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# UNIFORM BOUNDEDNESS AND UNIFORM ULTIMATE BOUNDEDNESS OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH UNBOUNDED DELAY*

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**Abstract**

In this paper, the author uses the Liapunov direct method, comparison method and "forgetful" functional to prove four general theorems about the uniform boundedness of the solutions of functional differential equations with infinite delay. Also, these results are effectively applied to Volterra integral-differential equations.

## § 1. Introduction

Boundedness of solutions of differential equations is a subject that has been paid close attention by many scholars. This is not only because it is important in stability theory, but also because many other nice properties of solutions can be obtained with the aid of it. For example, for ordinary differential equations, uniform boundedness and uniform ultimate boundedness imply the existence of periodic solutions (e.g. T. Yoshizawa's theorem on existence of periodic solutions). Therefore, how to realize the conditions for boundedness is a problem to be solved.

Also, many authors have tried to determine the uniform boundedness and uniform ultimate boundedness of solutions of functional differential equations. For functional differential equations with bounded delay, this problem was solved basically by T. Yoshizawa. However, there are very few results for functional differential equations with unbounded delay. Although some good results have been obtained recently in [1, 2, 3], the problem is still far from solved.

T. A. Burton has been interested in this subject and proposed an important open problem about it. It is as follows.

Consider the functional differential equations

$$x'(t)=F(t,\ x(s):\alpha\leqslant s\leqslant t),\ \alpha\geqslant-\infty, \tag{1}$$

where $F$ is a continuous functional taking value in $R^n$ whenever $t\in[\alpha,\ \infty)$ and $x$ is a continuous function of $C([\alpha,\ \infty)\to R^n)$. If $\alpha=-\infty$, then it should be understood

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that $x:(-\infty, \infty)\to R^n$.

Suppose that there is a Liapunov functional $V(t, x(\cdot))$ and wedges $W_i(r)$ $(i=1, 2, 3, 4)$ and $K>0$ such that

(i) $W_1(|x(t)|)\leqslant V(t, x(\cdot))\leqslant W_2(|x(t)|)+W_3(\|x\|^{[\alpha, t]})$ for all $t\geqslant\alpha$ and $x(s) \in C([\alpha, t]\to R^n)$;

(ii) $V'_{(1)}(t, x(\cdot))\leqslant -W_4(|x(t)|)+K$ for all $t\geqslant t_0$ and a solution $x(t)$ of (1).

The problem is if solutions of (1) are uniform bounded and uniform ultimate bounded.

This paper gives some partial results.

At first, we improve the definition for the so-called "forgetful functional" in [1], and generalize the result on uniform boundedness in [3]. Then, a condition is refined and used to obtain a numbers of results on uniform ultimate boundedness.

We study the subject in two ways. One set of theorems is obtained by using a method combining Liapunov functional with comparison method. The other set of theorems is obtained by using "forgetful" Liapunov functional.

Consider the ordinary differential equation

$$y'(t)=G(t, y(t)), \tag{2}$$

where $G:[\alpha, \infty)\times R\to R$ is continuous.

Theorems 1 and 2 in this paper establish some relations between solutions of (1) and (2) with the aid of $V$ functional. The relations can be summarized as follows. If there exists a Liapunov functional $V(t, x(\cdot))$ with $V'_{(1)}(t, x(\cdot))\leqslant G(t, V)$, then solutions of (1) have the same properties (e.g. boundedness, stability and so on) as the solutions of (2) under some restrictions on $V(t, x(\cdot))$. So, the study on FDE can be transferred to the study on ODE which we know very well.

## § 2. Notations and Definitions

Some concrete equations of (1) are the Volterra integrodifferential equations

$$x'(t)=Ax(t)+\int_\alpha^t C(t, s)x(s)\,ds+f(t), \tag{3}$$

where $\alpha=0$ or $\alpha=-\infty$, $A$ is an $n\times n$ constant matrix and $x\in R^n$.

Theorems of existence and uniqueness for solutions of (1) can be found in [4].

We use the following notations.

(a) If $x\in R^n$, then $|x|=\sum_{i=1}^{n}|x_i|$.

(b) If $x$: $[a, b]\to R^n$, then $\|x\|^{[a, b]}=\sup_{a\leqslant t\leqslant b}|x(t)|$.

(c) $V(t, x(\cdot))\in L$. Lip $x$ means functional $V(t, x(\cdot))$ is locally Lipschitz in $x$.

**Definition 1.** *Function $W(r)$: $[0, \infty)\to[0, \infty)$ is called a wedge if $W(r)$ is* *continuous and strictly increasing with* $W(0)=0$ *and* $W(r)\to\infty$ *as* $r\to\infty$.

**Definition 2.** $V(t, x(\cdot))$ *is called a Liapunov functional if* $V(t, x(\cdot))$ *is continuous in* $t$ *and* $V\in L$. Lip $x$.

**Definition 3.** *Let* $W(r)$ *be a wedge and* $l>0$. $V(t, x(\cdot))$ *is said to be forgetful if* $0\leqslant V(t, x(\cdot))\leqslant W(\|x\|^{[\alpha,t]})$, *and for any* $D>0$, $\sigma>0$ *and* $t_0\geqslant\alpha$ *there exists an* $S>0$ such that $[\|x\|^{[\alpha,t_0]}\leqslant D$; $|x(t^*)|\geqslant\sigma$ *for* $t_0\leqslant t^*\leqslant t$; $t\geqslant t_0+S]$ *implies* $V(t, x(\cdot))\leqslant lW(\|x\|^{[t_0,t]})$.

$V(t, x(\cdot))$ is uniform forgetful, if $S$ is independent of $t_0$.

*Example* 1. Let $\Phi(t, s)=\int_t^\infty |C(u, s)|du$ for all $t\geqslant\alpha$,

and
$$V(t, x(\cdot))=(1/2k)|x|+\int_\alpha^t \Phi(t, s)|x(s)|ds.$$

Then $V(t, x(\cdot))$ is uniform forgetful, if

$$\int_\alpha^\infty \Phi(t, s)ds\leqslant M \text{ for some } M>0 \tag{4}$$

and for each $\rho>0$ there exists $s>0$ such that $[t_0\geqslant\alpha,\ t\geqslant t_0+S]$ implies

$$\int_\alpha^{t_0}\Phi(t, s)ds\leqslant\rho. \tag{5}$$

In fact, by (4), we have

$$0\leqslant V(t, x(\cdot))\leqslant(1/2k+M)\|x\|^{[\alpha,t]}\overset{\text{def.}}{=\!=}W(\|x\|^{[\alpha,t]}).$$

By (5), for each $\sigma/D>0$ there exists an $S>0$ such that $[t_0\geqslant\alpha$; $\|x\|^{[\alpha,t_0]}\leqslant D$; $|x(t^*)|\geqslant\sigma$ for $t_0\leqslant t^*\leqslant t$; $t\geqslant t_0+S]$ implies

$$\int_\alpha^{t_0}\Phi(t, s)ds\leqslant\sigma/D.$$

Now

$$\int_\alpha^t \Phi(t, s)|x(s)|ds=\int_\alpha^{t_0}\Phi(t, s)|x(s)|ds+\int_{t_0}^t \Phi(t, s)|x(s)|ds$$
$$\leqslant|x(t^*)|+M\|x\|^{[t_0,t]}\leqslant(1+M)\|x\|^{[t_0,t]}.$$

So

$$V(t, x(\cdot))\leqslant\frac{\max\{1/2k,(1+M)\}}{1/2k+M}W(\|x\|^{[t_0,t]})\overset{\text{def.}}{=\!=}lW(\|x\|^{[t_0,t]}).$$

Thus $V(t, x(\cdot))$ is uniform forgetful.

## § 3. Conclusions

**A section.**

Consider equation (2) and denote by $u(t, s, \alpha)$ and $r(t, s, \alpha)$ the maximal solutions of (2) through $(s, \alpha)$ to the right and to the left respectively.

**Lemma** (Junji Kato[5]). *Suppose that* $D^+v(t)\leqslant G(t, v(t))$ *if* $v(s)\leqslant r(s, t, v(t))$ *for all* $s\in[t-\tau, t]$, *where* $\tau\geqslant0$. *Then we have*

$$v(t)\leqslant u(t,\ s,\ \alpha)\quad\text{for all } t\geqslant\alpha$$

*whenever* $v(s)\leqslant r(s,\ \alpha,\ \alpha)$ *for* $s\in[\alpha-\tau,\ \alpha]$.

**Theorem 1.** *Let $V$ be a Liapunov functional and suppose there are wedges $W(r)$, $W_1(r)$ such that*

(i) $W(|x(t)|)\leqslant V(t,\ x(\cdot))\leqslant W_1(|x\|^{[\alpha,t]})$ *for all* $t\geqslant\alpha$ *and* $x(s)\in C([\alpha,\ t]\to R^n)$;

(ii) $V'_{(1)}(t,\ x(\cdot))\leqslant G(t,\ V)$ *for all* $t\geqslant t_0$ *and any solution* $x(t)$ *of* (1), *while the solutions of* (2) *are uniform bounded (uniform ultimate bounded).*

*Then the solutions of* (1) *are uniform bounded (uniform ultimate bounded).*

The proof is almost the same as the one of Theorem 2, thus omitted.

**Theorem 2.** *Suppose the condition* (ii) *of Theorem 1 is satisfied, and there are wedges* $W(r)$, $W_1(r)$ $(W(r),\ W_i(r)(i=1,\ 2))$ *and a positive constant* $U$ *such that*

(i) $0\leqslant V(t,\ x(\cdot))\leqslant W_1(\|x\|^{[\alpha,t]})$ *for all* $t\geqslant\alpha$ *and* $x(s)\in C([\alpha,\ t]\to R^n)$;

(ii)* $V'_{(1)}(t,\ x(\cdot))\leqslant -W'_{(1)}(|x(t)|)\,(V'_{(1)}(t,\ x(\cdot))\leqslant -|W'_{(1)}(|x(t)|)|-W_2(|x(t)|))$ *for all* $t\geqslant t_0$ *and* $|x(t)|\geqslant U$.

*Then the solutions of* (1) *are uniform bounded (uniform ultimate bounded).*

*Proof* We only prove the case for uniform ultimate boundedness.

Let $x(t,\ t_0,\ \varphi)$ be a solution of (1) satisfying $\|\varphi\|^{[\alpha,t_0]}\leqslant H$ for given $H>0$ and $t_0\geqslant\alpha$, and $y(t,\ t_0,\ y_0)$ be the maximal solution of (2) to the right with $y_0=V(t_0,\ \varphi)$.

By (i), we have $y_0=V(t_0,\ \varphi)\leqslant W_1(\|\varphi\|^{[\alpha,t_0]})\leqslant W_1(H)$.

By (ii), there exists a $B_1>0$ such that for $W_1(H)>0$ there exists $T=T(H)>0$ such that $y_0\leqslant W_1(H)$, $t\geqslant t_0+T$ imply $|y(t,\ t_0,\ y_0)|\leqslant B_1$. From Lemma, we have

$$V(t,\ x(\cdot))\leqslant y(t,\ t_0,\ y_0)\leqslant B_1\ \text{for all } t\geqslant t_0+T.$$

As $V'_{(1)}(t,\ x(\cdot))\leqslant -W_2(|x(t)|)$ if $|x(t)|\geqslant U$, we see that if $t_0+T\leqslant t<\infty$, then there is a positive $P$ such that $|x(t)|$ can remain larger than $U$ on an interval of length at most $P$. Hence, there exists the first $t_1\geqslant t_0+T$ with $|x(t_1)|=U$. We may assume $t_1>t_0+T$.

Now for $x(t,\ t_0,\ \varphi)$ either

(A) $|x(t)|<U$ for $t_0+T\leqslant t<t_1$, or

(B) $|x(t)|>U$ for $t_0+T\leqslant t<t_1$.

If (B) holds, then by (ii)* we have

$$V(t_1)-V(t)\leqslant -W(|x(t)|)+W(|x(t_1)|),$$

$$|x(t)|\leqslant W^{-1}(W(U)+B_1)\overset{\text{def.}}{=\!=}B>U\ \text{for } t_0+T\leqslant t<t_1,$$

and also

$$|x(t)|\leqslant B\ \text{for all } t\geqslant t_1.$$

In fact, if $|x(t^*)|>U$, then there must exist $t_2$ with $t_1\leqslant t_2<t^*$, $|x(t_2)|=U$ and $|x(t)|\geqslant U$ on $[t_2,\ t^*]$, and $|x(t^*)|\leqslant B$ by using (ii)*.

Thus

$$|x(t)|\leqslant B\ \text{for all } t\geqslant t_0+T.$$

This is uniform ultimate boundedness and the proof is complete.

**Corollary 1.** *Let $V$ be a Liapunov functional and $\beta$, $\eta:[\alpha, \infty)\to R^+$ are continuous, and suppose there are wedges $W(r)$, $W_1(r)$ such that*

(i) $W(|x(t)|)\leqslant V(t, x(\cdot))\leqslant W_1(\|x\|^{[\alpha,t]})$ *for all* $t\geqslant\alpha$ *and* $x(s)\in C([\alpha, t]\to R^n)$;

(ii) $V'_{(1)}(t, x(\cdot))\leqslant -\eta(t)V^\nu(t, x(\cdot))+\beta(t)$ *for all* $t\geqslant t_0$ *and a solution* $x(t)$ *of* (1), *where* $\beta(t)\leqslant\beta$ *for some* $\beta>0$, $\eta(t)\geqslant\eta$ *for some* $\eta>0$ *and* $0<\nu\leqslant 1$.

*Then all solutions of* (1) *are uniform bounded and uniform ultimate bounded.*

**Corollary 2.** *Suppose the condition* (ii) *of Corollary* 1 *is satisfied and there are wedges $W(r)$, $W_1(r)$, $W_2(r)$ and a positive $U$ such that*

(i) $0\leqslant V(t, x(\cdot))\leqslant W_1(\|x\|^{[\alpha,t]})$ *for all* $t\geqslant\alpha$ *and* $x(s)\in C([\alpha, t]\to R^n)$;

(ii) $V'_{(1)}(t, x(\cdot))\leqslant -W_2(|x(t)|)-|W'_{(1)}(|x(t)|)|$ *if* $|x(t)|\geqslant U$. *Then the solutions of* (1) *are uniform bounded and uniform ultimate bounded.*

**Remark.** Some results of [2] [Chapter 7, Section 2] are special cases of Corollaries 1 and 2.

*Example* 2 Consider the integrodifferential equation (3).

If $A$ is stable, then there is a unique positive definite matrix $B$ with $A^TB+BA=-I$.

There are also positive constants $K$, $k$ and $h$ with

$$h|x|\leqslant[x^TBx]^{1/2}\leqslant(1/2k)|x|,\ |Bx|\leqslant K[x^TBx]^{1/2}. \tag{7}$$

Furthermore, if there is a $\overline{K}>K$ and $\overline{k}>0$ with

$$0<\overline{k}\leqslant k-\overline{K}\int_t^\infty|C(u, t)|du, \tag{8}$$

then we can define

$$V(t, x(\cdot))=[x^TBx]^{1/2}+\overline{K}\int_\alpha^t\int_t^\infty|C(u, s)||x(s)|du\,ds.$$

We require that (4) hold, and

$$|f(t)|\leqslant m \text{ for some } m>0 \text{ and } t\geqslant\alpha, \tag{9}$$

and there be a continuous function $\lambda(t)$ with $\lambda\leqslant\lambda(t)\leqslant 1$ for some $\lambda>0$ with

$$|C(t, s)|\geqslant\lambda(t)\int_t^\infty|C(u, s)|du. \tag{10}$$

Then

$$h|x|\leqslant V(t, x(\cdot))\leqslant(1/2k+M\overline{K})\|x\|^{[\alpha,t]}\overset{\text{def.}}{=\!=}W_1(\|x\|^{[\alpha,t]}).$$

Taking the derivative of $V$ along a solution of (3) we obtain

$$V'_{(3)}(t, x(\cdot))\leqslant-\overline{k}|x|-(\overline{K}-K)\int_\alpha^t|C(t, s)||x(s)|ds+K|f(t)|$$

$$\leqslant-(\overline{k}/2)|x|-h_1|\,|x|'| \quad \text{for} \quad |x(t)|\geqslant\frac{4Km}{\overline{k}}\overset{\text{def.}}{=\!=}U,$$

where $h_1=\min\{\overline{k}/2, (\overline{K}-K)\}$, and also

$$V'_{(3)}(t, x(\cdot)) \leqslant -2k\bar{k}\lambda(t)[x^T Bx]^{1/2} - \frac{\bar{K}-K}{\bar{K}}\lambda(t)\int_a^t\int_t^\infty \bar{K}|C(u, s)||x(s)|du\,ds$$
$$+K|f(t)|$$
$$= -\eta(t)V(t, x(\cdot)) + K|f(t)|,$$

where $\eta(t) = \lambda(t)\min\{2k\bar{k}, (\bar{K}-K)/\bar{K}\} \geqslant \lambda \min\{2k\bar{k}, (\bar{K}-K)/\bar{K}\} \overset{\text{def.}}{=\!=} \eta > 0$.

We define $W(r) = hr$, $W_1(r) = (1/2k + \bar{K}M)r$, $W_2(r) = (\bar{k}/2)r$, $W_3(r) = h_1 r$, $\beta(t) = K|f(t)| \leqslant Km \overset{\text{der.}}{=\!=} \beta$.

The conditions of Corollaries 1 and 2 are satisfied and we have the following proposition.

**Proposition 1.** *If $A$ is stable and* (4), (8), (9) *and* (10) *are satisfied, then all solutions of* (3) *are uniform bounded and uniform ultimate bounded.*

**B section.**

**Theorem 3.** *Let $V$ be a Liapunov functional and suppose there are wedges $W(r)$, $W_1(r)$ and a positive constant $U$ such that*

(i) $0 \leqslant V(t, x(\cdot)) \leqslant W_1(\|x\|^{[\alpha, t]})$ *for all* $t \geqslant \alpha$ *and* $x(s) \in C([\alpha, t] \to R^n)$;

(ii) $V'_{(1)}(t, x(\cdot)) \leqslant -W'_{(1)}(|x(t)|)$ *for all* $t \geqslant t_0$ *and* $|x(t)| \geqslant U$;

(iii) $\lim\limits_{r\to\infty}(W(r) - W_1(r)) = +\infty$.

*Then all solutions of* (1) *are uniform bounded.*

*Proof* Let $x(t, t_0, \varphi)$ be a solution satisfying $\|\varphi\|^{[\alpha, t_0]} \leqslant H$ for given $H > 0$ and $t_0 \geqslant \alpha$.

Let $\gamma = \max\{H, U\}$.

By (iii), there exists an $M \geqslant \gamma$ such that $r \geqslant M$ implies

$$W(r) - W_1(r) > W(\gamma). \tag{*}$$

Now for $x(t, t_0, \varphi)$ either

(A) $|x(t)| < M$ for all $t \geqslant t_0$, or

(B) there exists a first $t_1 \geqslant t_0$ with $|x(t_1)| = M$.

If (B) holds, then either

(A′) $|x(t)| > \gamma$ for all $t \geqslant t_1$, or

(B′) there exists the first $t_2 > t_1$ with $|x(t_2)| = \gamma$.

If (A′) holds, then $V'_{(1)}(t, x(\cdot)) \leqslant -W'_{(1)}(|x(t)|)$ so that

$$V(t) - V(t_1) \leqslant -W(|x(t)|) + W(|x(t_1)|) \quad \text{for } t > t_1.$$

Hence

$$|x(t)| \leqslant W^{-1}(W(M) + W_1(M)) \overset{\text{def.}}{=\!=} D.$$

If (B′) holds, let
$$|x(\bar{t}_1)| = \max_{t_1 \leqslant t \leqslant t_2}|x(t)| \geqslant M,$$
then
$$|x(t)| \leqslant |x(\bar{t}_1)| \quad \text{for all } t \geqslant t_2.$$

In fact, if it is false, then there is the first interval $[t_3, t_4]$ with $|x(t_3)| = \gamma$, $|x(t_4)| = |x(\bar{t}_1)|$, $|x(t)| \geqslant \gamma$ for $t \in [t_3, t_4]$.

By (ii), we have

$$V(t_4)-V(t_3)\leqslant -W(|x(t_4)|)+W(|x(t_3)|),$$

so that we have $0\leqslant V((t_4)\leqslant -W(|x(\bar{t}_1)|)+W_1(|x(\bar{t}_1)|)+W(\gamma)$, a **contradiction** to (*).

Now, we consider the interval $[t_1, t_2]$. By (ii), we have

$$W(|x(\bar{t}_1)|)\leqslant W(|x(t_1)|)+V(t_1)\leqslant W(M)+W_1(M)$$

and so

$$|x(\bar{t}_1)|\leqslant W^{-1}(W(M)+W_1(M))\overset{\text{def.}}{=\!=} D.$$

Thus, in all cases, $|x(t)|\leqslant D$ for all $t\geqslant t_0$.

This is uniform boundedness and the proof is complete.

**Remark.** Let $\bar{V}(t, x(\cdot))=V(t, x(\cdot))+W(|x(t)|)$and $\bar{W}(r)=W_1(r)+W(r)$. Then Theorem 3 can be stated as follows.

Suppose there is a Liapunov functional $\bar{V}(t, x(\cdot))$ and wedges $W(r)$, $\bar{W}(r)$, $\bar{\bar{W}}(r)$ with $W(|x|)\leqslant \bar{V}(t, x(\cdot))\leqslant \bar{W}(\|x\|^{[\alpha,t]})$ and $\lim\limits_{r\to\infty}(\bar{\bar{W}}(r)-\bar{W}(r))=+\infty$. If there exists a $U>0$ with $\bar{V}'_{(1)}(t, x(\cdot))\leqslant 0$ for $|x(t)|\geqslant U$, then the solutions of (1) are uniform bounded.

This shows that the counterpart of Theorem 0 in [1] for (1) can be obtained under a restriction on $b(r)$ (cf. Burton [1]).

*Example* 3 Consider again equation (3) in Example 2 with (4), (7), (8) and (9) as before.

Let $V(t, x(\cdot))$ be as in Example 2. Then

$$0\leqslant V(t, x(\cdot))\leqslant(1/2k+M\bar{K})\|x\|^{[\alpha,t]}\overset{\text{def.}}{=\!=}W_1(\|x\|^{[\alpha,t]}),$$

and

$$V'_{(3)}(t, x(\cdot))\leqslant -\bar{k}|x|-(\bar{K}-K)\int_\alpha^t|C(t, s)||x(s)|ds+K|f(t)|$$

$$\leqslant -\mu||x|'|\quad\text{for}\quad |x(t)|\geqslant\frac{2Km}{\bar{k}}\overset{\text{def.}}{=\!=}U,$$

where $\mu=\min\{\bar{k}/2, (\bar{K}-K)\}$.

We define

$$W(r)=\mu r.$$

In summary then, we need $\lim\limits_{r\to\infty}(W(r)-W_1(r))=+\infty$ or

$$\mu>1/2k+\bar{K}M. \tag{11}$$

By Theorem 3 this will yield the following proposition.

**Proposition 2.** *Let constants be defined by* (4), (7), (8) *and* (9). *If $A$ is stable and* (11) *is satisfied, then all solutions of* (3) *are uniform bounded.*

**Theorem 4.** *Let $V$ be a Liapunov functional and suppose there are wedges $W(r)$, $W_1(r)$, $W_2(r)$ and a positive $U$, and a continuous function $\beta$: $[\alpha, \infty)\to[0, \infty)$ such that*

(i) *$V$ is uniform forgetful;*

(ii) $V'_{(1)}(t, x(\cdot)) \leqslant -W_2(|x(t)|) - |W'_{(1)}(|x(t)|)|$ *for all* $t \geqslant t_0$ *and* $|x(t)| \geqslant U$;

(iii) $V'_{(1)}(t, x(\cdot)) \leqslant \beta(t)$ *for all* $t \geqslant t_0$ *and a solution* $x(t)$ *of* (1), *where* $\lim\limits_{t\to\infty} \beta(t) = 0$;

(iv) $\lim\limits_{r\to\infty}(W(r) - h(l)W_1(r)) = +\infty$, *where*

$$h(l) = \begin{cases} l, & l > 1, \\ 1, & l \leqslant 1. \end{cases}$$

*Then all solutions of* (1) *are uniform ultimate bounded.*

*Proof* Let $H>0$ be given. We must find $D>0$ such that $[t_0 \geqslant \alpha;\ \|\varphi\|^{[\alpha, t_0]} \leqslant H;\ t \geqslant \alpha]$ implies $|x(t, t_0, \varphi)| \leqslant D$.

As $V'_{(1)}(t, x(\cdot)) \leqslant -W_2(|x(t)|)$, there must be $P>0$ such that the inequality $|x(t)| \geqslant U$ must fail on any interval of length $P$.

By (i), there is an $l>0$ such that for the above $D>0$ and $2U$ there is an $S = S(D, 2U) > 0$ such that $[t_0 \geqslant \alpha;\ \|x\|^{[\alpha, t_0]} \leqslant D;\ |x(t^*)| \geqslant 2U$ for $t_0 \leqslant t^* \leqslant t;\ t \geqslant t_0 + S]$ implies $V(t, x(\cdot)) \leqslant lW_1(\|x\|^{[t_0, t]})$.

For $\rho = (W(2U) - W(U))/2S(D, 2U) > 0$, there is an $m>0$ such that $[t_0 \geqslant \alpha;\ t \geqslant t_0 + m]$ implies $\beta(t) \leqslant \rho$ as $\lim\limits_{t\to\infty} \beta(t) = 0$.

Consider the interval sequence

$$I_i = [t'_0 + (i-1)L,\ t'_0 + iL] \triangleq [t'_{i-1}, t'_i] \text{ for } i = 1, 2, \cdots,$$

where $t'_0 = t_0 + m$ and $L = P + S$.

Now for $x(t, t_0, \varphi)$, on $I_1$, either

(A) $|x(t)| \leqslant 2U$; or

(B) there is a $\tilde{t}_1 \in I_1$ with $|x(\tilde{t}_1)| > 2U$.

If (A) holds, then either

(A′) $|x(t)| < 2U$ for all $t \geqslant t'_0$, or

(A″) there is the first $t^* \geqslant t'_0$ with $|x(t^*)| = 2U$. In this case

$$V(t, x(\cdot)) \leqslant lW_1(\|x\|^{[t_0, t]}) \text{ for } t \geqslant \min\{t^*, t'_0 + S\}.$$

By (iv), there is an $R > 2U$ such that $r \geqslant R$ implies

$$W(r) - lW_1(r) > W(2U). \tag{**}$$

Now for $x(t, t_0, \varphi)$ either

($A''_1$) $|x(t)| < R$ for all $t \geqslant t'_1$, or

($A''_2$) there is the first $t^*_1 > t'_1$ with $|x(t^*_1)| = R$.

If ($A''_2$) holds, then $|x(t)| \leqslant W^{-1}(W(R) + lW_1(R)) \overset{\text{def.}}{=\!=} B$ for $t \geqslant t'_1$ (See the proof of Theorem 3).

Thus, in all cases $|x(t)| \leqslant B$ for $t \geqslant t'_1$.

If (B) holds, then for $x(t, t_0, \varphi)$ either

(B′) there at least is $[t^{(1)}, t^{(2)}] \subseteq I_1$ with $\|x\|^{[t^{(1)}, t^{(2)}]} \leqslant 2U$ and $t^{(2)} - t^{(1)} \geqslant S$, or

(B″) $t^{(2)}-t^{(1)}<S$ for each $[t^{(1)}, t^{(2)}]$ with $\|x\|^{[t^{(1)}, t^{(2)}]}\leqslant 2U$.

If (B′) holds, recalling the discussion for case $(A)$, we have $|x(t)|\leqslant B$ for all $t\geqslant t^{(1)}$, certainly for $t\geqslant t_1'$.

If (B″) holds, we may suppose that there at least exists an interval $[t^{(1)}, t^{(3)}]$ and $t^{(4)}>t^{(3)}$ with $|x(t^{(1)})|=2U$, $|x(t^{(2)})|=|x(t^{(3)})|=U$, $U\leqslant|x(t)|^{[t^{(1)}, t^{(2)}]}\leqslant 2U$ and $\|x\|^{[t^{(3)}, t^{(4)}]}<2U$, where $t^{(4)}$ is the first point greater than $t^{(3)}$ with $|x(t^{(4)})|=2U$ and $t^{(3)}$ is the biggest number with $|x(t)|=U$.

As $V'_{(1)}(t, x(\cdot))\leqslant-|W'_{(1)}(|x(t)|)|$ for $t\in[t^{(1)}, t^{(2)}]$,

$$V(t^{(2)})-V(t^{(1)})\leqslant-W(2U)+W(U)\triangleq-\delta<0,$$

so that $V$ decreases at least by $\delta$.

As $V'_{(1)}(t, x(\cdot))\leqslant\rho$ and $t^{(3)}-t^{(2)}<S$, $V(t)-V(t^{(2)})\leqslant S\rho<\delta/2$ for $t^{(2)}\leqslant t\leqslant t^{(3)}$, so that $V$ increases at most by $\delta/2$.

This shows that $V$ decreases at least by $\delta/2$ on $I_1$, and so, case (B″) happens at most $N$ times continuously for some integer $N$ as $V(t)\leqslant W_1(D)$.

Thus, if $t\geqslant t_0+m+(N+1)L$, then we will have $|x(t)|\leqslant B$ forever after. Take $T=m+(N+1)L$.

This completes the proof.

*Example* 4. Consider the system (3) with (4), (5), (7) and (8) as before.

We suppose

$$\lim_{t\to\infty}f(t)=0 \tag{12}$$

and

$$h_1>l,\ \text{where } l=\max\{1/2k,\ (1+\overline{K}M)\}. \tag{13}$$

Then it is easy to see that the conditions of Theorem 4 are satisfied and we have the following proposition.

**Proposition 3.** *If $A$ is stable and* (4), (5), (8), (12) and (13) *are satisfied, then all solutions of* (3) *are uniform ultimate bounded.*

## References


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