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NOTES ON LIE ALGEBRA $\Sigma(n, m, \underset{\sim}{r}, G)$*

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Abstract

Supplementary discussions are given to the Lie algebra $\Sigma(n, m, \underset{\sim}{r}, G)$. Minor errors in some formulas of a previous paper (see Chin. Ann. of Math., 4B(3), 1983, 329—346) are corrected.

All notations used in [2] retain their meanings in the present article.

(I) In [2] a class of Lie algebras $\Sigma=\Sigma(n, m, \underset{\sim}{r}, G)$ is constructed over a field F of characteristic $p>0$, from which three classes of simple Lie algebras, $\bar{\Sigma}$, Σ^* and $\tilde{\Sigma}$, are derived and Σ^* and $\tilde{\Sigma}$ are shown to be new if $m, n>0$. Then what about $\bar{\Sigma}$?

Proposition. *$\bar{\Sigma}$ is isomorphic to a simple Lie algebra associated with a nodal noncommutative Jordan algebra.*

Proof Let G be generated by $r_1=1, r_2, \cdots, r_m$. Then [2, (1.36)] becomes

$$\partial_0'=I-\sum_{i=1}^{n}\left(\mu_i x_{i0}\frac{\partial}{\partial x_{i0}}+\nu_i y_{i0}\frac{\partial}{\partial y_{i0}}\right)-z_1\frac{\partial}{\partial z_1}-\sum_{i=2}^{m} r_i z_i\frac{\partial}{\partial z_i}. \tag{1}$$

We have

$$\left(I-z_1\frac{\partial}{\partial z_1}\right)(z_1 f)=z_1\left(-z_1\frac{\partial f}{\partial z_1}\right),\quad f\in\mathfrak{A}. \tag{2}$$

Let $\delta_0'=-\sum_{i=1}^{n}\left(\mu_i x_{i0}\frac{\partial}{\partial x_{i0}}+\nu_i y_{i0}\frac{\partial}{\partial y_{i0}}\right)-\sum_{i=1}^{m} r_i z_i\frac{\partial}{\partial z_i}$, which is a derivation of $\mathfrak{A}$. Then, by (2),

$$\partial_0'(z_1 f)=z_1(\delta_0' f),\quad f\in\mathfrak{A}. \tag{3}$$

On the other hand

$$D_i(z_1 f)=z_1 D_i(f),\ i=0, 1, \cdots, n;\ D_i'(z_1 f)=z_1 D_i'(f),\ i=1, \cdots, n. \tag{4}$$

We have

$$[z_1 f, z_1 g]=z_1(f\circ g),\quad f, g\in\mathfrak{A}, \tag{5}$$

where

$$f\circ g=z_1((D_0 f)(\delta_0' g)-(D_0 g)(\delta_0' f)+\sum_{i=1}^{n}((D_i f)(D_i' g)-(D_i g)(D_i' f)). \tag{6}$$

Let φ be the linear transformation of $\mathfrak{A}$: $f\mapsto z_1 f$. Then by (5)

$$[\varphi f, \varphi g]=\varphi(f\circ g). \tag{7}$$

Thus under $\circ$, $\mathfrak{A}$ becomes a Lie algebra $A=A(n, m, \underset{\sim}{r}, G)$ and φ is an isomorphism

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of A onto Σ. By (6), it is easily seen that A is the antisymmetric algebra B^- associated with a Nodal noncommutative Jordan algebra B. Since $\varphi(1)=z_1(=z)$, we have $\overline{\Sigma}\cong(A/F1)'$, which is simple.

(II) Let char $F=2$. The proof of the simplicity of Σ^*, $\tilde{\Sigma}$ and $\overline{\Sigma}$ in [2, Theorems 2.1—2.3] is still valid. Especially when $G=0$, $\varepsilon_0=0$, $\varepsilon_i=\varepsilon_i'=0$, $i=1, \cdots, n$, we obtain simple Lie algebras which are isomorphic to Lie algebras of Cartan type $K(2n+1)$ of characteristic 2 defined by the Pfaffian forms $dx_0+\sum_{i=1}^{n}(\mu_i x_i dx_{n+i}+\nu_i x_{n+i} dx_i)$, $\mu_i+\nu_i=1$, $i=1, \cdots, n$ (for a detailed discussion, see [1]), while the usual normalized Lie algebra of Cartan type $K(2n+1)$ defined by $dx_0+\sum_{i=1}^{n}(x_i dx_{n+i}+x_{n+i}dx_i)$ is not simple[4].

(III) If char $F=0$, the construction of Σ $(n, m, \underset{\sim}{r}, G)$ is also possible, and two classes of simple graded Lie algebras $\overline{\Sigma}$ and Σ^* are obtained. But now G is infinite if $G\neq 0$ and all the grading spaces are infinite dimensional. In particular, the zero-grade term $\Sigma_0\cong P\otimes Fx_0\oplus(P\otimes \mathrm{sp}(n))$, where P is the algebra of Laurent polynomials in n variables $z_1, \cdots, z_n$, $P\otimes\mathrm{sp}(n)$ is a generalized loop algebra and x_0 acts as a derivation $\frac{\partial}{\partial z}:=\Sigma r_i z_i\frac{\partial}{\partial z_i}$ on P, i.e., $[g\otimes x_0, f\otimes y]=g\frac{\partial f}{\partial z}\otimes y$, $g, f\in P$, $y\in\mathrm{sp}(n)$ $\left(\text{for simplicity, here we set } \mu_i=\nu_i=\frac{1}{2}\right)$.

(IV) There are some minor errors in [2, (1.18)—(1.19)']. They should read:

$$([2], 1.18)\qquad [x^{\bar{k}}y^{\bar{l}}, x^{\bar{k}'}y^{\bar{l}'}]=k_0^* l_0' \overline{X}_0\overline{Y}_0'-k_0'^* l_0\overline{X}_0'\overline{Y}_0+\sum_{i=1}^{u}(k_i^* l_i'^*\overline{X}_i\overline{Y}_i'-k_i'^* l_i^*\overline{X}_i'\overline{Y}_i),$$

where

$$([2], 1.18)'\quad \overline{X}_i=x^{\bar{k}-\bar{e}_i}x^{\bar{k}'},\ \overline{X}_i'=x^{\bar{k}}x^{\bar{k}'-\bar{e}_i},\ \overline{Y}_i=y^{\bar{l}-\bar{e}_i}y^{\bar{l}'},\ \overline{Y}_i'=y^{\bar{l}}y^{\bar{l}'-\bar{e}_i},$$

$i=0, 1, \cdots, n$ (Note that $\overline{Y}_0=\overline{Y}_0'$).

([2], 1.19) Either $\overline{X}_i(\overline{X}_i')=0$ or $\overline{X}_i(\overline{X}_i')=x^{\bar{a}}$ with $\bar{a}\equiv\bar{k}+\bar{k}'-\bar{e}_i \pmod p$,

([2], 1.19)$'$ Either $\overline{Y}_i(\overline{Y}_i')=0$ or $\overline{Y}_i(\overline{Y}_i')=y^{\bar{b}}$ with $\bar{b}\equiv\bar{l}+\bar{l}'-\bar{e}_i \pmod p$,

$i=0, 1, \cdots, n$.

And [2, (1.26), (1.27), (1.35)'] should take corresponding changes. However, the main formulas (1.26)$'$ and (1.35) remain true and the validity of the discussions and results following them is not affected.

(V) If we start from a divided power algebra $\mathfrak{D}=\mathfrak{D}[x_0, x_1, \cdots, x_n, y_1, \cdots, y_n]$ of $2n+1$ indeterminates, then make adjunctions y_{0i}, z_i, satisfying [2, (1.5)], [2, (1.30)] respectively, in $\mathfrak{D}$, and let D_i, $i=0, 1, \cdots, n$, D_i', $i=1, \cdots, n$, be the special derivations of $\mathfrak{D}$, we obtain a Lie algebra similar to Σ, whose multiplication table is close to that of $K(2n+1, \underset{\sim}{r})$. When $G=0$, it is just $K(2n+1, \underset{\sim}{r})$ itself. This shows that $K(2n+1, \underset{\sim}{r})$ can be viewed as a "twisted form" of a Lie algebra of Cartan type H (i.e., the set of invariant elements of a derivation D of the form [2, (1.20)]).

Similar twisting process can be applied to Lie algebras of Cartan type W, S and K and we obtain algebras of Cartan type S, W and H respectively.

(VI) Another erratum of [2]: The formula in the last line of p. 343 and the first line of p. 344 should be

$$\begin{aligned}\mathscr{L}_i=&\langle \Sigma \alpha_r z^r \mid p_t(\alpha)=0,\ t=1, \cdots, i+1\rangle\\ &\oplus \sum_{j=1}^{i+1}\langle x\Sigma \alpha_r z^r \mid d(x)=j,\ p_t(\alpha)=0,\ t=0,\ 1, \cdots, i+1-j\rangle\\ &\oplus \langle xz^r \mid d(x)\geqslant i+2\rangle,\end{aligned}$$

where $x=x^k y^l$ and $d(x)=\sum_{i=0}^{n} k_i+\sum_{i=1}^{n} l_i$.

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