_s(\tilde{\omega})), d\xi_s(\omega)\rangle \\ -\frac{1}{2}\int_0^t |B^*(s, \xi_s(\omega))\varphi(s, \beta_s(\tilde{\omega}))|^2 ds\Big)\, dP(\tilde{\omega})\end{array}\right\}}. \tag{2}$$

Remark 1. Usually we take the probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ as follows:

$\Omega = C^{(n)}_{[0,T]}$-all continuous map from $[0, T]\to R^n$,

$\mathscr{F}=\mathscr{B}^c$-the Borel field generated by all open sets in $C^{(n)}_{[0,T]}$, where the norm is uniform convergence norm,

$\mathscr{F}_t = \mathscr{B}_t^c = \{f\in\Omega: f(s)\in A, s\in[0, t], A\text{-arbitrary Borel set in } R^n\}$,

P=some given Wiener measure on $(C^{(n)}_{[0,T]}, \mathscr{B}^c)$.

Remark 2. A_1 can be discontinuous, e.g. $A_1(t, x)=-x/|x|$, as $x\neq 0$; and $A_1(t, x)=0$, as $x=0$. Then $A_1(t, x)$ satisfies $\langle A_1(t, x)-A_1(t, y), x-y\rangle \leqslant 0$.

Proof By applying [19] and [6] it is not difficult to see that there exists a weak solution (β, ξ) for (1). Now if (β^i, ξ^i), $i=1, 2$, are two solutions of (1) in the same probability space with the same B. M. $(w^{(1)}, w)$, then by condition (iii) and Ito formula it is easily seen that $\beta^1=\beta^2$. Denote $\beta=\beta^1=\beta^2$. By Gronwall inequality we have

$$E\sup_{t\leqslant T}|\beta_t|^2 \leqslant k, \quad k \text{ is a constant.}$$

And the following matrix operation holds: (Denote $A^2=A\cdot A^*$.)

$$A^2-B^2=(A+B)(A^*-B^*)-(BA^*-AB^*)=(A+B)(A^*-B^*)$$
$$-B(A^*-B^*)-(B-A)B^*,$$

$$|A^2C-B^2C|\leqslant k|A^2-B^2|\cdot|C|. \tag{3}$$

Hence, applying Ito formula to ξ_t, by (3) and the 2nd equation of (1) as well as assumption (iii) we see that P–a.s.

$$\xi_t^1=\xi_t^2, \text{ for all } t\in[0, T].$$

Therefore by [7] (1) has a pathwise unique strong solution (β_t, ξ_t), which is $\mathscr{F}_t^{w^{(1)},w}$ -measurable. Furthermore, by condition (iii) it can be shown that β_t is $\mathscr{F}_t^{w^{(1)}}$-measurable. Hence β_t is independent of w. Moreover, since B_1 is uniformly non-degenerated, $w_t^{(1)}$ is measurable with respect to (w. r. t.) $\sigma(\beta_s, 0\leqslant s\leqslant t)$. Hence ξ_t is $\mathscr{F}_t^{\beta,w}$-measurable. Now let us introduce the following lemma without proof for saving pages.

Lemma 1. *Denote*

$$\Phi(t, \beta)=\varphi(t, \beta_t).$$

Then for any $b\in C^{(n)}_{[0,T]}$ the following S. D. E. (4) *and* (5) *have pathwise unique strong solutions ξ_t and η_t, respectively*:

$$\begin{cases} d\xi_t=B(t, \xi_t)B^*(t, \xi_t)\Phi(t, b)dt+B(t, \xi_t)dw_t, \\ \xi_0=0, \quad t\in[0, T], \end{cases} \tag{4}$$

$$\begin{cases} d\eta_t=B(t, \eta_t)dw_t, \\ \eta_0=0, \quad t\in[0, T]. \end{cases} \tag{5}$$

Denote μ_η–the solution measure on $C^{(n)}_{[0,T]}$ generated by η, etc. Then

$$\mu_{\xi^b}\sim\mu_\eta,$$

$$(d\mu_{\xi^b}/d\mu_\eta)(\eta)=\exp\left(\int_0^T\langle\Phi(s, b), d\eta_s\rangle-\frac{1}{2}\int_0^T|B^*(s, \eta_s)\Phi(s, b)|^2 ds\right). \tag{6}$$

Now let us return to prove that formula (2) holds. Since $\mu_{\xi^b}\sim\mu_\eta$, by the same approach as in the proofs of Lemmas 7.6 and 7.7 in [6] it is derived that

$$\mu_{\beta,\xi}\sim\mu_\beta\times\mu_\eta, \quad \mu_\beta\times\mu_\eta\text{-a.s.},$$

$$\mu_\xi\sim\mu_\eta, \quad \mu_\eta\text{ -a.s.},$$

and

$$(d\mu_{\beta,\xi}/d(\mu_\beta\times\mu_\eta))(b, x)=(d\mu_{\xi^b}/d\mu_\eta)(x), \quad \mu_\beta\times\mu_\eta\text{ -a.s..} \tag{7}$$

Therefore for any $A\in\mathscr{B}(C^{(n)}_{[0,T]})$,

$$\begin{aligned}
\int_{(\xi\in A)} g_T(\beta(\omega), \xi(\omega))dP(\omega)&=\int_A\int g_T(b, x)(d\mu_{\beta,\xi}/d(\mu_\beta\times\mu_\eta))(b, x)d\mu_\beta(b)d\mu_\eta(x)\\
&=\int_A\int g_T(b, x)(d\mu_{\xi^b}/d\mu_\eta)(x)d\mu_\beta(b)d\mu_\eta(x)\\
&=\int_A\int g_T(b, x)(d\mu_{\xi^b}/d\mu_\eta)(x)d\mu_\beta(b)(d\mu_\eta/d\mu_\xi)(x)d\mu_\xi(x)\\
&=\int_{(\xi\in A)}(d\mu_\eta/d\mu_\xi)(\xi(\omega))\,dP(\omega)\int g_T(\beta(\tilde\omega), \xi(\omega))(d\mu_{\xi^{\beta(\tilde\omega)}}/d\mu_\eta)(\xi(\omega))dP(\tilde\omega).
\end{aligned} \tag{8}$$

On the other hand

$$\int_{(\xi\in A)} g_T(\beta(\omega),\ \xi(\omega))dP(\omega)=\int_{(\xi\in A)} E(g_T(\beta,\ \xi)\,|\,\mathscr{F}_T^\xi)dP(\omega)$$

$$=\int_{(\xi\in A)} (d\mu_\eta/d\mu_\xi)(\xi(\omega))dP(\omega)E(g_T(\beta,\ \xi)\,|\,\mathscr{F}_T^\xi)\,|_{\xi=\xi(\omega)}$$

$$\times\int (d\mu_{\xi\beta(\tilde\omega)}/d\mu_\eta)(\xi(\omega))\,dP(\tilde\omega). \tag{9}$$

By (8) and (9) we have

$$E(g_T(\beta,\ \xi)\,|\,\mathscr{F}_T^\xi)=\left(\int g_T(\beta(\tilde\omega),\ \xi(\omega))\right)(d\mu_{\xi\beta(\tilde\omega)}/d\mu_\eta)(\xi(\omega))dP(\tilde\omega)$$

$$\times\left(\int (d\mu_{\xi\beta(\tilde\omega)}/d\mu_\eta)(\xi(\omega))dP(\tilde\omega)\right)^{-1}. \tag{10}$$

Since $\mu_{\xi\beta(\tilde\omega)}\sim\mu_\eta$, applying Lemma 4.10 in [6] and substituting

$$(d\mu_{\xi\beta(\tilde\omega)}/d\mu_\eta)(\xi(\omega))=\exp\left(\int_0^T\langle\varphi(s,\ \beta_s(\tilde\omega),\ d\xi_s(\omega)\rangle\right.$$

$$\left.-\frac{1}{2}\int_0^T|B^*(s,\ \xi_s(\omega))\varphi(s,\ \beta_s(\tilde\omega))|^2ds\right)$$

into (10) we arrive at formula (2).

Consider a more general system: $t\in[0,\ T]$,

$$\begin{cases} d\beta_t=A_1(t,\ \beta_t)dt+B_1(t,\ \beta_t)dw_t^{(1)},\ \beta_0=\beta_0(\omega)-\mathscr{F}_0 \text{ measurable},\\ d\xi_t=B(t,\ \xi_t)B^*(t,\ \xi_t)\varphi(t,\ \beta_t,\ \xi_t)dt+B(t,\ \xi_t)dw_t,\quad \xi_0=0. \end{cases} \tag{1$'$}$$

Corollary 1. *Under assumptions* (ii) *and* (iv) *in Theorem* 1 *and*

(i)$'$ $|A_1(t,\ x)|\leqslant k_0(1+|x|)$, $|B(t,\ x)|\leqslant k_0$, $|B(t,\ y)|\leqslant k_0$,

$|\varphi(t,\ x,\ y)|\leqslant k_0(1+|x|+|y|)$, *for all* $x,\ y\in R^n$, $t\in[0,\ T]$,

(iii)$'$ b) $|B(t,\ x)-B(t,\ y)|^2+2\langle x-y,\ B(t,\ x)B^*(t,\ x)\varphi(t,\ z,\ x)$

$-B(t,\ y)B^*(t,\ y)\varphi(t,\ z,\ y)\rangle\leqslant c_3^N(t)\rho_3^N(|x-y|^2)$, *for arbitrary* $z\in R^n$,

as $|x|\leqslant N$, $|y|\leqslant N$, $t\in[0,\ T]$, $N=1,\ 2,\ \cdots$,

where $c_3^N(t)$ *and* $\rho_3^N(u)$ *have the same property as that in* (iii) *of Theorem* 1,

(vi) *there exists an* $\varepsilon>0$ *such that*

$$E\exp(\varepsilon|\beta_0|^2)<\infty,$$

(vii) (1)$'$ *has a pathwise unique strong solution and* β *is independent of* w, *Bayes formula* (2) *then still holds.*

Corollary 1 can be proved similarly as Theorem 1. Let us prove our main theorem.

Theorem 2. *Assume that conditions* (i)—(iv) *in Theorem* 1 *are fulfilled, and*

(v) φ'_t, φ'_x, φ''_{xx} *exist, and*

$$|\varphi'_x(t,\ x)|,\ |\varphi'_t(t,\ x)|,\ |\varphi''_{xx}(t,\ x)|\leqslant k_0(1+|x|).$$

Then we have

1) *S. D. E.* (1) *has a pathwise unique strong solution* ξ_t, *which is* $\mathscr{F}_t^{w,\beta}$-*measurable, for all* $t\in[0,\ T]$;

2) $\bar{w}_t=\int_0^t B^{-1}(s, \xi_s)\,d\xi_s-\int_0^t B^*(s, \xi_s)\bar{\varphi}(s, \xi)ds$ *is a B. M. on the probability space* $(\Omega, \mathscr{F}, \mathscr{F}_t^\xi, P)$, *satisfying* $E(d\bar{w}_t dw_t^{(1)})=0$, *where* $\bar{\varphi}(t, \xi)=E(\varphi(t, \beta_t)|\mathscr{F}_t^\xi)$. *Therefore* ξ_t *satisfies S. D. E. for all* $t\in[0, T]$

$$d\xi_t=B(t, \xi_t)B^*(t, \xi_t)\bar{\varphi}(t, \xi)dt+B(t, \xi_t)d\bar{w}, \quad \xi_0=0, \tag{11}$$

3) $\mathscr{F}_t^\xi=\mathscr{F}_t^{\bar{w}}$, *for all* $t\in[0, T]$.

Remark 3. Conclusion 3) in Theorem 2 means that the process $\bar{w}$ carries the same "information" as the process ξ_t does. Usually we call $\bar{w}_t$ an innovation process.

Remark 4. Theorem 2 implies Theorem 1 got in [3] in some sense, since the observation process in [3] is assumed to be

$$d\xi_t=\beta_t\,dt+dw_t,\ \beta_0=0,\ t\in[0, T],$$

and it is considered in 1-dimensional space. Here our observation is as

$$d\xi_t=B(t, \xi_t)B^*(t, \xi_t)\varphi(t, \beta_t)dt+B(t, \xi_t)dw_t,\ \xi_0=0,\ t\in[0, T],$$

where the diffusion coefficient is not necessarily Lipschitz continuous and the processes are considered in n-dimensional space. The assumption of our theorem is also relaxed from [2] in some sense, where the Lipschitz continuities for B, A_1, B_1 and also φ, indeed, are assumed.

Remark 5. Let $a^j(t)$, $b(t)$ be $n\times n$, $1\times n$ matrixes, respectively, which are non-random and both of which have continuous first derivatives w. r. t, $t\in[0, T]$. And assume that all conditions in Theorem 1 for A_1, B_1 and B except φ are satisfied. Let $\varphi=(\varphi_1, \varphi_2, \cdots, \varphi_n)$, and for all $t\in[0, T]$, $x\in R^n$ let

$$\varphi_i(t, x)=(a^1(t)x)_i+(a^2(t)x)_i\sin x_i+(a^3(t)x)_i e^{-x_i}+(b(t))_i.$$

Then all conclusions in Theorem 2 hold.

Proof 1) is true by Theorem 1.

For

$$d\bar{w}_t=B^{-1}(t, \xi_t)d\xi_t-B^*(t, \xi_t)\bar{\varphi}(t, \xi)dt,$$

applying Ito formula we have

$$e^{i\langle z,\bar{w}_t\rangle}=e^{i\langle z,\bar{w}_s\rangle}+i\Big\langle z, \int_s^t e^{i\langle z,\bar{w}_u\rangle}B_u^{-1}(\xi_u)d\xi_u\Big\rangle$$

$$-i\Big\langle z, \int_s^t e^{i\langle z,\bar{w}_u\rangle}B_u^*(\xi_u)\bar{\varphi}_u(\xi)du\Big\rangle-\frac{|z|^2}{2}\int_s^t e^{i\langle z,\bar{w}_u\rangle}du.$$

Therefore

$$E(e^{i\langle z,\bar{w}_t\rangle}|\mathscr{F}_s^\xi)=\exp\Big(\frac{-|z|^2}{2}(t-s)\Big).$$

Hence $\bar{w}_t$ is a B. M. on $(\Omega, \mathscr{F}, \mathscr{F}_t^\xi, P)$. $Ed\bar{w}_t dw_t^{(1)}=0$ is evident. So 2) is proved. Let us prove 3). For this we only need to prove that the pathwise uniqueness holds for (11). Then applying Corollary 1 in (i) of [7] from 2) we can see that ξ_t is a pathwise unique strong solution for (11). That is to say ξ_t is a $\mathscr{F}_t^{\bar{w}}$-measurable function. And 3) holds. Assume that there are two solutions ξ_t^1 ahd ξ_t^2 satisfying (11) on the same probability space with the same B. M.. Denote this probability space by $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$, and

$$G_N=\inf(t:\ |\xi_s^i|>N,\ i=1,2),\ I_{G_N}(s)=I_{(s<G_N)}.$$

By (2)

$$\overline{\varphi}(s, \xi^i)=f(s, \xi^i(\omega))/g(s, \xi^i(\omega)),$$

where

$$f(t, \xi^i(\omega))=\int_{\widetilde{\Omega}}\varphi(t, \beta(\widetilde{\omega}))\gamma(t, \xi^i(\omega), \beta(\widetilde{\omega}))dP(\widetilde{\omega}),\quad i=1, 2,$$

$$g(t, \xi^i(\omega))=\int_{\widetilde{\Omega}}\gamma(t, \xi^i(\omega), \beta(\widetilde{\omega}))dP(\widetilde{\omega}),$$

$$\gamma(t, \xi^i(\omega), \beta(\widetilde{\omega}))=\exp\Big(\int_0^t\langle\varphi(s, \beta_s(\widetilde{\omega})), d\xi_s^i(\omega)\rangle$$
$$-\frac{1}{2}\int_0^t|B^*(s, \xi_s^i(\omega))\varphi(s, \beta_s(\widetilde{\omega}))|^2ds\Big).$$

If we can show that there exist constants k_1^N and k_2^N (depending on N only) such that as $t\leqslant G_N(\omega)$, $(k_1^N, k_2^N>0)$

$$k_1^N\leqslant|g(t, \xi^i(\omega))|\leqslant k_2^N,\quad |f(t, \xi^i(\omega))|\leqslant k_2^N, \tag{12}$$

then

$$E|\xi_{t\wedge G_N}^1-\xi_{t\wedge G_N}^2|^2\leqslant k_3^N\Big(\int_0^t\rho(E|\xi_s^1-\xi_s^2|^2I_{G_N(\omega)}(s))ds+I_2+I_3\Big), \tag{13}$$

where

$$\begin{cases}I_2=\int_0^t E|f(s, \xi^1(\omega))-f(s, \xi^2(\omega))|^2I_{G_N}(s)ds,\\ I_3=\int_0^t E|g(s, \xi^1(\omega))-g(s, \xi^2(\omega))|^2I_{G_N}(s)ds.\end{cases} \tag{14}$$

Here we take $c_i(t)=1$ in (iii) for simplicity, otherwise use the time change technique. As a matter of fact

$$d\varphi(t, \beta_t(\widetilde{\omega}))=\varphi_t'(t, \beta_t(\widetilde{\omega}))dt+\varphi_x'A_1(t, \beta_t(\widetilde{\omega}))dt+\varphi_x'B_1(t, \beta_t(\widetilde{\omega}))dw_1(t, \widetilde{\omega})$$
$$+\frac{1}{2}\sum_{i,j,k=1}^n\varphi_{x^ix^j}''B_1^{ik}B_1^{jk}(t, \beta_t(\widetilde{\omega}))dt,$$

where

$$\beta=(\beta^1, \beta^2, \cdots, \beta^n),\ B_1=[B_1^{ij}]_{i,j=1,2,\cdots,n}.$$

Introduce now a product probability space $(\Omega\times\widetilde{\Omega}, \mathscr{F}\times\widetilde{\mathscr{F}}, \mathscr{F}_t\times\widetilde{\mathscr{F}}_t, P\times\widetilde{P})$. Denote

$$\xi_t^i(\omega, \widetilde{\omega})=\xi_t^i(\omega),\ \overline{w}(t, \omega, \widetilde{\omega})=\overline{w}(t, \omega),$$
$$\widetilde{\beta}_t(\omega, \widetilde{\omega})=\beta_t(\widetilde{\omega}),\ w_1(t, \omega, \widetilde{\omega})=w_1(t, \widetilde{\omega})=w^{(1)}(t, \widetilde{\omega}).$$

Noting that $E_{P\times\widetilde{P}}dw(t)\,dw_1(t)=0$, by Ito formula it is not difficult to prove that there exist $\Lambda\subset\Omega$, $\widetilde{\Lambda}\subset\widetilde{\Omega}$ such that for $(\omega, \widetilde{\omega})\in(\Omega-\Lambda)\times(\widetilde{\Omega}-\widetilde{\Lambda})$

$$\langle\varphi(t, \beta_t(\widetilde{\omega})), d\xi_t^i(\omega)\rangle=d\langle\varphi(t, \beta_t(\widetilde{\omega})), \xi_t^i(\omega)\rangle-\langle d\varphi(t, \beta_t(\widetilde{\omega})), \xi_t^i(\omega)\rangle,$$

where $P(\Lambda)=\widetilde{P}(\widetilde{\Lambda})=0$. Hence as $(\omega, \widetilde{\omega})\in(\Omega-\Lambda)\times(\widetilde{\Omega}-\widetilde{\Lambda})$,

$$0 \leqslant g(t, \xi^i(\omega)) = \int_{\tilde{\Omega}} \exp\left\{I_t^i(\omega, \tilde{\omega}) - \frac{1}{2}\int_0^t |B^*(s, \xi_s^i(\omega))\varphi(s, \beta_s(\tilde{\omega}))|^2 ds\right\} dP(\tilde{\omega})$$
$$\leqslant \int_{\tilde{\Omega}} \exp(I_t^i(\omega, \tilde{\omega}))dP(\tilde{\omega}),$$

where

$$I_t^i(\omega, \tilde{\omega}) = I_t^{i1}(\omega, \tilde{\omega}) - I_t^{i2}(\omega, \tilde{\omega}),$$
$$I_t^{i1}(\omega, \tilde{\omega}) = \langle \varphi(t, \beta_t(\tilde{\omega})), \xi_t^i(\omega)\rangle - \int_0^t [\langle \varphi_s', \xi_s^i(\omega)\rangle + \langle \varphi_x' A_1(s, \beta_s(\tilde{\omega})), \xi_s^i(\omega)\rangle$$
$$+ (1/2)\sum_{i,k,j=1}^n \langle (\partial^2\varphi/\partial x^k \partial x^j) B_1^{ki} B_1^{ji}(s, \beta_s(\tilde{\omega})), \xi_s^i(\omega)\rangle] ds,$$
$$I_t^{i2}(\omega, \tilde{\omega}) = \int_0^t \langle \varphi_x' B_1(s, \beta_s(\tilde{\omega}))dw_1(s, \tilde{\omega}), \xi_s^i(\omega)\rangle.$$

By Theorem 4.7 in [6] there exists a $\delta > 0$ such that

$$\sup_{t\leqslant T} \tilde{E} \exp(\delta|\beta_t|^2) = \sup_{t\leqslant T} \int_{\tilde{\Omega}} \exp(\delta|\beta_t|^2) dP(\tilde{\omega}) < +\infty. \tag{15}$$

Therefore for any constant $k' > 0$ we see that as $t \leqslant G_N$,

$$\int_{\tilde{\Omega}} \exp(k'|\langle \varphi(t, \beta_t(\tilde{\omega})), \xi_t^i(\omega)\rangle| d\tilde{P}(\tilde{\omega}) \leqslant \int_{\Omega} \exp(k'|\varphi(t, \beta_t(\tilde{\omega}))||\xi_t^i(\omega)|) d\tilde{P}(\tilde{\omega})$$
$$\leqslant \exp(k'k_0N^2/\delta) \sup_{t\leqslant T} \tilde{E} \exp((\delta/2k_0)|\varphi(t, \beta_t)|^2) < \infty.$$

The same argument shows that as $t \leqslant G_N(\omega)$ for all $(\omega, \tilde{\omega}) \in (\Omega - \Lambda) \times (\tilde{\Omega} - \Lambda)$

$$\int_{\tilde{\Omega}} \exp(I_t^{i1}(\omega, \tilde{\omega})) dP(\tilde{\omega}) \leqslant k_N < +\infty. \tag{16}$$

On the other hand by Girsanov theorem as $t \leqslant G_N$,

$$\int_{\tilde{\Omega}} \exp(-I_t^{i2}(\omega, \tilde{\omega})) - \frac{1}{2}\int_0^t |\xi_s^i(\omega)^* \varphi_x' B_1(s, \beta_s(\tilde{\omega}))|^2) dP(\tilde{\omega}) \leqslant 1. \tag{16$_1$}$$

So applying Holder inequality, by (16) and $(16)_1$, we get that as $t \leqslant G_N$

$$g(t, \xi^i(\omega)) \leqslant k_N' < +\infty. \tag{17}$$

Moreover, by (17) for $0 < p < 1$, $q = p/(p-1) < 0$ as $t \leqslant G_N$,

$$\left(\int_{\tilde{\Omega}} |\exp(-I_t^{i2}(\omega, \tilde{\omega}))|^q dP(\tilde{\omega})\right)^{1/q} \geqslant 1/\exp(|q|^2 k_N')^{1/|q|}. \tag{18}$$

Note that there exist N_0, $r_0 > 0$ such that

$$\tilde{P}(\sup_{t\leqslant T} |\varphi(t, \beta_t(\tilde{\omega}))|^2 \leqslant N_0) \geqslant r_0 > 0.$$

Hence by the inverse Holder inequality (see [18]) it can be derived that

$$\left(\int_{\tilde{\Omega}} \left|\exp\left(I_t^{i1}(\omega, \tilde{\omega}) - \frac{1}{2}\int_0^t |B^*(s, \xi_s^i(\omega))\varphi(s, \beta_s(\tilde{\omega}))|^2 ds\right)\right|^p dP(\tilde{\omega})\right)^{1/p} \geqslant k_N'' > 0. \tag{19}$$

(12) is derived by (16)—(19). Hence (13) is obtained. Applying now

$$\sup_{t\leqslant T} \tilde{E}|\varphi(t, \beta_t(\tilde{\omega}))|^4 \leqslant c_4$$

(by (15)) and $|e^x - e^y| \leqslant |e^x + e^y| \cdot |x - y|$ we have

$$
\begin{aligned}
&E|f(t, \xi^1(\omega))-f(t, \xi^2(\omega))|^2\cdot I_{G_N}(t)\\
&\leqslant k_N^*\Big(E\int_{\tilde\Omega}\Big|\int_0^t\langle\varphi(s, \beta_s(\tilde\omega)), d\xi_s^1(\omega)-d\xi_s^2\ )(\omega)\rangle\Big|^2\cdot I_{G_N}(t)d\tilde P(\tilde\omega)\\
&\quad+\frac{1}{2}\int_{\tilde\Omega}E\Big|\int_0^t(|B(s, \xi_s^1(\omega))\varphi(s, \beta_s(\tilde\omega))|^2\\
&\quad-|B(s, \xi_s^2(\omega))\varphi(s, \beta_s(\tilde\omega))|^2)ds\Big|^2 I_{G_N}(t)d\tilde P(\tilde\omega)\Big)\\
&\leqslant k_N^*\Big(E\int_{\tilde\Omega}\Big|\int_0^t|B^2(s, \xi_s^1(\omega))\\
&\quad-B^2(s, \xi_s^2(\omega))||\bar\varphi(s, \xi^1)||\varphi(s, \beta_s(\tilde\omega))|ds\Big|^2 dP(\tilde\omega)I_{G_N}(t)\\
&\quad+E\int_{\tilde\Omega}\Big|\int_0^t|\bar\varphi(s, \xi^1(\omega))-\bar\varphi(s, \xi^2(\omega))||\varphi(s, \beta_s(\tilde\omega))|ds\Big|^2 d\tilde P(\tilde\omega)I_{G_N}(t)\\
&\quad+E\int_{\tilde\Omega}\Big|\int_0^t\langle(B(s, \xi_s^1(\omega))-B(s, \xi_s^2(\omega))(d\bar w(s, \omega),\varphi(s, \beta_s(\tilde\omega))\rangle I_{G_N}(t)\Big|^2 d\tilde P(\tilde\omega)\Big)\\
&\quad+k_N^1E\int_0^t\rho(E|\xi_s^1(\omega)-\xi_s^2(\omega)|^2I_{G_N}(t))ds=k_N^*(I_1''+I_2'+I_3''+I_4').
\end{aligned}
$$

It is easily seen that

$$I_1'\leqslant k_N^2\int_0^t\rho(E|\xi_s^1(\omega)-\xi_s^2(\omega)|^2\cdot I_{G_N}(s))ds,$$

$$I_2'\leqslant k_N^3(I_2+I_3),\ I_2 \text{ and } I_3\text{–defined in (14).}$$

By Fubini theorem

$$I_3'\leqslant\int_{\tilde\Omega}E\int_0^t|B(s, \xi_s^1(\omega))-B(s, \xi_s^2(\omega))|^2|\varphi(s, \beta_s(\tilde\omega))|^2dsI_{G_N}(t)d\tilde P(\tilde\omega)\leqslant k_N^4I_4'.$$

Hence

$$E|f(t, \xi^1(\omega))-f(t, \xi^2(\omega))|^2\cdot I_{G_N}(t)\leqslant k_N^5(I_2+I_3+I_4').$$

The same argument shows that the same inequality holds for $E|g(t, \xi^1(\omega))-g(t, \xi^2(\omega))|^2I_{G_N}(t)$. Applying Gronwall inequality (Lemma 4.15 in [6]) we get

$$
\begin{aligned}
&E|f(t, \xi^1(\omega))-f(t, \xi^2(\omega))|^2I_{G_N}(t)+E|g(t, \xi^1(\omega))-g(t, \xi^2(\omega))|^2\cdot I_{G_N}(t)\\
&\leqslant k_N^6\Big(I_4'+k_N^7\int_0^t\exp(k_N^7(t-s))\int_0^s\rho(E|\xi_u^1-\xi_u^2|^2\cdot I_{G_N}(u))ds\Big)ds\leqslant k_N^8I_4'.
\end{aligned}
$$

At last from (13) we get

$$E|\xi_{t\wedge G_N}^1(\omega)-\xi_{t\wedge G_N}^2(\omega)|^2\leqslant\bar k_N\int_0^t\rho(E|\xi_{s\wedge G_N}^1(\omega)-\xi_{s\wedge G_N}^2(\omega)|^2)ds.$$

Hence by Lemma 3 in [1] (or by [7]) $E|\xi_{s\wedge G_N}^1(\omega)-\xi_{s\wedge G_N}^2(\omega)|^2=0$. From this it is easily derived that the pathwise uniqueness holds for (11).

Corollary 2. *Assume that $\varphi(t, x, y)=\varphi(t, x)$ does not depend on y, and conditions* (i), (ii) *and* (iv) *in Theorem* 1 *and* (iii)′ b), (vi), (vii) *in Corollary* 1 *are satisfied. Then all the conclusions in Theorem* 2 *hold.*

§ 3. An Application to Optimal control (n-Dimensional Case)

Let us consider the optimal control problem for partially observed process

$$\begin{cases} d\beta_t = u_t\,dt + B_1(t)\,dw_t^{(1)}, \ \beta_0 = \beta_0(\omega) - \mathscr{F}_0 \text{ measurable}, \\ d\xi_t = B(t)B^*(t)(A_{20}(t)\beta_t + A_{21}(t))dt + B(t)\,dw_t, \ \xi_0 = 0, \ t\in[0, T], \end{cases} \tag{20}$$

where the performence function is (we denote the solution of (20) by β_t^u for u_t)

$$J(u) = E\int_0^t |\beta_t^u|^2 dt, \tag*{(20)$_1$}$$

and the admissible control set is

$\mathscr{U} = \{u$: $u(t) - \mathscr{F}_t^\xi$-measurable, for all $t\in[0, T]$, and optional, $|u_t|\leqslant 1$, such that (20) has a pathwise unique strong solution$\}$.

The optimal control problem is to find $u_0\in\mathscr{U}$ such that

$$J(u_0) = \min_{u\in\mathscr{U}} J(u). \tag*{(20)$_2$}$$

For definiteness let us take $\mathscr{F}_t = \mathscr{F}_t^{w^{(1)}, w}$ and $\mathscr{F} = \mathscr{F}_T^{w^{(1)}, w}$. Assume that

(i) A_{20}, A_{21}, B_1 and B are all bounded $n\times n$ matrixes, which do not depend on x, and there exists a $\delta>0$ such that for all $t\in[0, T]$ and any $\lambda\in R^n$,

$$\langle B(t)\lambda, \lambda\rangle \geqslant \delta|\lambda|^2,$$

and so do $B_1(t)$ and $A_{20}(t)$.

(ii) $w^{(1)}$ is independent of w.

(iii) the initial distribution $P(\beta_{01}\leqslant a_{01}, \cdots, \beta_{0n}\leqslant a_{0n})$ is Guassian, where $\beta_0 = (\beta_{01}, \cdots, \beta_{0n})$, $a = (a_{01}, \cdots, a_{0n})$, with mean vector $m_0 = E(\beta_0)$ and covariance matrix $\gamma_0 = E((\beta_0 - m_0)(\beta_0 - m_0)^*)$ such that tr. $\gamma_0 < \infty$ and γ_0 is positive definite.

(iv) there exists an $\varepsilon>0$ such that

$$E(\exp(\varepsilon|\beta_0|^2)) < \infty.$$

Applying Corollary 2 of Theorem 2, we see that for each $u\in\mathscr{U}$,

$$\bar{w}_t = \int_0^t B^{-1}(s)\,d\xi_s - \int_0^t B^*(s)(A_{20}(s)\hat{\beta}_s + A_{21}(s))ds, \tag{21}$$

where $\hat{\beta}_s = E(\beta_s|\mathscr{F}_s^\xi)$ is a B. M. under the probability space $(\Omega, \mathscr{F}, \mathscr{F}_t^\xi, P)$. Denote for each u,

$$m_t = E(\beta_t|\mathscr{F}_t^\xi), \ \gamma_t = E((\beta_t - m_t)(\beta_t - m_t)^*|\mathscr{F}_t^\xi).$$

Applying Theorem 12.7 in [6] we see that m_t, γ_t are the unique continuous $\mathscr{F}_t^\xi$-measurable solution for $S.\ D.\ E.$

$$\begin{cases} dm_t = u_t dt + \gamma_t A_{20}^*(t) B^*(t)^{-1} d\bar{w}_t, \\ m_0 = E(\beta_0), \quad t\in[0, T], \end{cases} \tag{22}$$

$$\begin{cases} \dot{\gamma}_t = B_1(t)B_1^*(t) - \gamma_t A_{20}^*(t)(B(t)B^*(t))^{-1}A_{20}(t)\gamma_t, \\ \gamma_0 = E(\beta_0 - m_0)(\beta_0 - m_0)^*, \quad t\in[0, T], \end{cases} \tag{23}$$

and γ_t is positive definite for $t\in[0, T]$.

From (22), (21) and (20) we get

$$\begin{cases} d(\beta_t-m_t)=B_1(t)dw_t^{(1)}-\gamma_t A_{20}^*(t)(A_{20}(t)(\beta_t-m_t)dt+B(t)dw_t), \\ \beta_0-m_0=\beta_0-E(\beta_0), \quad t\in[0, T]. \end{cases} \tag{24}$$

Therefore β_t-m_t does not depend on $u\in\mathscr{U}$. Since

$$d(\beta_t-m_t)=B_1(t)dw_t^{(1)}-\gamma_t A_{20}^*(t)B^*(t)^{-1}d\bar{w}_t,$$

we have

$$d\bar{w}_t=B^*(t)A_{20}^*(t)^{-1}\gamma_t^{-1}(B_1(t)dw_t^{(1)}-d(\beta_t-m_t)), \tag{25}$$

i.e. $\bar{w}_t$ does not depend on $u\in\mathscr{U}$ either. Therefore

$$\bar{w}_t^u=\bar{w}_t^0, \text{ for all } u\in\mathscr{U},$$

where $\bar{w}_t^0$ is the innovation process corresponding to control $u=0$. Hence

$$\mathscr{F}_t^{\bar{w}^u}=\sigma(\bar{w}_s^u,\ s\leqslant t)=\sigma(\bar{w}_s^0,\ s\leqslant t)=\mathscr{F}_t^{\bar{w}^0},$$

where all σ-algebras are completed by the P-nullsets from $\mathscr{F}$. From now on, we always write $\bar{w}_t$ as $\bar{w}_t^0$.

Now let us consider the separated problem.

Discuss (22) for all $u\in\mathscr{U}'$, where

$$\mathscr{U}'=\{u:\ u_t \text{ is optional, } \mathscr{F}_t^{\bar{w}}\text{-measurable for all } t\in[0, T] \text{ such that } |u_t|\leqslant 1\}. \tag{26}$$

For such admissible control, consider the performence function as

$$J_1(u)=E\int_0^T |m_t|^2\,dt.$$

And we are going to find out a $u_0\in\mathscr{U}'$ such that

$$J_1(u_0)=\min_{u\in\mathscr{U}'} J_1(u). \tag{27}$$

We have the following lemma.

Lemma 2. *Set*

$$U(y)=\begin{cases} -y/|y|, \text{ as } y\neq 0, \\ 0, \text{ as } y=0. \end{cases} \tag{28}$$

Then S. D. E.

$$\begin{cases} dm_t^\circ=U(m_t^\circ)dt+\gamma_t C(t)^*d\bar{w}_t, \ (C(t)^*=A_{20}^*(t)B^*(t)^{-1}) \\ m_0^\circ=E\beta_0, \quad t\in[0, T] \end{cases} \tag{29}$$

has a pathwise unique strong solution m_t°, which is $\mathscr{F}_t^{\bar{w}}$-measurable. Moreover

$$u_t^\circ=U(m_t^\circ)$$

is the optimal control for the separated problem (22), (26), (27), i.e.

$$J_1(u^\circ)=\min_{u\in\mathscr{U}'} J_1(u).$$

For proving this lemma we need the following proposition. Set

$$\tau_t=\int_0^t |\gamma_s C_s^*|^2\,ds.$$

Then τ_t strictly increases to infinity as $t\uparrow\infty$. Denote

$$T_t=(\tau_t)^{-1}.$$

Consider the following stochastic optimal control problem: For

$$M_t=m_0^\circ+\int_0^t v_s|\gamma_s C_s^*|^{-2}ds+\tilde{w}_t, \quad t\in[0, T_1], \tag{30}$$

where

$$\tilde{w}_t=\int_0^{T_t}\gamma_s C_s^*\, d\bar{w}_s \tag{31}$$

is a B. M. adapted to

$$\tilde{\mathscr{F}}_t=\mathscr{F}^{\bar{w}}_{T_t}, \quad t\in[0, T_1],$$

where $T_1=\tau_T$,

$$\mathscr{U}''=(v\colon |v|\leqslant 1,\ v_t \text{ is } \tilde{\mathscr{F}}_t\text{-measurable and optional}), \tag{32}$$

$$J_2(v)=E\int_0^{T_1}|M_t^v|^2\,dt, \tag{33}$$

find $v^\circ\in\mathscr{U}''$ such that

$$J_2(v^\circ)=\min_{v\in\mathscr{U}''}J_2(v). \tag{34}$$

We have the following proposition.

Proposition 1. *$v_t^\circ=U(M_t^\circ)$, where $U(.)$ is defined as (28) and M_t° satisfies S. D. E.*

$$M_t^\circ=m_0^\circ+\int_0^t U(M_s^\circ)|\gamma_s C_s^*|^{-2}ds+\tilde{w}_t, \quad t\in[0, T_1], \tag{35}$$

(actually M_t° is the pathwise unique strong solution of (35)), is the optimal control for stochastic optimal control problem (30)—(34).

Proof By Example 2 of [22], (35) has a pathwise unique strong solution. By [1] it can be shown similarly that the pathwise uniqueness holds for the following S. D. E. in 1-dimensional space:

$$\begin{cases} dx(t)=2(x(t)^+)^{1/2}dw_1(t)+(n-2|\gamma_s C_s^*|^{-2}(x(t)^+)^{1/2})dt, \\ x(0)=x_0, \quad t\in[0, T_1]. \end{cases}$$

Therefore the proof of Theorem 2.1 of Chapter VI in [10] is applied.

Proposition 2. *For any $v\in\mathscr{U}''$, set*

$$u_t=v_{\tau_t},$$

then $u\in\mathscr{U}'$. Conversely, for any $u\in\mathscr{U}'$, set

$$v_t=u_{T_t},$$

then $v\in\mathscr{U}''$.

Now let us return to the proof of Lemma 2. By (35) and (31) we have

$$m_t^\circ=m_0^\circ+\int_0^t U(m_s^\circ)\,ds+\int_0^t\gamma_s C_s^*\,d\bar{w}_s,$$

where we have applied $m_s=M_{\tau(s)}$; and P-a.s.

$$E|m_t^\circ|^2\leqslant E|m_t^u|^2, \text{ for all } t\in[0, T],$$

for any $u\in\mathscr{U}'$. The proof is completed.

Now let us present the following theorem.

Theorem 3. *$u_t^\circ = U(m_t^\circ)$ is the optimal control for the stochastic optimal control problem* (20)—(20)$_2$, where *m_t° is the pathwise unique strong solution of S. D. E.* (29).

For proving Theorem 3 we need some more preparation.

Consider for $t\in[0, T]$,

$$\begin{cases} d\bar\beta_t^u = u_t\,dt, \ \bar\beta^0 = 0, \\ d\bar\xi_t^u = B(t)B^*(t)A_{20}(t)\bar\beta_t^u\,dt, \quad \bar\xi_0^u = 0, \end{cases} \tag{36}$$

$$\begin{cases} d\tilde\beta_t = B_1(t)dw_t^{(1)}, \ \tilde\beta_0 = \beta_0, \\ d\tilde\xi_t = B_1(t)B^*(t)(A_{21}(t) + A_{20}(t)\tilde\beta_t)dt + B(t)dw_t, \quad \tilde\xi_0 = 0. \end{cases} \tag{37}$$

Then it is apparent that

$$\beta_t^u = \bar\beta_t^u + \tilde\beta_t, \quad \xi_t^u = \bar\xi_t^u + \tilde\xi_t.$$

Applying Theorem 2, we get an important relation, which will be used frequently in the following

$$\mathscr{F}_s^{\tilde\xi} = \mathscr{F}_s^{\xi^0} = \mathscr{F}_s^{\bar w^0} = \mathscr{F}_s^{\bar w^u} \subset \mathscr{F}_s^{\xi^u}.$$

Denote

$$\hat{\hat u}_t = E(u_t\,|\,\mathscr{F}_t^{\tilde\xi}) = E(u_t\,|\,\mathscr{F}_t^{\bar w}),\ \hat u_t = E(u_t\,|\,\mathscr{F}_t^{\xi^u}),\ \hat\beta_t^u = E(\beta_t^u\,|\,\mathscr{F}_t^{\xi^u}), \text{ etc;}$$

and

$$\bar w_t = \bar w_t^0.$$

We have the following lemma.

Lemma 3. $E(\bar\beta_t^u\,|\,\mathscr{F}_t^{\bar w}) = \bar\beta_t^{\hat{\hat u}}$.

Proof By $\bar\beta_t^u = \int_0^t u_s\,ds$ we have

$$E(\bar\beta_t^u\,|\,\mathscr{F}_t^{\bar w}) = \int_0^t E(u_s - \hat{\hat u}_s\,|\,\mathscr{F}_t^{\bar w})ds + \bar\beta_t^{\hat{\hat u}}.$$

It is easily seen that $E(\xi_s(\bar w_t - \bar w_r)) = 0$, as $t > r \geqslant s$. Now since for $u\in\mathscr{U}$, $u_s - \hat{\hat u}_s$ is $\mathscr{F}_s^{\xi^u}$-measurable and $E((u_s - \hat{\hat u}_s)\bar w_r) = 0$, as $r \leqslant s$ we have

$$E(u_s - \hat{\hat u}_s\,|\,\mathscr{F}_t^{\bar w}) = 0, \text{ as } t \geqslant s.$$

Lemma 4. *For $u\in\mathscr{U}$ we have $\hat{\hat u}\in\mathscr{U}'$ and*

$$J(\hat{\hat u}) \leqslant J(u).$$

Proof $J(u) - J(\hat{\hat u}) = E\int_0^T(|\beta_s^u|^2 - |\beta_s^{\hat{\hat u}}|^2)ds = E\int_0^T\langle\bar\beta_s^u + \tilde\beta_s + \bar\beta_s^{\hat{\hat u}} + \tilde\beta_s,\ \bar\beta_s^u - \bar\beta_s^{\hat{\hat u}}\rangle$

$$= E\int_0^T\langle\bar\beta_s^u,\ \bar\beta_s^u - \bar\beta_s^{\hat{\hat u}}\rangle ds + 2E\int_0^T\langle\tilde\beta_s,\ \bar\beta_s^u - \bar\beta_s^{\hat{\hat u}}\rangle ds$$

$$+ E\int_0^T\langle\bar\beta_s^{\hat{\hat u}},\ \bar\beta_s^u - \bar\beta_s^{\hat{\hat u}}\rangle ds = I_1 + I_2 + I_3.$$

Evidently by Lemma 3 we have

$$I_3 = 0,$$

$$I_1 = E\int_0^T\langle\bar\beta_s^u - \bar\beta_s^{\hat{\hat u}},\ \bar\beta_s^u - \bar\beta_s^{\hat{\hat u}}\rangle ds + I_3 \geqslant 0,$$

$$I_2=2E\int_0^T E(\langle\tilde{\beta}_s, \bar{\beta}_s^u-\hat{\bar{\beta}}_s^u\rangle|\mathscr{F}_s^{\xi^u})\,ds=2E\int_0^T\langle\hat{\beta}_s, \bar{\beta}_s^u-\hat{\bar{\beta}}_s^u\rangle ds.$$

By $\bar{w}_t^u=\bar{w}_t^0$ and the nonsingularity of $B^*(t)$, $A_{20}(t)$ we can show tnat

$$\hat{\hat{\beta}}_s=\hat{\hat{\bar{\beta}}}_s.$$

Hence $I_2=0$. And we come to the conclusion.

Now we are in a position to prove Theorem 3.

Proof of Theorem 3. First of all it is obviously that

$$J(u)=E\int_0^T(m_t m_t^*)dt+\mathrm{Tr.}\int_0^T\gamma_t\,dt=J_1(u)+\mathrm{Tr.}\int_0^T\gamma_t\,dt.$$

By Lemma 2 for $u^\circ=U(m_t^\circ)$ we have

$$J_1(u^\circ)\leqslant J_1(u),\ \text{for all } u\in\mathscr{U}'.$$

Applying Lemma 4 we get

$$J(u^\circ)=\min_{u\in\mathscr{U}'}J(u)=\min_{u\in\mathscr{U}}J(u).$$

At last, let us show that

$$u^\circ\in\mathscr{U}.$$

Since u_t° is $\mathscr{F}_t^{\bar{w}^\circ}=\mathscr{F}_t^{\tilde{\xi}}$-measurable, but by (37) $\tilde{\xi}_t$ is $\mathscr{F}_t^{w^{(1)},w}$-measurable, we see that u_u° is $\mathscr{F}_t^{w^{(1)},w}$-measurable. From (20) for $t\in[0, T]$,

$$\begin{cases} d\beta_t^{u^\circ}=u_t^\circ\,dt+B_1(t)dw_t^{(1)},\ \beta_0^{u^\circ}=\beta_0,\\ d\xi_t^{u^\circ}=B(t)B^*(t)(A_{20}\beta_t^{u^\circ}+A_{21}(t))dt+B(t)dw_t,\ \xi_0^{u^\circ}=0.\end{cases}\tag{20$'$}$$

Hence $\beta_t^{u^\circ}$ and $\xi_t^{u^\circ}$ are $\mathscr{F}_t^{w^{(1)},w}$-measurable, i.e. (20) for u_t° has a pathwise unique strong solution $\beta_t^{u^\circ}$, $\xi_t^{u^\circ}$. Moreover, u_t° is $\mathscr{F}_t^{\bar{w}^\circ}$-measurable. Since

$$\mathscr{F}_t^{\bar{w}^\circ}=\mathscr{F}_t^{\bar{w}^{u^\circ}}\subset\mathscr{F}_t^{\xi^{u^\circ}},$$

we have $u^\circ\in\mathscr{U}$.

An almost step by step approach as section 2 here generalizes the above result to a more general partially observed process system (in n-dimensional space)

$$\begin{cases} d\beta_t=(\beta_t+u_t)dt+B_1(t)dw_t^{(1)},\ \beta_0=\beta_0(\omega)-\mathscr{F}_0\text{-measurable},\\ d\xi_t=B(t)B^*(t)(A_{20}(t)\beta_t+A_{21}(t))dt+B(t)dw_t,\ \xi_0=0,\ t\in[0, T].\end{cases}\tag{38}$$

We state the result here.

Theorem 4. *Under assumptions* (I)—(Iv) *in this section* $u_t^\circ=U(m_t^\circ)$ *is the optimal control for the stochastic optimal control problem* (38), $(20)_1$—$(20)_2$, *where* $U(y)$ *is defined in* (28) *and* m_t° *is the pathwise unique strong solution of the following S. D. E.*

$$\begin{cases} dm_t^\circ=(m_t^\circ+U(m_t^\circ))dt+\gamma_t C(t)^*d\bar{w}_t,\\ m_0^\circ=E\beta_0,\quad t\in[0, T],\end{cases}\tag{39}$$

where γ_t *satisfies the prior ordinary differential equation*

$$\begin{cases} \dot{\gamma}_t=2\gamma_t+B_1(t)B_1^*(t)-\gamma_t A_{20}^*(t)(B(t)B^*(t))^{-1}A_{20}(t)\gamma_t,\\ \gamma_0=E((\beta_0-m_0)(\beta_0-m_0)^*),\quad t\in[0, T].\end{cases}\tag{40}$$

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