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A NOTE ON THE REGULARITY OF SOLUTIONS TO A NONLINEAR ELLIPTIC SYSTEM FROM ELASTICITY-PLASTICITY THEORY

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Abstract

The regularity of the weak solutions to an elliptic system from elasticity-plasticity theory is studied. Although this system is a nonlinear elliptic system with discontinuous coefficients, $C^{1,\alpha}$-everywhere regularity for its weak solutions is proved.

§ 1. Introduction

In this note we prove the $H^{2,r}(r>2)$ regularity of the weak solution to the Dirichlet problem for a nonlinear elliptic system with discontinuous coefficients of the form

$$\partial^T(D_{ep}(\partial u)\partial u)+F(x)=0 \quad \text{in } \Omega, \tag{1.1}$$

$$u=0 \quad \text{on } \partial\Omega. \tag{1.2}$$

Here Ω is a bounded domain in $\mathbb{R}^2$, u is a vector in $\mathbb{R}^2$, ∂ is the differential operator matrix:

$$\partial=\begin{bmatrix} \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix}, \tag{1.3}$$

$$D_{ep}(\partial u)=D_e-D_eA\alpha(P)A^TD_e, \tag{1.4}$$

$$D_e=\frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \tag{1.5}$$

is the elastic matrix,

$$A=[A_1A_2A_3]^T,$$

$$P=A^TD_e\partial u, \quad \alpha(P)=\begin{cases} 0 & \text{if } P<0, \\ \alpha_0 & \text{if } P\geqslant 0, \end{cases} \quad \alpha_0=\frac{1}{h_0+A^TD_eA}. \tag{1.6}$$

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Here, $h_0>0$, $E>0$, $0<\mu<1$ and A_i $(i=1, 2, 3)$ are given constants.

This system arises from a class of elastic-plastic problems (see [1, 2, 4]).

It is known[3] that $D_{ep}(\partial u)$ is a positive definite matrix:

$$\xi^T D_{ep}(\partial u)\xi \geq \lambda|\xi|^2,\ \lambda>0,\quad \forall \xi\in\mathbb{R}^3,$$

and for any $\eta\in\mathbb{R}^2$, $\eta\neq 0$, the 2×2 matrix

$$\begin{bmatrix} \eta_1 & 0 \\ 0 & \eta_2 \\ \eta_2 & \eta_1 \end{bmatrix}^T D_{ep}(\partial u)\begin{bmatrix} \eta_1 & 0 \\ 0 & \eta_2 \\ \eta_2 & \eta_1 \end{bmatrix}$$

is a positive definite matrix, too. So, we say that system (1.1) is a strongly elliptic system in Visik-Nirenberg sense.

§ 2. Preliminaries

In this section we shall state some notations and well known lemmas which will be needed in the following.

We shall denote by $H^{m,p}(\Omega, \mathbb{R}^N)$ the Cartesian product

$$H^{m,p}(\Omega)\times H^{m,p}(\Omega)\times\cdots\times H^{m,p}(\Omega)=(H^{m,p}(\Omega))^N,$$

$H^{m,p}(\Omega)$ being the standard Sobolev spaces. A similar meaning holds for $C^{m,\alpha}(\Omega, \mathbb{R}^N)$ and so on.

Moreover we shall denote by $|\Omega|$ the Lebesgue measure of Ω and by

$$⨍_\Omega f\,dx$$

the average of f on Ω:

$$⨍_\Omega f\,dx=\frac{1}{|\Omega|}\int_\Omega f\,dx.$$

Let us now recall some well known lemmas we shall use in the following.

Lemma 2.1[5]. *Let $\theta(t)$, $-\infty<t<\infty$, be a Lipschitz function whose derivative $\theta'(t)$ exists except at finitely many points $\{a_1, \cdots, a_M\}$ and let $u\in H^{1,s}(\Omega)$, $1\leq s<\infty$. Then*

$$\theta(u)\in H^{1,s}(\Omega)$$

and

$$(\theta(u))_{x_i}=\theta'(u)u_{x_i}\quad \textit{(in the sense of distributions)}$$

with the convention that both sides are zero when $x\in\bigcup_j\{y: u(y)=a_j\}$.

Lemma 2.2[6,7] (Reverse Hölder Inequality). *Let Q be an n-cube. Assume that g, f are non-negative functions on Q and that*

$$g\in L^q(Q),\ q>1,\ f\in L^s(Q),\ s>q.$$

Suppose

$$⨍_{Q_R(x^0)} g^q\,dx\leq b\Big(⨍_{Q_{2R}(x^0)} g\,dx\Big)^q+⨍_{Q_{2R}(x^0)} f^q\,dx+\theta⨍_{Q_{2R}(x^0)} g^q\,dx$$

for each $x^0 \in Q$ *and each* $R < \frac{1}{2}\operatorname{dist}(x^0, \partial Q) \wedge R_0$, *where*

$$Q_R(x^0) = \{x \in \mathbb{R}^n;\ |x_i - x_i^0| < R,\ i=1, 2, \cdots, n\},$$

and R_0, b, θ *are constants with* $b>1$, $R_0>0$, $0 \leqslant \theta < 1$. *Then*

$$g \in L^p_{\mathrm{loc}}(Q) \quad \text{for } p \in [q, q+\varepsilon),$$

moreover

$$\left(\fint_{Q_R} g^p\,dx\right)^{\frac{1}{p}} \leqslant C\left\{\left(\fint_{Q_{2R}} g^q\,dx\right)^{\frac{1}{q}} + \left(\fint_{Q_{2R}} f^p\,dx\right)^{\frac{1}{p}}\right\}$$

for $Q_{2R} \subset Q$, $R < R_0$, *where* C *and* ε *are positive constants depending only on* n, b, θ, q *and* s.

§ 3. $H^{2,r}_{\mathrm{loc}}$ Regularity

In [3], the authors proved the existence, uniqueness and $H^{2,2}(\Omega, \mathbb{R}^2)$ regularity of the weak solution for the Dirichlet problem of system (1.1). Here we would like to show that this weak solution is actually in $H^{2,r}(\Omega, \mathbb{R}^2)$ for some $r>2$ and thus in $C^{1,\delta}(\overline{\Omega}, \mathbb{R}^2)$ for some δ, $0<\delta<1$.

Definition. *A vector* $u(x)$ *is called the weak solution of problem* (1.1), (1.2), *if* $u \in H^{1,2}(\Omega, \mathbb{R}^2)$ *and satisfies the following integral identity*:

$$\int_\Omega (\partial\varphi)^T D_{ep}(\partial u)\partial u\,dx - \int_\Omega \varphi^T F dx = 0, \quad \forall \varphi \in C_0^\infty(\Omega, \mathbb{R}^2). \tag{3.1}$$

We begin with proving the regularity in the interior. The main result in this section is the following theorem.

Theorem 3.1. *If* $F \in L^s(\Omega, \mathbb{R}^2)$, $s>2$, *then there exists an exponent* $r>2$ *such that if* $u \in H^{1,2}(\Omega, \mathbb{R}^2)$ *is the weak solution of problem* (1.1), (1.2), *then* $u \in H^{2,r}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$, *where* r *depends only on* λ, D_e, A, h_0, s.

Proof The first step consists in proving that $u_{x_k} = \dfrac{\partial u}{\partial x_k}$ $(k=1, 2)$ satisfies

$$\int_\Omega (\partial\varphi)^T D_{ep}(\partial u)\partial u_{x_k}\,dx + \int_\Omega \varphi^T_{x_k} F\,dx = 0, \quad \forall \varphi \in C_0^\infty(\Omega, \mathbb{R}^2),\ k=1, 2. \tag{3.2}$$

We begin by remarking that from (1.4)—(1.6) one can rewrite (3.1) as follows

$$\int_\Omega (\partial\varphi)^T (D_e\partial u - D_e A\gamma(P))\,dx = \int_\Omega \varphi^T F\,dx, \quad \forall \varphi \in C_0^\infty(\Omega, \mathbb{R}^2), \tag{3.3}$$

where $\gamma(P) = \begin{cases} 0, & P<0 \\ \alpha_0 P, & P \geqslant 0 \end{cases}$ is a Lipschitz function of P.

Therefore, for any $\psi \in C_0^\infty(\Omega, \mathbb{R}^2)$, inserting $\varphi = \psi_{x_k}$ $(k=1, 2)$ in (3.3), we have

$$\int_\Omega (\partial\psi_{x_k})^T (D_e\partial u - D_e A\gamma(P))\,dx = \int_\Omega \psi^T_{x_k} F\,dx, \quad \forall \psi \in C_0^\infty(\Omega, \mathbb{R}^2),\ k=1, 2. \tag{3.4}$$

Since $u \in H^{2,2}(\Omega, \mathbb{R}^2)$[3], using Lemma 2.1, it is easy to show that $\gamma(P) \in H^{1,2}(\Omega)$. Hence, the vector $D_e\partial u - D_e A\gamma(P)$ belongs to $H^{1,2}(\Omega, \mathbb{R}^3)$ and it can be differentiated in the weak sense, i.e.

$$\int_\Omega (\partial\psi_{x_k})^T(D_e\partial u - D_e A\gamma(P))\,dx$$

$$= -\int_\Omega (\partial\psi)^T(D_e\partial u_{x_k} - D_e A\gamma'(P)A^T D_e\partial u_{x_k})\,dx, \quad \forall\psi\in C_0^\infty(\Omega, \mathbb{R}^2),\ k=1,2. \tag{3.5}$$

Then, from (3.4) and (3.5), we have

$$\int_\Omega (\partial\psi)^T(D_e - D_e A\gamma'(P)A^T D_e)\partial u_{x_k}\,dx + \int_\Omega \psi_{x_k}^T F\,dx = 0, \quad \forall\psi\in C_0^\infty(\Omega, \mathbb{R}^2),\ k=1,2.$$

Therefore, (3.2) is proved. Moreover, (3.2) holds for every $\varphi\in H_0^{1,2}(\Omega, \mathbb{R}^2)$.

The second step is now to get the result of this theorem from (3.2).

Let Q be an n-cube. $Q\subset\subset\Omega$. For each $x^0\in Q$ and each $R<\frac{1}{2}\operatorname{dist}(x^0, \partial Q)\wedge R_0$, where $R_0>0$ is a constant, we construct a cut-off function $\eta(x)$:

$$\eta\in C_0^\infty(Q_{2R}(x^0)), \quad 0\leqslant\eta\leqslant 1, \ \eta\equiv 1 \text{ on } Q_R(x^0), \ |D\eta|\leqslant\frac{C}{R}, \tag{3.6}$$

and choose as test vector in (3.2) $\varphi=\varphi_k=\eta^2(u_{x_k}-(u_{x_k})_{2R})$, $k=1, 2$, where

$$(u_{x_k})_{2R} = ⨍_{Q_{2R}(x^0)} u_{x_k}\,dx,$$

then we get

$$\int_\Omega [\partial(\eta^2(u_{x_k}-(u_{x_k})_{2R}))]^T D_{ep}(\partial u)\partial(u_{x_k}-(u_{x_k})_{2R})\,dx$$

$$+\int_\Omega [\eta^2(u_{x_k}-(u_{x_k})_{2R})]_{x_k}^T F\,dx = 0, \quad k=1, 2.$$

By some computations we have

$$\int_\Omega [\partial(\eta(u_{x_k}-(u_{x_k})_{2R}))]^T D_{ep}(\partial u)\partial(\eta(u_{x_k}-(u_{x_k})_{2R}))\,dx$$

$$=\int_\Omega [\partial(\eta(u_{x_k}-(u_{x_k})_{2R}))]^T D_{ep}(\partial u)(\partial^T(\eta I))^T(u_{x_k}-(u_{x_k})_{2R})\,dx$$

$$-\int_\Omega (u_{x_k}-(u_{x_k})_{2R})^T\partial^T(\eta I)D_{ep}(\partial u)\partial(\eta(u_{x_k}-(u_{x_k})_{2R}))\,dx$$

$$+\int_\Omega (u_{x_k}-(u_{x_k})_{2R})^T\partial^T(\eta I)D_{ep}(\partial u)(\partial^T(\eta I))^T(u_{x_k}-(u_{x_k})_{2R})\,dx$$

$$-\int_\Omega \eta[\eta(u_{x_k}-(u_{x_k})_{2R})]_{x_k}^T F\,dx - \int_\Omega \eta_{x_k}\eta(u_{x_k}-(u_{x_k})_{2R})^T F\,dx,$$

where I is a 3×3 unit matrix.

Since $D_{ep}(\partial u)$ is a positive definite matrix, elements of $D_{ep}(\partial u)$ are bounded, and those bounds depend only on D_e, A, h_0, we have

$$\sum_{k=1}^2\int_\Omega |\partial(\eta(u_{x_k}-(u_{x_k})_{2R}))|^2\,dx \leqslant C\sum_{k=1}^2\int_\Omega |\partial^T(\eta I)|^2|u_{x_k}-(u_{x_k})_{2R}|^2\,dx$$

$$+C\int_\Omega \eta^2|F|^2\,dx,$$

where C depends on λ, D_e, A, h_0.

By Korn's inequality[8], we obtain

$$\sum_{k=1}^{2}\int_{\Omega}|D(\eta(u_{x_k}-(u_{x_k})_{2R}))|^2dx\leqslant\sum_{k=1}^{2}\int_{\Omega}|D\eta|^2|u_{x_k}-(u_{x_k})_{2R}|^2dx+C\int_{\Omega}\eta^2|F|^2dx,$$

and from (3.6) we get

$$\int_{Q_R(x^0)}|D^2u|^2dx\leqslant\frac{C}{R^2}\int_{Q_{2R}(x^0)}|Du-(Du)_{2R}|^2dx+C\int_{Q_{2R}(x^0)}|F|^2dx. \tag{3.7}$$

Using Sobolev-Poincaré inequality[7], we obtain

$$\int_{Q_{2R}(x^0)}|Du-(Du)_{2R}|^2dx\leqslant C\Big(\int_{Q_{2R}(x^0)}|D^2u|^{\frac{2n}{n+2}}dx\Big)^{\frac{n+2}{n}}. \tag{3.8}$$

Combining (3.7) and (3.8), we have

$$⨍_{Q_R(x^0)}|D^2u|^2dx\leqslant C\Big(⨍_{Q_{2R}(x^0)}|D^2u|^{\frac{2n}{n+2}}dx\Big)^{\frac{n+2}{n}}+C⨍_{Q_{2R}(x^0)}|F|^2dx.$$

Now, choosing $g=|D^2u|^{\frac{2n}{n+2}}$, $f=|F|^{\frac{2n}{n+2}}$, $q=\frac{n+2}{n}$ and $\theta=0$ in Lemma 3.2, we get

$$|D^2u|^{\frac{2n}{n+2}}\in L^p_{loc}(Q)\quad\text{for}\quad p\in\Big[\frac{n+2}{n},\ \frac{n+2}{n}+\varepsilon\Big),$$

moreover

$$\Big(⨍_{Q_R}|D^2u|^{\frac{2np}{n+2}}dx\Big)^{\frac{1}{p}}\leqslant C\Big(⨍_{Q_{2R}}|D^2u|^2dx\Big)^{\frac{n}{n+2}}+C\Big(⨍_{Q_{2R}}|F|^{\frac{2np}{n+2}}dx\Big)^{\frac{1}{p}}$$

for $Q_{2R}\subset\subset Q\subset\Omega$, $R<R_0$, where C and ε are positive constants depending only on n, λ, D_e, A, h_0, s.

If we let $r=\frac{2n}{n+2}p$, then $|D^2u|\in L^r_{loc}(Q)$ for $r\in\Big[2,\ 2+\frac{2n}{n+2}\varepsilon\Big)$. Here, $n=2$, so $r=p\in[2, 2+\varepsilon)$ and $u\in H^{2,r}_{loc}(Q, \mathbb{R}^2)$ for any $Q\subset\subset\Omega$. Therefore, we have

$$u\in H^{2,r}_{loc}(\Omega, \mathbb{R}^2)\quad\text{for}\quad r\in[2, 2+\varepsilon).$$

§ 4. Regularity up to the Boundary

Suppose $\Omega\in C^2$. In this section we are going to prove the regularity of the weak solution up to the boundary.

For each $x^0\in\partial\Omega$, without loss of generality, we suppose $x^0=0$, $U(0)$ is a neighborhood of 0 and $\partial\Omega\cap U(0)$ can be expressed as

$$x_2+\xi(x_1)=0,$$

and under a transformation

$$\begin{cases} y_1=x_1,\\ y_2=x_2+\xi(x_1),\end{cases} \tag{4.1}$$

the image of the domain $\Omega\cap U(0)$ lies on the upper half plane $\mathbb{R}^2_+$ and includes a half-cube $Q^+=Q\cap\mathbb{R}^2_+$, where $Q=\{y\in\mathbb{R}^2: |y_i|<R_1,\ i=1, 2\}$. In the sequel we shall set D for the image of $U(0)$ and D^+ for the image of $\Omega\cap U(0)$.

From (3.1), we know that if u is the weak solution of problem (1.1), (1.2), then $u\in H_0^{1,2}(\Omega, \mathbb{R}^2)$ satisfies

$$\int_{\Omega\cap U(0)}(\partial\varphi)^T D_{ep}(\partial u)\partial u\,dx=\int_{\Omega\cap U(0)}\varphi^T F\,dx, \quad \forall\varphi\in C_0^\infty(U(0), \mathbb{R}^2),$$

and under the transformation (4.1), we have

$$\int_{D^+}(\partial\tilde{\varphi})^T D_{ep}(\partial\tilde{u})\,\partial\tilde{u}\,dy=\int_{D^+}\tilde{\varphi}^T\tilde{F}\,dy, \quad \forall\tilde{\varphi}\in C_0^\infty(D, \mathbb{R}^2), \tag{4.2}$$

where $\tilde{u}(y)=u(x)$, $\tilde{\varphi}(y)=\varphi(x)$, $\tilde{F}(y)=F(x)$. ∂ still stands for the differential opertor matrix with respect to x (see (1.3)).

Now, for any $\tilde{\psi}\in C_0^\infty(D, \mathbb{R}^2)$, inserting $\tilde{\varphi}=\dfrac{\partial\tilde{\psi}}{\partial y_1}$ in (4.2), we have

$$\int_{D^+}\Big[\partial\Big(\frac{\partial\tilde{\psi}}{\partial y_1}\Big)\Big]^T D_{ep}(\partial\tilde{u})\partial\tilde{u}\,dy=\int_{D^+}\Big(\frac{\partial\tilde{\psi}}{\partial y_1}\Big)^T\tilde{F}\,dy, \quad \forall\tilde{\psi}\in C_0^\infty(D, \mathbb{R}^2). \tag{4.3}$$

It is easy to show

$$\partial\Big(\frac{\partial\tilde{\psi}}{\partial y_1}\Big)=\frac{\partial}{\partial y_1}\,\partial\tilde{\psi}-\xi''(y_1)\tilde{\partial}_y\tilde{\psi}, \tag{4.4}$$

where

$$\tilde{\partial}_y=\begin{bmatrix}\dfrac{\partial}{\partial y_2} & 0\\ 0 & 0\\ 0 & \dfrac{\partial}{\partial y_2}\end{bmatrix}.$$

Therefore, we obtain

$$\int_{D^+}\Big(\frac{\partial}{\partial y_1}\partial\tilde{\psi}\Big)^T D_{ep}(\partial\tilde{u})\partial\tilde{u}\,dy-\int_{D^+}\xi''(y_1)(\tilde{\partial}_y\tilde{\psi})^T D_{ep}(\partial\tilde{u})\partial\tilde{u}dy$$
$$=\int_{D^+}\Big(\frac{\partial\tilde{\psi}}{\partial y_1}\Big)^T\tilde{F}\,dy, \quad \forall\tilde{\psi}\in C_0^\infty(D, \mathbb{R}^2). \tag{4.5}$$

Using Lemma 2.1 and (4.4), we have

$$\int_{D^+}\Big(\frac{\partial}{\partial y_1}\partial\tilde{\psi}\Big)^T D_{ep}(\partial\tilde{u})\partial\tilde{u}\,dy$$
$$=-\int_{D^+}(\partial\tilde{\psi})^T D_{ep}(\partial\tilde{u})\Big[\partial\Big(\frac{\partial\tilde{u}}{\partial y_1}\Big)+\xi''(y_1)\tilde{\partial}_y\tilde{u}\Big]dy, \quad \forall\tilde{\psi}\in C_0^\infty(D, \mathbb{R}^2). \tag{4.6}$$

Combining (4.5) and (4.6), we get

$$\int_{D^+}(\partial\tilde{\psi})^T D_{ep}\partial\Big(\frac{\partial\tilde{u}}{\partial y_1}\Big)dy+\int_{D^+}(\partial\tilde{\psi})^T D_{ep}\xi''(y_1)\tilde{\partial}_y\tilde{u}\,dy$$
$$+\int_{D^+}\xi''(y_1)(\tilde{\partial}_y\tilde{\psi})^T D_{ep}\partial\tilde{u}\,dy+\int_{D^+}\Big(\frac{\partial\tilde{\psi}}{\partial y_1}\Big)^T\tilde{F}dy=0, \quad \forall\tilde{\psi}\in H_0^{1,2}(D, \mathbb{R}^2). \tag{4.7}$$

For any $y^0\in Q$ and $R<\dfrac{1}{4}\operatorname{dist}(y^0, \partial Q)$ we have three possibilities:

1. $Q_{3R}(y^0)\cap Q^+=\varnothing$,
2. $Q_{3R}(y^0)\cap Q^-=\varnothing$,
3. $Q_{3R}(y^0)\cap Q^+\neq\varnothing$, $Q_{3R}(y^0)\cap Q^-\neq\varnothing$.

In case 2, as we have seen, we have

$$\int_{Q_R(y^0)} \left|D\left(\frac{\partial \tilde{u}}{\partial y_1}\right)\right|^2 dy \leqslant C\left(\int_{Q_{2R}(y^0)} \left|D\left(\frac{\partial \tilde{u}}{\partial y_1}\right)\right|^{\frac{2n}{n+2}} dy\right)^{\frac{n+2}{n}} + C\int_{Q_{2R}(y^0)} |\tilde{F}|^2 dy. \quad (4.8)$$

In case 3, inserting $\tilde{\psi}=\eta^2 \frac{\partial \tilde{u}}{\partial y_1} \in H_0^{1,2}(D, \mathbb{R}^2)$ into (4.7), where η is a cut-off function

$$\eta(y) \in C_0^\infty(Q_{2R}(y^0)),\ 0\leqslant\eta\leqslant 1,\ \eta\equiv 1 \text{ in } Q_R(y^0),\ |D\eta| \leqslant \frac{C}{R}, \quad (4.9)$$

we have

$$\int_{D^+} \left[\partial\left(\eta^2 \frac{\partial \tilde{u}}{\partial y_1}\right)\right]^T D_{ep} \partial\left(\frac{\partial \tilde{u}}{\partial y_1}\right) dy + \int_{D^+} \left[\partial\left(\eta^2 \frac{\partial \tilde{u}}{\partial y_1}\right)\right]^T D_{ep} \xi''(y_1) \tilde{\partial}_y \tilde{u}\, dy$$
$$+ \int_{D^+} \xi''(y_1) \left[\tilde{\partial}_y\left(\eta^2 \frac{\partial \tilde{u}}{\partial y_1}\right)\right]^T D_{ep} \partial \tilde{u}\, dy + \int_{D^+} \left[\frac{\partial}{\partial y_1}\left(\eta^2 \frac{\partial \tilde{u}}{\partial y_1}\right)\right]^T \tilde{F}\, dy = 0.$$

By some computations, we get

$$\int_{Q_{2R}^+(y^0)} \left[\partial\left(\eta \frac{\partial \tilde{u}}{\partial y_1}\right)\right]^T D_{ep} \left[\partial\left(\eta \frac{\partial \tilde{u}}{\partial y_1}\right)\right] dy$$
$$= -\int_{Q_{2R}^+(y^0)} \left(\frac{\partial \tilde{u}}{\partial y_1}\right)^T \partial^T(\eta I) D_{ep} \partial\left(\eta \frac{\partial \tilde{u}}{\partial y_1}\right) dy$$
$$+ \int_{Q_{2R}^+(y^0)} \left[\partial\left(\eta \frac{\partial \tilde{u}}{\partial y_1}\right)\right]^T D_{ep} [\partial^T(\eta I)]^T \frac{\partial \tilde{u}}{\partial y_1}\, dy$$
$$+ \int_{Q_{2R}^+(y^0)} \left(\frac{\partial \tilde{u}}{\partial y_1}\right)^T \partial^T(\eta I) D_{ep} [\partial^T(\eta I)]^T \frac{\partial \tilde{u}}{\partial y_1}\, dy$$
$$- \int_{Q_{2R}^+(y^0)} \eta \left[\partial\left(\eta \frac{\partial \tilde{u}}{\partial y_1}\right)\right]^T D_{ep} \xi''(y_1) \tilde{\partial}_y \tilde{u}\, dy$$
$$- \int_{Q_{2R}^+(y^0)} \eta \left(\frac{\partial \tilde{u}}{\partial y_1}\right)^T \partial^T(\eta I) D_{ep} \xi''(y_1) \tilde{\partial}_y \tilde{u}\, dy$$
$$- \int_{Q_{2R}^+(y^0)} \eta \xi''(y_1) \left[\tilde{\partial}_y\left(\eta \frac{\partial \tilde{u}}{\partial y_1}\right)\right]^T D_{ep} \partial \tilde{u}\, dy$$
$$- \int_{Q_{2R}^+(y^0)} \eta \xi''(y_1) \left(\frac{\partial \tilde{u}}{\partial y_1}\right)^T \tilde{\partial}_y^T(\eta I) D_{ep} \partial \tilde{u}\, dy$$
$$- \int_{Q_{2R}^+(y^0)} \eta \left[\frac{\partial}{\partial y_1}\left(\eta \frac{\partial \tilde{u}}{\partial y_1}\right)\right]^T \tilde{F}\, dy - \int_{Q_{2R}^+(y^0)} \eta \frac{\partial \eta}{\partial y_1} \left(\frac{\partial \tilde{u}}{\partial y_1}\right)^T \tilde{F}\, dy.$$

Using Korn's inequality, we have

$$\int_{Q_R^+(y^0)} \left|D\left(\eta \frac{\partial \tilde{u}}{\partial y_1}\right)\right|^2 dy \leqslant C \int_{Q_{2R}^+(y^0)} \left(|\partial^T(\eta I)|^2 + |\tilde{\partial}_u^T(\eta I)|^2 + \left|\frac{\partial \eta}{\partial y_1}\right|^2\right) \left|\frac{\partial \tilde{u}}{\partial y_1}\right|^2 dy$$
$$+ C \int_{Q_{2R}^+(y^0)} (|\partial \tilde{u}|^2 + |\tilde{\partial}_y \tilde{u}|^2 + |\tilde{F}|^2) dy, \quad (4.10)$$

where C depends only on λ, D_e, A, h_0, $\sup|\xi''(y_1)|$.

From (4.10) and (4.9), we get

$$\int_{Q_R^+(y^0)} \left|D\left(\frac{\partial \tilde{u}}{\partial y_1}\right)\right|^2 dy \leqslant \frac{C}{R^2} \int_{Q_{2R}^+(y^0)} \left|\frac{\partial \tilde{u}}{\partial y_1}\right|^2 dy + C \int_{Q_{2R}^+(y^0)} (|\partial \tilde{u}|^2 + |\tilde{\partial}_y \tilde{u}|^2 + |\tilde{F}|^2) dy.$$

Since $u=0$ on $\partial\Omega$, we have $\frac{\partial \tilde{u}}{\partial y_1}=0$ on $\partial Q_{2R}^+(y^0) \cap \{y_2=0\}$. Using Sobolev–Poincaré inequality and then dividing it by R^n, we obtain

$$\fint_{Q_R^+(y^0)}\left|D\left(\frac{\partial\tilde u}{\partial y_1}\right)\right|^2dy\leqslant C\left(\fint_{Q_{2R}^+(y^0)}\left|D\left(\frac{\partial\tilde u}{\partial y_1}\right)\right|^{\frac{2n}{n+2}}dy\right)^{\frac{n+2}{n}}$$
$$+C\fint_{(Q_{2R}^+(y^0)}(|\partial\tilde u|^2+|\tilde\partial_y\tilde u|^2+|\tilde F|^2)dy.$$

Now, let

$$g=\begin{cases}\left|D\left(\frac{\partial\tilde u}{\partial y_1}\right)\right|^{\frac{2n}{n+2}} & \text{in } Q^+,\\ 0 & \text{in } Q^-,\end{cases}$$
$$f=\begin{cases}(|\partial\tilde u|^2+|\tilde\partial_y\tilde u|^2+|\tilde F|^2)^{\frac{n}{n+2}} & \text{in } Q^+,\\ 0 & \text{in } Q^-,\end{cases}$$
$$q=\frac{n+2}{n}.$$

It is easy to verify that $g\in L^q(Q)$, $q>1$, $f\in L^{r^*}(Q)$, $q<r^*\leqslant\frac{qs}{2}$, and

$$\fint_{Q_R(y^0)}g^q\,dy\leqslant C\left(\fint_{Q_{2R}(y^0)}g\,dy\right)^q+C\fint_{Q_{2R}(y^0)}f^q\,dy. \tag{4.11}$$

Therefore, in each case, the inequality (4.11) holds.

Using reverse Hölder inequality again, we have

$$g\in L^p_{loc}(Q)\quad\text{for } p\in[q,\ q+\varepsilon)$$

and

$$\left(\fint_{Q_R}g^p\,dy\right)^{\frac1p}\leqslant C\left\{\left(\fint_{Q_{2R}}g^q\,dy\right)^{\frac1q}+\left(\fint_{Q_{2R}}f^p\,dy\right)^{\frac1p}\right\}$$

for $Q_{2R}\subset Q$, where C and ε are positive constants depending only on n, λ, D_e, A, h_0, $\sup|\xi''|$, s, which implies

$$\left|D\left(\frac{\partial\tilde u}{\partial y_1}\right)\right|^{\frac{2n}{n+2}}\in L^p(Q_R^+)\quad\text{for}\quad p\in[q,\ q+\varepsilon),\ Q_{2R}\subset Q.$$

Setting $r=\frac{2n}{n+2}p$, we get $\left|D\left(\frac{\partial\tilde u}{\partial y_1}\right)\right|\in L^r(Q_R^+)$ for $r\in\left[2,\ 2+\frac{2n}{n+2}\varepsilon\right)$. Since $D\left(\frac{\partial\tilde u}{\partial y_1}\right)=\left\{\frac{\partial}{\partial x_\alpha}\left(\frac{\partial\tilde u_i}{\partial y_1}\right)\right\}_{\alpha=1,2;\,i=1,2}$ and $\frac{\partial}{\partial x_1}=\frac{\partial}{\partial y_1}+\xi'(y_1)\frac{\partial}{\partial y_2}$, $\frac{\partial}{\partial x_2}=\frac{\partial}{\partial y_2}$, we have

$$\left|\frac{\partial^2\tilde u}{\partial y_2\partial y_1}\right|,\quad\left|\frac{\partial^2\tilde u}{\partial y_1^2}\right|\in L^r(Q_R^+).$$

In order to show that $u\in H^{2,r}(\Omega,\mathbb{R}^2)$, it remains to prove that $\left|\frac{\partial^2\tilde u}{\partial y_2^2}\right|\in L^r(Q_R^+)$. To do this, we will use system (1.1).

It is not difficult to check that under transformation (4.1), system (1.1) can be written as follows:

$$\begin{bmatrix}\xi'(y_1) & 0\\ 0 & 1\\ 1 & \xi'(y_1)\end{bmatrix}^T D_{ep}\begin{bmatrix}\xi'(y_1) & 0\\ 0 & 1\\ 1 & \xi'(y_1)\end{bmatrix}\frac{\partial^2\tilde u}{\partial y_2^2}+\cdots=\tilde F\ \text{a.e. in } D^+,$$

where the notation "…" is used to express all of the terms which belong to $L^r(Q_R^+)$.

By elliptic condition (1.7) we have

$$\det\left(\begin{bmatrix} \xi'(y_1) & 0 \\ 0 & 1 \\ 1 & \xi'(y_1) \end{bmatrix}^T D_{ep} \begin{bmatrix} \xi'(y_1) & 0 \\ 0 & 1 \\ 1 & \xi'(y_1) \end{bmatrix}\right) \neq 0.$$

Therefore, we get

$$\frac{\partial^2 \tilde{u}}{\partial y_2^2} \in L^r(Q_R^+).$$

Using the standard technique we deduce that $|D^2u| \in L^r$ near $\partial\Omega$.

From this conclusion and the regularity in the interior, we have

Theorem 4.1. *If* $\Omega \in C^2$, $f \in L^s(\Omega, \mathbb{R}^2)$, $s>2$, *then the weak solution* u *of the problem* (1.1), (1.2) *belongs to* $H^{2,r}(\Omega, \mathbb{R}^2)$ *for some* $r>2$.

A consequence of this theorem is the following theorem.

Theorem 4.2. *Under the assumptions of Theorem* 4.1, *the weak solution* u *of the problem* (1.1), (1.2) *belongs to* $C^{1,\delta}(\overline{\Omega}, \mathbb{R}^2)$, *where* $0<\delta\leqslant 1-\frac{2}{r}$, $r>2$ *is the exponent in Theorem* 4.1.

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References

[1] Havner, K. S., Patel, H. P., On convergence of the finite-element method for a class of elastic-plastic solids, *Quarterly of Appl. Math.*, **34**: 1 (1976), 59—68.

[2] Jiang, L. S., Wu, L. C., Wang, Y. D. & Ye, Q. X., On the existence, uniqueness of a class of elastic-plastic problem and the convergence of the approximate solutions, *Acta Mathematicae Applicatae Sinica*, **4**: 2 (1981), 166—174.

[3] Jiang, L. S. & Wu, L. C., A class of nonlinear elliptic systems with discontinuous coefficients, *Acta Mathematicae Sinica*, **26**: 6 (1983), 660—668.

[4] Jiang, L. S., On an elastic-plastic problem, *Journal of Differential Equations*, **51**: 1 (1984), 97—115.

[5] Kinderlehrer, D. & Stampacchia, G., An introduction to variational inequalities and their applications, Academic Press, 1980.

[6] Giaquinta, M. & Modica, G., Regularity results for some classes of higher order nonlinear elliptic systems, *Journal für die reine und angewandte Mathematik*, 311/312, (1979), 145—169.

[7] Giaquinta, M., Multiple integrals in the calculus of variations and nonlinear elliptic systems, Princeton University Press, 1983.

[8] Fichera, G., Existence theorem in elasticity, in "Handbuch der Physik", Band VI a/2, 347—389.