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SURFACE IN R^3 WITH PRESCRIBED GAUSS CURVATURE

HONG JIAXING (洪家兴)*

Abstract

Assume that a smooth function $K(x, y)$ vanishes on a simple closed smooth curve Λ and $dK|_\Lambda \neq 0$. Then the result of the present paper shows that there is a sufficiently smooth surface $z=z(x, y)$ defined in a neighbourhood of Λ with curvature K.

Let K be given C^∞ function of $(x, x_2) \in R^2$. Is there a local graph $z=z(x_1, x_2)$ near some point p, in R^2 with curvature K? As is well known, the problem was solved in [1, 2], for the case $K(p) \neq 0$, and in [3] for the case $K(p)=0$, $dK(p) \neq 0$. As a continuation of [3], the present paper is devoted to the study of the case:

$K=0$ and $dK \neq 0$ on a simple smooth closed curve Λ:

$$x_i = x_i(u)\ (i=1, 2),\ \ \dot{x}_1^2(u)+\dot{x}_2^2(u) \neq 0, \quad u \in [0, 2\pi]. \tag{0.0}$$

Our main result is the following theorem.

Theorem. *Let* (0.0) *be fulfilled. Then there is a sufficiently smooth graph* $z=z(x_1, x_2)$ *defined in a neighbourhood of* Λ *with curvature* K.

The Problem discussed here is very closely related to the local solvability of Monge Ampére equation with mixed type. The section 1 is devoted to reduction of Monge Ampére equation to the sum of an ordinary differential equation of Fuchsian type and a nonlinear perturbation. The technique used in this section and in section 3 is similar to that in [3]. To author's knowledge, the theory of the positive symmetric system given by [4] and later generalized by [5], so far, has been the most powerful tool to attack the mixed equation. In section 2, based on this theory, a priori estimates for solutions to linearized equation are given. In section 3, the application of Nash–Moser methods completes the proof of existence.

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§ 1. Fundamental Equations

Without loss of generality, we assume that $K=K(u, v)$ is smooth in $[0, 2\pi] \times$

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* Institute of Mathematics, Fudan University, Shanghai, China.

$[-\pi, \pi]$ with period 2π in u. Throughout this paper, we always suppose

$$K(u, \pi/2)=0, \ K_v(u, \pi/2)>0. \tag{1.1}$$

In order to find the graph with curvature K, one may seek "generalized Torus"

$$r(u, v)=(r_0\cos u, r_0\sin u, \rho(u, v)\sin v) \tag{1.2}$$

with $r_0=(1+\rho(u, v)\cos v)$ as the object to be investigated. Now we proceed to find the equation $\rho(u, v)$ satisfies.

The metric on (1.1)

$$dr^2=(r_0^2+\rho_u^2)du^2+2\rho_u\rho_v du\,dv+(\rho_v^2+\rho^2)dv^2. \tag{1.3}$$

The equation we expect may be written in the form

$$\begin{aligned} 0&=\mathscr{F}(u, v, \rho, \rho_u, \rho_v, \rho_{uu}, \rho_{uv}, \rho_{vv}) \\ &=(\rho_{vv}-2\rho_v^2\rho^{-1}-\rho)(\rho_{uu}-r_0\cos v-2\rho_u^2r_0^{-1}\cos v-\rho_v r_0\rho^{-1}\sin v) \\ &\quad-(\rho_{uv}-\rho_u\rho_v\rho^{-1}-\rho_u r_0^{-1}(\rho_v\cos v-\rho\sin v))^2-K(u, v)\Phi(v, \rho, \rho_u, \rho_v), \end{aligned} \tag{1.4}$$

where Φ is a smooth function of its arguments and

$$\Phi(\pi/2, \rho, 0, 0)=\rho^2. \tag{1.5}$$

Take a change of variables

$$u=x, \ v=\varepsilon^2 y+\pi/2, \ \rho=\rho_0+\frac{1}{8}(2v-\pi)^2\rho_2(u)+\varepsilon^5 W, \tag{1.6}$$

where ρ_0 and $\rho_2(u)$ are to be determined. Denoting by $\mathscr{F}(w)$, after changing the variables (1.6), the right hand side in (1.4), we have

$$\mathscr{F}(0)=-((\rho_2(x)-\rho_0)^2+K_v(x, \pi/2)\rho_0^3)\varepsilon^2 y\rho_0^{-1}+\varepsilon^4 I(\varepsilon, x, y). \tag{1.7}$$

Here $I(\varepsilon, x, y)$ is a smooth function of ε, x, y. Particularly, choice of

$$\rho_0=-\frac{1}{8}, \ \rho_2(x)=\rho_0\left(1+\sqrt{K_v(x, \pi/2)/8}\right) \tag{1.8}$$

gives

$$\mathscr{F}(0)=\varepsilon^4 I(\varepsilon, x, y). \tag{1.9}$$

With

$$\overline{\mathscr{F}}_\rho=\int_0^1\frac{\partial\mathscr{F}}{\partial\rho}(\lambda w)d\lambda$$

we rewrite (1.6) in the form

$$\begin{aligned} &\varepsilon\overline{\mathscr{F}}_{\rho_{vv}}w_{yy}+\varepsilon^3\overline{\mathscr{F}}_{\rho_{vu}}w_{xy}+\varepsilon^5\overline{\mathscr{F}}_{\rho_{uu}}w_{xx}+\varepsilon^5\overline{\mathscr{F}}_{\rho_u}w_x+\varepsilon^3\overline{\mathscr{F}}_{\rho_v}w_y \\ &\qquad \varepsilon^5\overline{\mathscr{F}}_\rho w+\mathscr{F}(0)=0. \end{aligned} \tag{1.10}$$

The direct computation yields

$$\overline{\partial\mathscr{F}}/\partial\rho_v=-(\rho_2(x)-\rho_0)\rho_0^{-1}+\varepsilon F_2, \tag{1.11}$$

$$\overline{\partial\mathscr{F}}/\partial\rho_{uu}=(\rho_2(x)-\rho_0)+\varepsilon F_{11}, \tag{1.12}$$

$$\overline{\partial\mathscr{F}}/\partial\rho_{uv}=\varepsilon^2 F_{12}, \tag{1.13}$$

$$\overline{\partial\mathscr{F}}/\partial\rho_{vv}=\varepsilon^2 y(1-\rho_2(x)/\rho_0)+\varepsilon^3 F_{22}. \tag{1.14}$$

Here and later we always take F_i, F_{ij} to express some smooth functions

$$\text{of } \varepsilon, x, y, \varepsilon^2 w, \varepsilon^2 w_x, \varepsilon w_{xy}, \varepsilon^2 w_{xx}, w_y, w_{yy}. \tag{1.15}$$

After inserting (1.11)—(1.14) into (1.10), dividing both sides by ε^3 and noting (1.1), (1.8), (1.9) we obtain

$$\mathscr{L}(w)=(y+\varepsilon F_{22})w_{yy}+\varepsilon^2F_{12}w_{xy}+(|\rho_0|+\varepsilon F_{11})\varepsilon^2 w_{xx}$$
$$+\varepsilon^2F_1w_x(1+\varepsilon F_2)w_y+\varepsilon^2Fw+\varepsilon I(\varepsilon,x,y)=0. \tag{1.10'}$$

It is easy to see that the derivative operator of $\mathscr{L}(w)$

$$\lim_{t\to 0}\frac{\mathscr{L}(u+tw)-\mathscr{L}(u)}{t}=\mathscr{L}'(u)w$$
$$=(y+\varepsilon F_{22})w_{yy}+\varepsilon^2F_{12}w_{xy}+(|\rho_0|+\varepsilon F_{11})\varepsilon^2w_{xx}+\varepsilon^2F_1w_x$$
$$+(1+\varepsilon F_2)w_y+\varepsilon^2Fw+\varepsilon I(\varepsilon,x,y), \tag{1.16}$$

where, after u is replaced by w, F_i, F_{ij} are also the smooth functions like (1.15).

§ 2. A Priori Estimates for Solution to Linearized Equations

The purpose of this section is to establish a priori estimates for solutions to the linearized equation of (1.10′). The framework of the positive symmetric system given in [4] is adequate. Let G denote the bounded domain $\{(x,y)\,|\,0\leqslant x\leqslant 2\pi,\ -2<y<2\}$. Consider a symmetric system defined in G of the form

$$RV=A\frac{\partial V}{\partial y}+B\frac{\partial V}{\partial x}+CV=f, \tag{2.1}$$

where

$$A=A^0(x,y)+\varepsilon A^1(\varepsilon,x,y),$$
$$B=B^0(x,y)+\varepsilon B^1(\varepsilon,x,y),$$
$$C=C^0(x,y)+\varepsilon C^1(\varepsilon,x,y)$$

and A^i, B^i, C^i $(i=0,1)$ are smooth matrices of period 2π in x. It is evident that the following differential operators

$$\mathscr{D}=\{D=I,\ D_1=\partial/\partial x,\ D_2=\alpha_2(y)\partial/\partial y,\ D_3=\alpha_3(y)(y-2)\partial/\partial y\} \tag{2.2}$$

form a complete system of tangential differential operators on G if $\alpha_2+\alpha_3=1$ on G and $\alpha_2=1$ as $y<1/2$, $\alpha_3=1$ as $y>1$ (see [4, 5]).

Assume that

$$\det A^0(x,2)\neq 0 \quad \text{as} \quad x\in[0,2\pi], \tag{2.3}$$

which implies the boundary $y=2$ is not characteristics for (2.1) if ε is small enough. In the sequel, we always impose this restriction on ε. Under this assumption, by the results in [4, 5], one can find smooth matrices $P_{\sigma\tau}$, t_σ such that the $1-st$ enlarged system of (2.1)

$$RD_\sigma=-\sum_\tau P_{\sigma\tau}D_\tau+(D_\sigma-t_\sigma)R. \tag{2.4}$$

Similarly, the argument of induction gives immediately the following lemma.

Lemma 2.1. *Let (2.3) be fulfilled and let ε be small enough. Then the s-th enlarged system of* (2.1)

$$RD_{\sigma_1}\cdots D_{\sigma_s}=-\sum_p P_{\sigma_p\tau}D_{\sigma_1}\cdots D_{\sigma_{p-1}}D_\tau D_{\sigma_{p+1}}\cdots D_{\sigma_s}$$
$$+\sum_{r\leqslant s-1}D^{q_1}t_{\tau_1}\cdots D^{q_{l-1}}t_{\tau_{l-1}}D^{q_l}P_{\tau_l\tau}D_\tau D_{\tau_{l+1}}\cdots D_{\tau_r}+\prod_{i=1}^{s}(D_{\sigma_i}-t_{\sigma_i})R \tag{2.5}$$

with the integers satisfying $q_1+\cdots+q_l\geqslant 1$, $q_1+\cdots+q_l+1+r-l\geqslant s$. According to the definition in [4], a symmetric system like (2.1) is said to be positive definite if and only if on the region discussed $\gamma=C+C^t-\partial A/\partial y-\partial B/\partial x$ is positive definite.

Lemma 2.2. *Let* (2.3) *be fulfilled. If there exists a positive integers such that the s-th enlarged system of* $R^0=A^0\partial/\partial y+B^0\partial/\partial x+C^0$ *is positive definite, it follows that the s-th enlarged system of* (2.1) *is positive definite too, when* $|\varepsilon|\leqslant\varepsilon_s$ *for some positive constant* ε_s.

Proof First we have to find the relation between both of the s-th enlarged systems of R^0 and R. In doing so we split $P_{\sigma\tau}$ into two parts

$$P_{\sigma\tau}=P_{\sigma\tau}(x,\ y)+\varepsilon\bar{P}_{\sigma\tau}(\varepsilon,\ x,\ y),$$
$$t_\sigma=t_\sigma(x,\ y)+\varepsilon\bar{t}_\sigma(\varepsilon,\ x,\ y),$$

where the matrices occurring in the right hand side of the last two equalities are all smooth in their arguments. Denote by $\underset{s}{R^0}$ and $\underset{s}{R}$ the s-th enlarged systems of R^0 and R. Thus

$$\underset{s}{R^0}=\underset{s}{A^0}\frac{\partial}{\partial y}+\underset{s}{B^0}\frac{\partial}{\partial x}+\underset{s}{C^0},$$
$$\underset{s}{R}=\underset{s}{A}\frac{\partial}{\partial y}+\underset{s}{B}\frac{\partial}{\partial x}+(\underset{s}{C^0}+\varepsilon\underset{s}{\bar{C}}).$$

Here $\underset{s}{A^0}(\underset{s}{B^0},\ \underset{s}{A},\ \underset{s}{B})$ is the matrix with A^0 (B^0, A, B) as diagonal elements and $\underset{s}{C^0}$, $\underset{s}{\bar{C}}$ are smooth in x, y. Hence

$$\underset{s}{\gamma}=(\underset{s}{C^0}+(\underset{s}{C^0})^t-\partial\underset{s}{A^0}/\partial y-\partial B^0/\partial x)+O(\varepsilon)=\underset{s}{\gamma^0}+O(\varepsilon). \tag{2.6}$$

The positivity of $\underset{s}{\gamma^0}$ implies the positivity of $\underset{s}{\gamma}$ if ε is small enough. This completes the proof.

Now we turn to the linearized equation $\mathscr{L}'(u)w=g$. Consider a boundary value problem

$$\mathscr{L}'(u)w=g, \tag{2.7}$$
$$w(x,\ 2)=0,\ x\in[0,\ 2\pi]\text{ and }w(0,\ y)=w(2\pi,\ y),\ |y|\leqslant 2. \tag{2.8}$$

This is a boundary value problem of the equation of complicated mixed type. To specify the type of (2.7), we first make some assumption. Suppose that

$$|\varepsilon F_{22}|\leqslant 1,\ |\varepsilon F_{11}|\leqslant\frac{1}{16},\ |\varepsilon F_{12}|\leqslant\frac{1}{8},$$
$$\text{when }(x,\ y)\in\bar{G},\ |u|_{C_3}\leqslant 1\text{ and }|\varepsilon|\leqslant\varepsilon_0, \tag{2.9}$$

for some constant $\varepsilon_0>0$. Here and later, $|u|_{C_0}$, $|u|_{C_1}\cdots$ stand for the continuous norms of the function u and its derivatives. (2.9) guarantees that (2.7) is elliptic near $y=2$ and hyperbolic near $y=-2$. With the aid of a transformation of variables

$$V=\exp\left(\frac{1}{8}y\right)(\partial w/\partial y,\ w,\ \varepsilon\partial w/\partial x)^t=(v_0,\ v,\ v_1)^t \tag{2.10}$$

(2.7) may be reduced to a symmetric system

$$RV = A\frac{\partial V}{\partial y} + B\frac{\partial V}{\partial x} + CV = (g,\ 0,\ 0)^t, \tag{2.11}$$

where

$$A = \begin{bmatrix} y + \varepsilon F_{22} & 0 & 0 \\ 0 & -\delta & 0 \\ 0 & 0 & -(|\rho_0| + \varepsilon F_{11}) \end{bmatrix},$$

$$B = \begin{bmatrix} \varepsilon^2 F_{12} & 0 & (|\rho_0| + \varepsilon F_{11})\varepsilon \\ 0 & 0 & 0 \\ (|\rho_0| + \varepsilon F_{12})\varepsilon & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 - \frac{1}{8}y + \varepsilon\left(F_2 - \frac{1}{8}F_{22}\right) & \varepsilon^2 F & \varepsilon F_1 \\ \delta & \frac{1}{8}\delta & 0 \\ 0 & 0 & \frac{1}{8}(|\rho_0| + \varepsilon F_{11}) \end{bmatrix}$$

for some positive constant δ to be determined. Letting $\varepsilon = 0$ in (2.11) gives

$$RV = \begin{bmatrix} y & 0 & 0 \\ 0 & -\delta & 0 \\ 0 & 0 & -|\rho_0| \end{bmatrix}\frac{\partial V}{\partial y} + \begin{bmatrix} 1 - \frac{1}{8}y & 0 & 0 \\ \delta & \frac{1}{8}\delta & 0 \\ 0 & 0 & \frac{1}{8}|\rho_0| \end{bmatrix}V \tag{2.11'}$$

$$= (g,\ 0,\ 0)^t.$$

It should be emphasized that (2.11′) is independent of F_i, F_{ii}. For (2.11′)

$$\gamma^0 = \begin{bmatrix} 1 - \frac{1}{4}y & \delta & 0 \\ \delta & \frac{1}{4}\delta & 0 \\ 0 & 0 & \frac{1}{4}|\rho_0| \end{bmatrix}, \tag{2.12}$$

which is positive definite if δ is small enough. From now on we fix the constant δ which makes (2.12) positive definite. In fact, we can get much sharper result. We claim that for arbitrary s, the s-th enlarged system of (2.11′) is positive definite. In doing so, we evaluate the commutators

$$R^0 D_1 = D_1 R^0,$$

$$R^0 D_2 = -\frac{\partial A^0}{\partial y}D_2 + \cdots + \left(D_2 + \frac{\partial \alpha_2}{\partial y}\right)R^0, \tag{2.13}$$

$$R^0 D_3 = -\frac{\partial A^0}{\partial y}D_3 + \cdots + \left(D_3 + \frac{\partial}{\partial y}(\alpha_3(y-2))\right)R^0,$$

where the terms omitted, according to the remark in [[4], part II], will not affect the positivity of the 1-*st* enlarged system of (2.11′). From (2.4), (2.5), (2.13) it follows that the positivity of s-th enlarged system of (2.11′) is determined by

$$\operatorname{diag}(\gamma^0, \cdots, \gamma^0) + \operatorname{diag}\left(n_1 \frac{\partial A}{\partial y}, \cdots, n_p \frac{\partial A}{\partial y}\right)$$

with $p=3\cdot 4^s$ and $n_j \in [0, 1, \cdots, s]$, $1 \leqslant j \leqslant p$, which is positive definite.

Theorem 2.1. *Assume that $u \in C^\infty(\bar{G})$ and $|u|_{C_4} \leqslant 1$. Then for arbitrarys, there esist constants $\varepsilon_s \leqslant \varepsilon_0$, and $C_s > 0$ such that for any $g \in C^\infty(\bar{G})$, (2.7), (2.8) admits a solution W in $H_s(G)$ satisfying*

$$\|W\|_s \leqslant C_s \|g\|_s \quad (s=0, 1, 2) \tag{2.14}$$

and

$$\|W\|_s \leqslant C_s(\|g\|_s + \|u\|_{s+3}\|g\|_2), \quad s \geqslant 3. \tag{2.15}$$

Proof Let us first study the boundary value problem of (2.11) with

$$v(x, 2) = v_1(x, 2) = 0, \quad V(0, y) = V(2\pi, y), \quad |y| \leqslant 2. \tag{2.16}$$

Under the assumption (2.9), (2.16) is the stable admissible boundary condition of (2.11) (Refer to [5]). In order to get the differentiable solutions to (2.11), (2.16), we have to investigate the positivity of (2.11). Application of (2.5) to (2.11) gives

$$\begin{aligned}\prod_{i=1}^{s}(D_{\sigma_i} - t_{\sigma_i})R = RD_{\sigma_1}\cdots D_{\sigma_s} + \sum_p P_{\sigma_p\tau} D_{\sigma_1}\cdots D_{\sigma_{p-1}} D_\tau D_{\sigma_{p+1}}\cdots D_{\sigma_s} \\ + \sum D^{q_1}t_{\tau_1}(0)\cdots D^{q_{l-1}}t_{\tau_{l-1}}(0) D^{q_l}P_{\tau_l\tau}(0) D_\tau D_{\tau_{l+1}}\cdots D_{\tau_r} \\ + \sum_{\substack{q_1+\cdots+q_1\geqslant 1 \\ q_1+\cdots+q_1+1+r-l\leqslant s}} [D^{q_1}t_{\tau_1}\cdots D^{q_{l-1}}t_{\tau_{l-1}} \, D^{q_l}P_{\tau_l\tau}]_0^\varepsilon D_\tau D_{\tau_{l+1}}\cdots D_{\tau_r}\end{aligned} \tag{2.17}$$

with $[F]_0^\varepsilon = F(\varepsilon) - F(0)$, $t_{\sigma_i} = t_{\sigma_i}(0) + \varepsilon \bar{t}_{\sigma_i}$, $p_{\sigma\tau} = p_{\sigma\tau}(0) + \varepsilon \bar{p}_{\sigma\tau}$. Evidently $t_\sigma(0)$, $P_{\sigma\tau}(0)$ are all determined by the enlarged system of R^0 and independent of u. As mentioned above, the positivity of (2.17) is completely determined by the first three terms of the right hand side of (2.17). Combining Lemma 2.2 and the fact that for any s, the s-th enlarged system of R^0 is positive, we can find a constant ε_s which only depends on the norm $|u|_{C_3}$ such that (2.17) is positive if $|\varepsilon| \leqslant \varepsilon_s$. The existence of solution in $H_s(G)$ to (2.11), (2.16) follows at once from the results in [5]. Thus $w = \exp\left(-\frac{1}{8}y\right) v$ is the solution to (2.7), (2.8) in $H_s(G) \subset C^2$ if $s \geqslant 4$.

Now we prove (2.14), (2.15). In order to do so, we introduce another kind of norm for V. Set

$$|\!|\!| V |\!|\!|^2 = \sum_{0\leqslant l\leqslant s} \|D_{\sigma_1}\cdots D_{\sigma_l} V\|^2.$$

If $l \leqslant 2$, the l-th enlarged system of (2.11) only depends on the derivatives $\partial^\alpha A$ $\partial^\beta B$ $\partial^\gamma C$ of order $|\beta|$, $|\alpha|$, $|\gamma| \leqslant 2$. From the energy inequality for the positive symmetric system and the assumption $|u|_{C_4} \leqslant 1$ it follows that

$$|\!|\!| V |\!|\!|_l \leqslant C_l |\!|\!| RV |\!|\!|_l \leqslant C_l \|g\|_l \quad (l=0, 1, 2). \tag{2.18}$$

From now on, unless stated otherwise, constants $C_1, C_2, \cdots, C_s, \cdots$ are all independent of ε and the derivatives $\partial^\alpha u$ of order $|\alpha| > 4$. The case of $l=0$ in (2.14) is the trivial consequence of (2.18), i.e.,

$$\|w\| = \left\|\exp\left(-\frac{1}{8}y\right)v\right\| \leqslant C\|V\| \leqslant C\|g\|.$$

Direct computation yields

$$\frac{\partial}{\partial x} = D_1, \tag{2.19}$$

$$\frac{\partial}{\partial y} = D_2 + \alpha_3 A^{-1}(R - CD_0 - BD_1). \tag{2.20}$$

Denote by ∂^s each of derivatives with respect to x, y of order $|\alpha| = s$. It is easily seen that (2.20), (1.19) can be rewritten as

$$\partial^1 = \sum E_\tau(x, y) D_\tau + \varepsilon \sum F_\tau D_\tau + F_4 R. \tag{2.21}$$

F_i ($i = 0, \cdots, 4$) only depend on the derivatives $\partial^\alpha u$ of order $|\alpha| \leqslant 3$. Furthermore

$$\partial^s = \sum_{|l| \leqslant s} E^s_{\tau_1 \cdots \tau_l} D_{\tau_1} \cdots D_{\tau_l} + \sum_{|\alpha| \leqslant s-1} E_\alpha(x, y) \partial^\alpha (\varepsilon \sum F_\tau D_\tau + F_4 R). \tag{2.21'}$$

By (2.21) and (2.18) we can derive that

$$\|W\|_l \leqslant \|V\|_l \leqslant C'_l (\|V\|_l + \|RV\|_{l-1}) \leqslant C_l \|g\|_l \quad (l = 1, 2). \tag{2.22}$$

To prove (2.15) we have first to estimate the norms $\|V\|_l$. As claimed previously, the system composed of the first three terms in the right hand side of (2.17) is positive when $|\varepsilon| \leqslant \varepsilon_s$. Analogously, application of the energy inequality for positive symmetric system yields

$$\|V\|_s \leqslant C_s \left(\left\| \prod_{i=1}^{s} (D_{\sigma_i} - t_{\sigma_i}) RV \right\| + \| [D^{q_1} t_{\tau_1} \cdots D^{q_{l-1}} t_{\tau_{l-1}} D^{q_l} P_{\tau_l \tau}]^{\varepsilon}_0 D_\tau D_\tau \cdots D_{\tau_r} V \| \right),$$

$$q_1 + \cdots + q_l \geqslant 1,$$

$$q_1 + \cdots + q_l + 1 + r - l \leqslant s.$$

Because the last terms in the last inequality are the sum of certain terms which are of the form

$$\varepsilon D^{p_1} F_1 \cdots D^{p_l} F_l D_\tau D_{l+1} \cdots D_r V$$

with F_i related to $\partial^\alpha u$ of order $|\alpha| \leqslant 4$ and integers satisfying $p_1 + \cdots p_l + r - l \leqslant s - 2$, we see that the term we are evaluating is pounded by

$$\begin{aligned} &\varepsilon \sum \prod_l |F_l|_{C_0}^{1 - p_l/(s-1)} \|F_l\|_{s-1}^{p_l/(s-1)} |V|_{C_0}^{(p_1 + \cdots + p_l)/(s-1)} \|V\|_{s-1}^{1 - \frac{p_1 + \cdots + p_l}{s-1}} \\ &\leqslant \varepsilon C'_s (\|u\|_{s+3} |V|_{C_0})^{(p_1 + \cdots p_l)/(s-1)} \|V\|_{s-1}^{1 - (p_1 + \cdots + p_l)/(s-1)} \\ &\leqslant \varepsilon C_s (\|V\|_{s-1} + \|u\|_{s+3} |V|_{C_0}). \end{aligned} \tag{2.23}$$

In getting the last inequality we have used Nirenberg inequality. From the similar argument, the inequality

$$\|\Pi (D_{\sigma_i} - t_{\sigma_i}) RV\| \leqslant C_s (\|g\|_s + \|u\|_{s+3} |g|_{C_0}), \tag{2.24}$$

follows without difficulty. Summing up, and using the embedded theorem and (2.22), we get

$$\|V\|_s \leqslant C_s (\|g\|_s + \|u\|_{s+3} \|g\|_2 + \varepsilon \|V\|_{s-1}), \quad s \geqslant 3. \tag{2.25}$$

Turn back to (2.21′). Dealing with the last term in (2.21′) in the same way as done in getting (2.23), one can obtain

$$\|V\|_s \leqslant C_s(\varepsilon\|V\|_s + \|g\|_s + \|u\|_{s+3}\|g\|_2). \tag{2.26}$$

Choice of a smaller constant ε_s yields (2.15). This proves Theorem 2.1.

§ 3. Proof of the Main Theorem

This section is to complete the proof of the main theorem by proving the existence of sufficient smooth solution of (1.10′) in the region $\{0\leqslant x\leqslant 2\pi,\ -1\leqslant y\leqslant 1\}$. If θ is a constant >8, then $\eta_n=(1-(\theta^{-1}+\cdots+\theta^{-(n-1)}))>1/2$ and $G_n=\{(x, y)\,|\,0\leqslant x\leqslant 2\pi,\ -2\eta_n\leqslant y\leqslant 2\eta_n\}\subset G$. Let $u\in C^\infty(\overline{G})$ be of period 2π in x. Then

$$u(x, y)=\sum_{j=-\infty}^{\infty} a_j(y)\exp(\sqrt{-1}\,jx) \tag{3.1}$$

with smooth coefficients $a_j(y)$. Denote the mollifier by

$$J_n u=\sum_{|j|\leqslant N_n}(j(\theta_n y)\theta_n^* a_j(y))\exp(\sqrt{-1}\,jx) \tag{3.2}$$

with $\theta_n=\theta^{\tau^n}\left(\frac{4}{3}<\tau<2\right)$, $N_n\geqslant\theta_n$ and $j(y)\in C^\infty(R^1)$ satisfying

$$\int j(y)\,dy=1,\ \int y^p j(y)\,dy=0,\quad 1\leqslant p\leqslant s$$

(s to be determined), and supp $j\subset(-1, 1)$. As well known, for any $u\in C^\infty(\overline{G})$,

$$\|J_n u\|_{H_{s_1}(G_{n+1})}\leqslant C(s_1, s_2)\theta_n^{s_1-s_2}\|u\|_{H_{s_2}(G_n)} \text{ if } s_2\leqslant s_1\leqslant s \tag{3.3}$$

and

$$\|(I-J)_n u\|_{H_{s_2}(G_{n+1})}\leqslant C(s_1, s_2)\theta_n^{s_2-s_1}\|u\|_{H_{s_1}(G_n)} \text{ if } s_2<s_1\leqslant s. \tag{3.4}$$

The constant in (3.3), (3.4) will not change when θ_n is increasing. Consider the boundary value problems of linearized equation (1.10′)

$$\mathscr{L}'(u_n)W_n=-\mathscr{L}(u_n), \tag{3.5}$$

$$W_n(x, 2\eta_n)=0,\ W_n(0, y)=w_n(2\pi, y)\ \ |y|\leqslant 2\eta_n \tag{3.6}$$

and

$$u_0=0,\ u_{n+1}=u_n+J_n w_n. \tag{3.7}$$

By Theorem 2.1 it is easily seen that (3.5), (3.6) admits a unique solution in $H_s(G_n)$ for given s if $|\varepsilon|\leqslant\varepsilon_s$. Moreover, the solution obtained satisfies (2.14), (2.15) for $g=g_n=-\mathscr{L}(u_n)$ and the integration related to the norms in (2.14), (2.15) are taken over G_n. Besides, the constants occurring in (2.14) (2.15) are independent of n. For the solutions to (3.5), (3.6), (3.7), we have the following estimates.

Lemma 3.1. *Let* $u_k|_{C_4(\overline{G}_k)}\leqslant 1\ (k=0, \cdots, n)$. *Then*

$$\|g_k\|_{H_s(G_k)}=\|-\mathscr{L}(u_k)\|_{H_s(G)}\leqslant C_s(\|g_0\|_s+\|u_k\|_{H_{s+2}(G_k)}) \tag{3.8}$$

and

$$\|u_{k+1}\|_{H_{s+3}(G_{k+1})}\leqslant C_s^{k+1}\theta_{k+1}^{\beta}\|g_0\|_s \tag{3.9}$$

for some constant $\beta\geqslant 9$ *and* $k=0, \cdots, n$. *Here and later the integration of* $\|u\|_s$ *is always taken over* G_1.

Proof It is easy to see that

$$\|g_k\|_{H_s(G_k)}=\|\mathscr{L}(u_k)-\mathscr{L}(u_0)+\mathscr{L}(u_0)\|_{H_s(G_k)}\leqslant(\|g_0\|_s+C_s\|u_k\|_{H_{s+2}(G_k)}).$$

Now we proceed to prove (3.9). From (3.7) it follows that if $k=0, \cdots, n$,

$$\|u_{k+1}\|_{H_{s+3}(G_{k+1})}\leqslant\|u_k\|_{H_{s+3}(G_{k+1})}+\|J_kw_k\|_{H_{s+3}(G_{k+1})}\leqslant\|u_k\|_{H_{s+3}(G_{k+1})}+C_s\theta_k^3\|w_k\|_{H_s(G_k)}$$
$$\leqslant C_s(\|u_k\|_{H_{s+3}(G_k)}+\|g_0\|_s)\theta_k^3. \tag{3.10}$$

In getting (3.10) we have used (2.14), (2.15), (3.8) and the assumption in this lemma. Hence

$$\|u_{k+1}\|_{H_{s+1}(G_{k+1})}\leqslant(k+1)C_s^{k+1}\theta_k^3\cdots\theta_0^3\|g_0\|_s. \tag{3.11}$$

Choosing eC_s as a new constant C_s and the constant $\beta\geqslant 9>3/(\tau-1)$ $(2>\tau>4/3)$, by (3.11) we can derive (3.9) without difficulty.

Lemma 3.2. *Let* $|u_k|_{C_4(\bar{G}_k)}\leqslant 1$ $(k=0, \cdots, n)$. *Then there exists a constant* $\chi>6/(2-\tau)$ *such that for any* $s^*>\beta+2+\chi\tau$, *the inequality*

$$\|g_{k+1}\|_{H(G_{k+1})}\leqslant\theta_{k+1}^{-\chi}\|g_0\|_{s^*}\quad(k=0, \cdots, n-1) \tag{3.12}$$

holds when $\theta\geqslant\theta^*$ *and* $|\varepsilon|\leqslant\varepsilon_{s^*}(\theta)$ *for some constants* θ^* *and* $\varepsilon_{s^*}(\theta)$.

Proof Obviously

$$-g_{k+1}=\mathscr{L}(u_{k+1})=\mathscr{L}(u_k)+\mathscr{L}'(u_k)J_kw_k+Q(u_k, J_kw_k)$$
$$=\mathscr{L}'(u_k)(J_k-I)w_k+Q(u_k, J_kw_k),$$

where Q is the quadratic form. Application of (3.3) gives, when $s^*>2$,

$$\|\mathscr{L}'(u_k)(J_k-I)w_k\|_{H(G_{k+1})}\leqslant C'\theta_k^{2-s^*}\|W_k\|_{H_{s^*}(G_k)}\leqslant C\theta_k^{2-s^*}(\|g_0\|_{s^*}+\|u\|_{H_{s+2}(G_k)}),$$

which is bounded by

$$C_{s^*}^k\theta_k^{2-s^*+\beta}\|g_0\|_{s^*}$$

since with the aid of (3.8), (3.9) the norms in last ineqality can be controlled by the norm in $H_{s^*}(G_k)$ of g_0. On the other hand, by Nirenberg inequality

$$\|Q(u_k, J_kw_k)\|_{H(G_{k+1})}\leqslant C'\sum_{|\alpha|\leqslant 2}\left(\int_{G_{k+1}}|\partial^\alpha J_kw_k|^4dxdy\right)^{\frac{1}{2}}\leqslant C'\|J_kw_k\|^2_{H_3(G_{k+1})}$$
$$\leqslant C''\theta_k^6\|w_k\|^2_{H(G_k)}\leqslant C\theta_k^6\|g_k\|^2_{H(G_k)}.$$

Summing up, we have obtained the estimate

$$\|g_{k+1}\|_{H(G_{k+1})}\leqslant C_{s^*}^k\theta_k^{2-s^*+\beta}\|g_0\|_{s^*}+C\theta_k^6\|g_k\|^2_{H(G)_k}. \tag{3.13}$$

With

$$d_{k+1}=\max(C, 1)\theta_{k+1}^{\chi}\|g_{k+1}\|_{H(G_{k+1})}$$

one has the inequality

$$d_{k+1}\leqslant d_k^2+\frac{1}{4}\|g_0\|_{s^*} \tag{3.14}$$

if

$$6+\chi(\tau-2)<0,\ s^*>2+\beta+\chi\tau \tag{3.15}$$

and

$$4\max(C, 1)C_{s^*}^k\leqslant\theta^{\tau^k(s^*-2-\beta-\chi\tau)}. \tag{3.16}$$

This is possible to choose first χ then s^* in (3.15) and later θ^* in (3.16). Noting that $g_0=-\mathscr{L}(u_0)=-\varepsilon I(\varepsilon, x, y)$. Therefore, Take such a small $\varepsilon_{s^*}(\theta)$ that if $|\varepsilon|\leqslant\varepsilon_{s^*}(\theta)$,

$$\|g_0\|_{s^*}=\|\varepsilon I(\varepsilon, x, y)\|_{s^*}\leqslant 1 \text{ and } \max{}^2(C, 1)\theta_0^{2\chi}\|g_0\|\leqslant\frac{1}{4},$$

which gives $d_0^2\leqslant\|g_0\|_{s^*}/4\leqslant 1/4$. Inserting that into (3.14) we get

$$d_{k+1}\leqslant\frac{1}{2}\|g_0\|_{s^*} \quad (k=0, \cdots, n),$$

which implies (3.12) immediately.

The end of the proof of the main theorem To prove the convergence of the iterating sequence constructed by Nash-Moser methods, it is necessary to control the norm in $C_4(\overline{G}_k)$ of $u_k\leqslant 1$, or the norm in $H_6(G_k)$,

$$\|u_k\|_{H_6(G_k)}\leqslant\Gamma \tag{3.17}$$

for some constant Γ. We shall next reason by induction. It is trivial that (3.17) holds when $k=0$ and ε is sufficient small. Now (3.17) is supposed to be true when $k\leqslant n$. If $6\leqslant s<s^*$, by (2.14), (2.15) and Lemma 3.2 one has

$$\|u_{n+1}\|_{H_s(G_{n+1})}\leqslant\sum_{k=0}^{u}\|J_kw_k\|_{H_s(G_{k+1})}\leqslant C'\sum_{0}^{n}\|w_k\|_{H_s(G_k)}\leqslant C''\sum_{0}^{n}\|w_k\|_{H_{s^*}(G_k)}^{s/s^*}\|w_k\|_{H(G_k)}^{1-s/s^*}$$

$$\leqslant C''\sum_{0}^{n}(\|g_0\|_{s^*}+\|u_k\|_{H_{s^*+3}(G_k)})^{s/s^*}(\theta_k^{-\chi}\|g_0\|_{s^*})^{1-s/s^*}$$

which is bounded by

$$C\sum_{0}^{n}C_{s^*}^k\theta_k^{-\left(\chi-\frac{s(\beta+\chi)}{s^*}\right)}\|g_0\|_{s^*} \tag{3.18}$$

for another C_{s^*}. If we choose $\beta=9$, $\tau=\frac{4}{3}+\delta$, $\chi=9+O(\delta)>\frac{6}{2-\tau}$ and $s^*>2+\beta+\chi\tau=23+O(\delta)$, $s<s^*/2$, when δ is small enough (3.18) converges and for some constant C independent of n,

$$\sum_{k=0}^{n}\|J_kw_k\|_{H_s(G_{k+1})}\leqslant C\|g_0\|_{s^*}, \tag{3.19}$$

provided that θ is big enough. This just completes the proof of the induction if we take a smaller ε_{s^*} so that $C\|g_0\|_{s^*}\leqslant\Gamma$ when $|\varepsilon|\leqslant\varepsilon_{s^*}\leqslant\varepsilon_{s^*}(\theta)$.

The conclusion we have reached means that u_n converges uniformly to a function u in $H_s(G_\infty)\subset H_{11}(G_\infty)\subset C_9(\overline{G}_\infty)$ if s^* satisfies the assumption mentioned above. So u is a solution in $C_9(\overline{G}_\infty)$ to (1.10') since (3.12) implies that $\mathscr{L}(u_n)\to 0=\mathscr{L}(u)$ in $H(G_\infty)$. Now we have the following theorem.

Theorem 3.1. *Let $K\in C^{s^*+2}$ satisfy* (1.1) ($s^*>23$). *Then* (1.4) *admits a local solution $\rho\in C^{s-2}$ of period 2π in u if $s<s^*/2$.*

In order to complete the proof of the main theorem in this paper we have to show that the "generalized Torus" (1.2) obtained is a graph. It is easy to get

$$\partial(x_1, x_2)/\partial(u, v)=+\rho(u, v) \text{ on } v=\frac{\pi}{2}$$

which does not vanish if ε is small enough. Finally, we should show how to determine the integer s involved in the construction of $j(y)$ as the kernel of mollifier. If we expect to find the solution $\in C^\sigma$, then we choose $s=2\sigma+8$.

We have proved the main theorem if Λ is a circte. The remainder of the proof is omitted since there is no principal difficulty for them.

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