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HÖLDER CONTINUITY OF THE GRADIENT OF THE SOLUTIONS OF CERTAIN DEGENERTE PARABOLIC EQUATIONS

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Abstract

This paper is concerned with the parabolic equation

$$\frac{\partial u}{\partial t}-\operatorname{div}(|\nabla u|^{p-2}\nabla u)=0, \quad x\in\Omega\subset\mathbb{R}^N,\ t>0$$

with $p>\max\left\{\frac{3}{2}, \frac{2N}{N+2}\right\}$ which is degenerate if $p<2$ or singular if $\frac{3}{2}<p<2$. Let $u(x, t)$ be any weak solution of the equation in $L^\infty[0, T; L^2(\Omega)]\cap L^p[0, T; W^{1,p}(\Omega)]$. The Hölder continuity of ∇u is established.

§ 1. Introduction

In this paper we consider the parabolic equation

$$\frac{\partial u}{\partial t}-\operatorname{div}(|\nabla u|^{p-2}\nabla u)=0 \tag{1.1}$$

with $p>\max\left\{\frac{3}{2}, \frac{2N}{N+2}\right\}$, where $\nabla=\operatorname{grad}_x$ and x varies in an open set $\Omega\subset\mathbb{R}^N$. The equation is degenerate if $p>2$ or singular if $\frac{3}{2}<p<2$.

By a solution u of (1.1) in the cylindrical domain $\Omega_T=\Omega\times(0, T]$ we mean a function, $u(x, t)$, defined in Ω_T and satisfying:

$$u\in L^\infty[0, T; L^2(\Omega)]\cap L^p[0, T; W^{1,p}(\Omega)] \tag{1.2}$$

and (1.1) holds in the weak sense

$$\iint_{\Omega_T}[u\varphi_t-|\nabla u|^{p-2}\nabla u\cdot\nabla\phi]\,dx\,dt=0, \quad \forall\phi\in C_0^1(\Omega_T). \tag{1.3}$$

The main result of this paper asserts that for any solution u of (1.1) with

$$\max\left\{\frac{3}{2}, \frac{2N}{N+2}\right\}<p<\infty,$$

∇u is local Hölder continuity in Ω_T. For any $\Omega'_T\subset\subset\Omega_T$, the Hölder coefficient and exponent of ∇u depend only on Ω'_T and on a bound on the norm of u with respect

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to the spaces in (1.2).

For the equation (1.1), N. D. Alikakos and L. C. Evans[1] obtained the continuity of ∇u with $p>2$. They applied Evan's method for elliptic equations in [2] to (1.1), but they could not manage non-uniformlity between space and time parts very well and so they had not found Hölder continuity of ∇u. E. Dibenedetto and A. Friedman[3] studied the corresponding parabolic systems, which contain the eguation (1.1) as a special case, but they established only the continuity of ∇u with modulus $\omega(r)=\left(\log\log\frac{A}{r}\right)^{-\sigma}$ for some constants A and σ. They asserted that their results hold for $p>\max\left\{1, \frac{2N}{N+2}\right\}$, but they made some mistake in the calculation. The coefficient of the first term on the right hand side of the inequality (iii) on p. 87 in [3] should be $(p-2)/4$ instead of $(p-2)/2$ and, in fact, their results hold only for $p>\max\left\{\frac{3}{2}, \frac{2N}{N+2}\right\}$.

The paper is organized as follows:

In § 2 we establish L^∞ estimate on ∇u. E. Dibenedetto and A. Friedman[3] emphasized that both the Moser and De Giorgi iteration on procedures must be used for it. In this paper we use only Moser iteration and so the proof is much shorter.

In § 3 we give some fundamental lemmas and finally we come to our conclusion in § 4.

§ 2. Boundedness of ∇u

Let Ω be a bounded open domain in $\mathbb{R}^N$ and set $\Omega_T=\Omega\times(0, T]$, $T>0$. Let u be a solution of

$$\frac{\partial u}{\partial t}-\operatorname{div}(|\nabla u|^{p-2}\nabla u)=0 \quad \text{in } \Omega_T, \tag{2.1}$$

where $p>\max\left\{\frac{3}{2}, \frac{2N}{N+2}\right\}$, and set

$$\|u\|_p=\operatorname*{ess\,sup}_{0<t\leq T}\|u(\cdot, t)\|_{2,\Omega}+\|\nabla u\|_{p,\Omega_T}, \tag{2.2}$$

where $\|\cdot\|_{q,G}$ is the norm in $L^q(G)$. By definition we have $\|u\|_p<\infty$. For any $s>0$, set

$$\Omega_{T,s}=\Omega_s\times(s, T],$$

where

$$\Omega_s=\{x\in\Omega \mid \operatorname{dist}\{x, \partial\Omega\}>s\}.$$

Theorem 2.1. *For any $s>0$ and $\Gamma>0$ there exists a constant C^*, depending only on N, p, s, T and Γ, such that, for any solution u of (2.1) with $\|u\|_p<\Gamma$,*

$$\|\nabla u\|_{\infty,\Omega_{T,s}}\leq C^*. \tag{2.3}$$

Proof Fix any point $P_0(x^0, t^0)\in\Omega_T$ and set

$$B(x^0, R)=\{x\in\mathbb{R}^N \mid |x-x^0|<R\},$$
$$Q(p_0, R)=B(x^0, R)\times(t^0-R^2, t^0]$$

for R such that $\overline{Q(p_0, R)}\subset\Omega_T$. For clarity we shall first establish (2.3) under the assumption that u, u_t, ∇u, D_x^2u are in suitable L^q spaces so that the calculations below are justified.

Differentiating (2.1) with respect to x_j we get

$$\frac{\partial}{\partial t}u_{x_j}-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u_{x_j}+\frac{\partial}{\partial x_j}|\nabla u|^{p-2}\nabla u\right)=0. \tag{2.4}$$

Set $v=|\nabla u|^2$ and apply to (2.4) the test function

$$\varphi=u_{x_j}v^\alpha\zeta^2 \quad \left(\alpha\geqslant\min\left\{0, \frac{p-2}{2}\right\}\right),$$

where ζ is the usual C^1 cut-off function with respect to $Q(p_0, R)$, $Q(p_0,(1-\sigma)R)$ $(0<\sigma<1)$. After some computation (cf. p. 87 in [3]) we get

$$\begin{aligned}&\frac{1}{2(\alpha+1)}\operatorname*{ess\,sup}_{t^0-R^2<t<t^0}\int_{B(R)}\zeta^2v^{\alpha+1}\,dx\,dt+\frac{p+2\alpha-2}{4}\iint_{Q(R)}\zeta^2v^{\frac{p+2\alpha-4}{2}}|\nabla v|^2\,dx\,dt\\&+\iint_{Q(R)}\zeta^2v^{\frac{p+2\alpha-2}{2}}|D^2u|^2\,dx\,dt+\frac{\alpha(p-2)}{2}\iint_{Q(R)}\zeta^2v^{\frac{p+2\alpha-6}{2}}(\nabla u\cdot\nabla v)^2\,dx\,dt\\&\leqslant C\iint_{Q(R)}\zeta v^{\frac{p+2\alpha-2}{2}}|\nabla\zeta|\,|\nabla v|\,dx\,dt+\frac{2}{2(\alpha+1)}\iint_{Q(R)}|\zeta_t|v^{\alpha+1}\,dx\,dt,\end{aligned} \tag{2.5}$$

where $B(R)=B(x^0, R)$, $Q(R)=Q(p_0, R)$. Here and after we always express constants depending on N and p by C.

Noting that

$$\begin{aligned}v^{\frac{p+2\alpha-2}{2}}|D^2u|^2&=v^{\frac{p+2\alpha-4}{2}}\sum_{k=1}^N\left(\sum_{i=1}^N u_{x_i}^2\cdot\sum_{i=1}^n u_{x_ix_k}^2\right)\\&\geqslant v^{\frac{p+2\alpha-4}{2}}\sum_{k=1}^N\left(\sum_{i=1}^N u_{x_i}u_{x_ix_k}\right)^2=\frac{1}{4}v^{\frac{p+2\alpha-4}{2}}|\nabla v|^2,\end{aligned} \tag{2.6}$$

we obtain from (2.5)

$$\begin{aligned}&\frac{1}{2(\alpha+1)}\operatorname*{ess\,sup}_{t^0-R^2<t<t^0}\int_{B(R)}\zeta^2v^{\alpha+1}\,dx\,dt+\frac{p+2\alpha-1}{4}\iint_{Q(R)}\zeta^2v^{\frac{p+2\alpha-4}{2}}|\nabla v|^2\,dx\,dt\\&+\frac{\alpha(p-2)}{2}\iint_{Q(R)}\zeta^2v^{\frac{p+2\alpha-6}{2}}(\nabla u\cdot\nabla v)^2\,dx\,dt\\&\leqslant C\iint_{Q(R)}\zeta v^{\frac{p+2\alpha-2}{2}}|\nabla\zeta|\,|\nabla v|\,dx\,dt+\frac{1}{2(\alpha+1)}\iint_{Q(R)}|\zeta_t|v^{\alpha+1}\,dx\,dt.\end{aligned} \tag{2.7}$$

If $p+2\alpha-1>0$ and $\alpha(p-2)>0$, we have, applying Cauchy inequality,

$$\begin{aligned}&\frac{1}{2(\alpha+1)}\operatorname*{ess\,sup}_{t^0-R^2<t<t^0}\int_{B(R)}\zeta^2v^{\alpha+1}\,dx+\frac{p+2\alpha-1}{8}\iint_{Q(R)}\zeta^2v^{\frac{p+2\alpha-4}{2}}|\nabla v|^2\,dx\,dt\\&\leqslant\frac{C}{p+2\alpha-1}\iint_{Q(R)}v^{\frac{p+2\alpha}{2}}|\nabla\zeta|^2\,dx\,dt+\frac{1}{2(\alpha+1)}\iint_{Q(R)}|\zeta_t|v^{\alpha+1}\,dx\,dt.\end{aligned} \tag{2.8}$$

We consider two cases:

(1) $p>2$.

Let $\alpha\geqslant 0$. Then it follows from (2.8) that, for any $\alpha\geqslant 0$,

$$\operatorname*{ess\,sup}_{t^0-R^2<t<t^0}\int_{B(R)}\zeta^2 v^{\alpha+1}dx+\iint_{Q(R)}\zeta^2\left|\nabla\left(v^{\frac{p+2\alpha}{4}}\right)\right|^2dx\,dt$$
$$\leqslant C\left[\iint_{Q(R)}v^{\frac{p+2\alpha}{2}}|\nabla\zeta|^2dx\,dt+\iint_{Q(R)}|\zeta_t|v^{\alpha+1}dx\,dt\right], \tag{2.9}$$

where C is independent of α.

Using Hölder inequality and Sobolev embedding theorem, we have

$$\iint_{Q(R-\sigma R)}v^{\frac{p-2}{2}+(\alpha+1)\left(1+\frac{2}{N}\right)}dx\,dt\leqslant\int_{t^0-(1-\sigma)^2R^2}^{t^0}\left(\int_{B(R-\sigma R)}v^{\alpha+1}dx\right)^{\frac{2}{N}}$$
$$\times\left(\int_{B(R-\sigma R)}v^{\frac{(p+2\alpha)N}{2(N-2)}}dx\right)^{\frac{N-2}{N}}dt$$
$$\leqslant\operatorname*{ess\,sup}_{t^0-(1-\sigma)^2R^2<t<t^0}\left(\int_{B(R-\sigma R)}v^{\alpha+1}dx\right)^{\frac{2}{N}}\iint_{Q(R-\sigma R)}\left(v^{\frac{p+2\alpha}{2}}+\left|\nabla\left(v^{\frac{p+2\alpha}{4}}\right)\right|^2\right)dx\,dt$$
$$\leqslant C\left[\iint_{Q(R)}v^{\frac{p+2\alpha}{2}}|\nabla\zeta|^2dx\,dt+\iint_{Q(R)}|\zeta_t|v^{\alpha+1}dx\,dt\right]^{1+\frac{2}{N}}. \tag{2.10}$$

Set $\varkappa=1+\dfrac{2}{N}$ and, for $l=0, 1, 2, \cdots$,

$$R_l=R\left(\frac{1}{2}+\frac{1}{2^{l+1}}\right),$$
$$\zeta_l(x)\in C_0^1(Q(R_l)),\ 0\leqslant\zeta_l(x)\leqslant 1,\ \zeta_l(x)\equiv 1\text{ in }Q(R_{l+1}), \tag{2.11}$$
$$\alpha+1=\varkappa^l.$$

Then it follows from (2.10) that

$$\left(\iint_{Q(R_{l+1})}v^{\frac{p-2}{2}+\varkappa^{l+1}}dx\,dt\right)^{\frac{1}{\varkappa}}\leqslant C\,\frac{4^l}{R}\left[\iint_{Q(R_l)}v^{\frac{p-2}{2}+\varkappa^l}dx\,dt+\iint_{Q(R_l)}v^{\varkappa^l}dx\,dt\right]. \tag{2.12}$$

Without loss of generality, we can assume that, for any $l\geqslant 0$,

$$\iint_{Q(R_l)}v^{\frac{p-2}{2}+\varkappa^l}dx\,dt\geqslant 1,$$

and so from (2.12) we have, noting $p>2$,

$$\left(\iint_{Q(R_{l+1})}v^{\frac{p-2}{2}+\varkappa^{l+1}}dx\,dt\right)^{\frac{1}{\varkappa}}\leqslant\frac{C4^l}{R^2}\iint_{Q(R_l)}v^{\frac{p-2}{2}+\varkappa^l}dx\,dt.$$

The standard Moser iteration procedure yields

$$\left(\iint_{Q(R_{l+1})}v^{\frac{p-2}{2}+\varkappa^{l+1}}dx\,dt\right)^{\frac{1}{\varkappa^{l+1}}}\leqslant\frac{C}{R^{N+2}}\iint_{Q(R)}v^{\frac{p}{2}}dx\,dt,\quad l=0, 1, 2, \cdots.$$

Let $l\to\infty$, we have

$$\operatorname*{ess\,sup}_{Q\left(\frac{R}{2}\right)}v\leqslant\frac{C}{R^{N+2}}\iint_{Q(R)}|\nabla u|^p dx\,dt. \tag{2.13}$$

(2) $2>p>\max\left\{\frac{3}{2}, \frac{2N}{N+2}\right\}$.

Let $\alpha\geqslant\frac{p-2}{2}$. Obviously we have $p+2\alpha-1>0$. When $\alpha\leqslant 0$, (2.8) holds since $\alpha(p-2)>0$. When $\alpha>0$, the third term of (2.7) can be bounded below as

$$\frac{\alpha(p-2)}{2}\iint_{Q(R)}\zeta^2 v^{\frac{p+2\alpha-6}{2}}(\nabla u\cdot\nabla v)^2\,dx\,dt\geqslant-\frac{\alpha(2-p)}{2}\iint_{Q(R)}\zeta^2 v^{\frac{p+2\alpha-4}{2}}|\nabla v|^2\,dx\,dt.$$

It follows from (2.7) that, for $\alpha\geqslant 0$,

$$\frac{1}{2(\alpha+1)}\operatorname*{ess\,sup}_{t^0-R^2<t<t^0}\int_{B(R)}\zeta^2 v^{\alpha+1}\,dx\,dt+\frac{(1+2\alpha)(p-1)}{4}\iint_{Q(R)}\zeta^2 v^{\frac{p+2\alpha-4}{2}}|\nabla v|^2\,dx\,dt$$

$$\leqslant C\iint_{Q(R)}\zeta v^{\frac{p+2\alpha-2}{2}}|\nabla\zeta|\,|\nabla v|\,dx\,dt+\frac{1}{2(\alpha+1)}\iint_{Q(R)}|\zeta_t|\,v^{\alpha+2}\,dx\,dt.$$

Similarly, (2.9) and then (2.10) hold for $\alpha\geqslant\frac{p-2}{2}$ and $2>p>\frac{3}{2}$. Now set

$$\alpha_0=\frac{N(2-p)}{4},\ \lambda=1-\alpha_0+\frac{p-2}{2},\ \alpha+1=\alpha_0+\lambda\varkappa^l,$$

where $\varkappa=1+\frac{2}{N}$. Instead of (2.12) we have

$$\left(\iint_{Q(R_{l+1})}v^{\alpha_0+\lambda\varkappa^{l+1}}\,dx\,dt\right)^{\frac{1}{\varkappa}}\leqslant\frac{C4^l}{R^2}\left[\iint_{Q(R_l)}v^{\frac{p-2}{2}+\alpha_0+\lambda\varkappa^l}\,dx\,dt+\iint_{Q(R)}v^{\alpha_0+\lambda\varkappa^l}\,dx\,dt\right].$$

Clearly, $p>\frac{2N}{N+2}$ implies $\lambda>0$. Without loss of generlity, we can assume

$$\iint_{Q(R_l)}v^{\alpha_0+\lambda\varkappa^l}\,dx\,dt\geqslant 1\quad\text{for any } l\geqslant 0,$$

and then we have, noting $p<2$,

$$\left(\iint_{Q(R_{l+1})}v^{\alpha_0+\lambda\varkappa^{l+1}}\,dx\,dt\right)^{\frac{1}{\varkappa}}\leqslant\frac{C4^l}{R^2}\iint_{Q(R_l)}v^{\alpha_0+\lambda\varkappa^l}\,dx\,dt.$$

Moser iteration procedure results in

$$\left(\iint_{Q(R_{l+1})}v^{\alpha_0+\lambda\varkappa^{l+1}}\,dx\,dt\right)^{\frac{1}{\varkappa^{l+1}}}\leqslant\frac{C}{R^{N+2}}\iint_{Q(R)}v^{\frac{p}{2}}\,dx\,dt.$$

Thus

$$\operatorname*{ess\,sup}_{Q(\frac{R}{2})}v\leqslant\frac{C}{R^{\frac{N+2}{2}}}\left(\iint_{Q(R)}|\nabla u|^p\,dx\,dt\right)^{\frac{1}{\lambda}}.\tag{2.14}$$

We have now completed the proof under the assumption that u, u_t, u_{x_j}, ∇u_{x_j} are in suitable L^q spaces so that the formal calculation can be justified. For the general case, we can average with respect to t and finite-difference wtih respect to x (refer to the explanation on p. 95 in [3]). The theorem has been proved.

As a by-product we have the following theorem:

Theorem 2.2. *For any* $\varepsilon>0$, $T>0$

$$\iint_{\Omega_{T,s}} |\nabla u|^{p-2}|D^2u|^2\,dx\,dt \leqslant C^{**} \quad if\ p>2, \tag{2.15}$$

and

$$\iint_{\Omega_{T,s}} |\nabla u|^{2(p-2)}|D^2u|^2\,dx\,dt \leqslant C^{**} \quad if\ \frac{3}{2}<p<2, \tag{2.16}$$

for any solution u of (2.1) with $\|u\|_p \leqslant \Gamma$, where C^{**} is a positive constant depending only on p N, T, s and Γ.

Proof If $p>2$, (2.5) with $\alpha=0$ gives (2.15).

If $2>p>\frac{3}{2}$, combining (2.5) with (2.6) and setting $\alpha=\frac{p-2}{2}$ we can obtain

$$\frac{1}{p}\operatorname*{ess\,sup}_{t^0-R^2<t\leqslant t^0}\int_{B(R)} \zeta^2 v^{\frac{p}{2}}\,dx\,dt + (2p-3)\iint_{Q(R)} \zeta^2 v^{p-2}|D^2u|^2\,dx\,dt$$

$$\leqslant C\iint_{Q(R)} \zeta v^{p-2}|\nabla\zeta|\,|\nabla v|\,dx\,dt + \frac{1}{p}\iint_{Q(R)} |\zeta_t|\, v^{\frac{p}{2}}\,dx\,dt.$$

It is easy to get (2.16) from this.

§ 3. Local Properties of ∇u

Suppose that $u(x, t)$ is a solution of (2.1) in some cylindrical domain

$$Q_\mu(R) = \left\{(x, t)\,\middle|\, |x|<R, -\frac{R^2}{\mu^{p-2}}<t\leqslant 0\right\}, \tag{3.1}$$

where $0<R\leqslant 1$ and $\mu>0$ are some constants.

Set

$$M^{\pm}_{j,\mu}(R) = \operatorname*{ess\,sup}_{Q_\mu(R)}(\pm u_{x_j}) \quad \text{for} \quad j=1, 2, \cdots, N,$$

$$M_\mu(R) = \max_{1\leqslant j\leqslant N} \operatorname*{ess\,sup}_{Q_\mu(R)} |u_{x_j}|.$$

Lemma 3.1. *Let μ satisfy*

$$2M^{+}_{1,\mu}(R) \geqslant \mu \geqslant M_\mu(R). \tag{3.2}$$

Then there exists ε_0, depending only on p and N, such that

$$⨍\!\!\!\!\iint_{Q_\mu(R)} [M^{+}_{1,\mu}(R))^\alpha - |u_{x_1}|^{\alpha-1}u_{x_1}]^2\,dx\,dt \leqslant \varepsilon_0 [M^{+}_{1,\mu}(R)]^{2\alpha} \tag{3.3}$$

implies

$$\operatorname*{ess\,inf}_{Q_\mu\left(\frac{R}{2}\right)} u_{x_1} \geqslant \frac{M^{+}_{1,\mu}(R)}{2^{\frac{1}{\alpha}}}, \tag{3.4}$$

where $\alpha=\min\{1, p-1\}$ and

$$⨍\!\!\!\!\iint_{Q_\mu(R)} f\,dx\,dt = \frac{1}{\operatorname{meas} Q_\mu(R)} \iint_{Q_\mu(R)} f\,dx\,dt.$$

Proof Assume for the moment u is a smooth solution of (2.1). Upon differentiating with respect to x_1, we derive the equation

$$\frac{\partial u_{x_1}}{\partial t} - (a_{ij}|\nabla u|^{p-2}u_{x_1x_j})_{x_i} = 0, \tag{3.5}$$

where

$$a_{ij}=\delta_{ij}+\frac{(p-2)u_{x_i}u_{x_j}}{|\nabla u|^2} \quad \text{if } |\nabla u|\neq 0. \tag{3.6}$$

Obviously we have

$$\min\{p-1,\ 1\}|\xi|^2\leqslant a_{ij}\xi_i\xi_j\leqslant\max\{p-1,\ 1\}|\xi|^2, \quad \forall\xi\in\mathbb{R}^N. \tag{3.7}$$

Set

$$v=|u_{x_1}|^{\alpha-1}u_{x_1}, \tag{3.8}$$

which means $u_{x_1}=|v|^{\frac{1}{\alpha}}\operatorname{sign} v$. The function v satisfies in the weak sense the equation

$$\frac{1}{\alpha}|v|^{\frac{1-\alpha}{\alpha}}\frac{\partial v}{\partial t}-\Big(\frac{1}{\alpha}a_{ij}|\nabla u|^{p-2}|v|^{\frac{1-\alpha}{\alpha}}v_{x_j}\Big)_{x_i}=0. \tag{3.9}$$

Let $\zeta(x)$ be the usual cut-off function in $Q_\mu(R)$ and set $M_1=M^+_{1,\mu}(R)$. Applying the test function $\zeta^2(v-k)^-=\zeta^2\max\{k-v,0\}$ to (3.9) for $\frac{M_1^\alpha}{2}\leqslant k\leqslant M_1^\alpha$ and setting

$$\chi_k(s)=\int_0^s|k-s|^{\frac{1-\alpha}{\alpha}}s\,ds \tag{3.10}$$

we have

$$\iint_{Q_\mu(R)}\zeta^2\frac{\partial\chi_k((v-k)^-)}{\partial t}\,dx\,dt+\iint_{Q_\mu(R)\cap\{v\leqslant k\}}\zeta^2a_{ij}\,|\nabla u|^{p-2}\,|v|^{\frac{1-\alpha}{\alpha}}v_{x_j}v_{x_i}\,dx\,dt$$
$$=2\iint_{Q_\mu(R)}\zeta a_{ij}|\nabla u|^{p-2}|v|^{\frac{1-\alpha}{\alpha}}(v-k)^-\zeta_{x_i}v_{x_j}\,dx\,dt. \tag{3.11}$$

Now define

$$\phi_k(s)=\begin{cases}0 & \text{if } s>k\\ k-s & \text{if } k\geqslant s\geqslant k-\dfrac{M_1^\alpha}{4},\\ \dfrac{M_1^\alpha}{4} & \text{if } s<k-\dfrac{M_1^\alpha}{4}.\end{cases} \tag{3.12}$$

Noting that $M_1^\alpha/2\leqslant k\leqslant M_1^\alpha$ and (3.2) we can get

$$\phi_k(v)\leqslant(v-k)^-\leqslant C\phi_k(v).$$

After simple computation we find

$$\iint_{Q_\mu(R)}\zeta^2\frac{\partial\chi_k(\phi_k(v))}{\partial t}\,dx\,dt+\frac{\min\{p-1,\ 1\}}{2}\iint_{Q_\mu(R)}\zeta^2|\nabla u|^{p-2}|v|^{\frac{1-\alpha}{\alpha}}|\nabla\phi_k(v)|^2\,dx\,dt$$
$$\leqslant C\iint_{Q_\mu(R)}|\nabla u|^{p-2}|v|^{\frac{1-\alpha}{\alpha}}[(v-k)^-]^2|\nabla\zeta|^2\,dx\,dt,$$

and then

$$\operatorname*{ess\,sup}_{-\frac{R^2}{\mu^{p-2}}<t<0}\int_{B(R)}\zeta^2\chi_k(\phi_k(v))\,dx+\iint_{Q_\mu(R)}\zeta^2|\nabla u|^{p-2}|v|^{\frac{1-\alpha}{\alpha}}|\nabla\phi_k(v)|^2\,dx\,dt$$
$$\leqslant C\Big[\iint_{Q_\mu(R)}|\nabla u|^{p-2}|v|^{\frac{1-\alpha}{\alpha}}\phi_k^2(v)|\nabla\zeta|^2\,dx\,dt+\iint_{Q_\mu(R)}|\zeta_t|\chi_k(\phi_k(v))\,dx\,dt\Big]. \tag{3.13}$$

Obviously we have, noting that $\frac{M_1^\alpha}{2}\leqslant k\leqslant M_1^\alpha$ and (3.2),

$$\frac{1}{C}\mu^{1-\alpha}\phi_k^2(v)\leqslant\chi_k(\phi_k(v))\leqslant C\mu^{1-\alpha}\phi_k^2(v),$$

$$|\nabla u|^{p-2}|v|^{\frac{1-\alpha}{\alpha}}\leqslant N^{\frac{p-2}{2}}\mu^{p-\alpha-1},$$

and, if $k\geqslant v\geqslant k-\frac{M_1^\alpha}{4}$,

$$|\nabla u|^{p-2}|v|^{\frac{1-\alpha}{\alpha}}\geqslant\frac{1}{C}\mu^{p-\alpha-1}.$$

Thus it follows from (3.11) that

$$\operatorname*{ess\,sup}_{-\frac{R^2}{\mu^{p-2}}\leqslant t\leqslant 0}\int_{B(R)}\zeta^2\phi_k^2(v)dx+\mu^{p-2}\iint_{Q_\mu(R)}|\nabla(\zeta\phi_k(v))|^2dx\,dt$$
$$\leqslant C\iint_{Q_\mu(R)}\phi_k^2(v)[\mu^{p-2}|\nabla\zeta|^2+|\zeta_t|]dx\,dt.$$

Let $\tau=t\mu^{p-2}$ and

$$\tilde{Q}(R)=\{(x,\tau)\,|\,|x|<R,\ -R^2<\tau<0\}.$$

Then we obtain

$$\operatorname*{ess\,sup}_{-R^2\leqslant\tau\leqslant 0}\int_{B(R)}\zeta^2\phi_k^2(v)dx+\iint_{\tilde{Q}(R)}|\nabla(\zeta\phi_k(v))|^2dx\,d\tau$$
$$\leqslant C\iint_{\tilde{Q}(R)}\phi_k^2(v)\Big[|\nabla\zeta|^2+\frac{1}{\mu^{p-2}}|\zeta_t|\Big]dx\,d\tau.$$

According to Ladyzenskaja et al ([4], p. 75) again, we find

$$\Big[\iint_{\tilde{Q}(R)}(\zeta\phi_k(v))^{\frac{2(N+2)}{N}}dx\,d\tau\Big]^{\frac{N}{N+2}}\leqslant C\iint_{\tilde{Q}(R)}\phi_k^2(v)\Big[|\nabla\zeta|^2+\frac{1}{\mu^{p-2}}|\zeta_t|\Big]dx\,d\tau.$$

Back to the original time variable t, we have

$$\Big[\iint_{Q_\mu(R)}(\zeta\phi_k(v))^{\frac{2(N+2)}{N}}dx\,dt\Big]^{\frac{N}{N+2}}\leqslant C\mu^{\frac{2(p-2)}{N+2}}\iint_{Q_\mu(R)}\phi_k^2(v)\Big[|\nabla\zeta|^2+\frac{1}{\mu^{p-2}}|\zeta_t|\Big]dx\,dt. \tag{3.14}$$

Set

$$k_l=M_1^\alpha\Big(\frac{1}{2}+\frac{1}{2^{l+1}}\Big),\quad R_l=R\Big(\frac{1}{2}+\frac{1}{2^{l+1}}\Big),\qquad l=0,1,2,\cdots$$

and let $\zeta_l(x,t)$ be the cut-off function in $(Q_\mu(R_l)$ satisfying

$$\zeta_l(x,t)\equiv 1\quad\text{in }Q_\mu(R_{l+1}),$$
$$0\leqslant\zeta_l(x,t)\leqslant 1,$$
$$|\nabla\zeta_l|^2\leqslant\frac{C4^l}{R^2},\quad|(\zeta_l)_t|\leqslant\frac{C4^l\mu^{p-2}}{R^2}.$$

Applying (3.14) in $Q_\mu(R_l)$ we get

$$\Big[\iint_{Q_\mu(R_l)}(\zeta\phi_{k_l}(v))^{\frac{2(N+2)}{N}}dx\,dt\Big]^{\frac{N}{N+2}}\leqslant\frac{C4^l\mu^{\frac{2(p-2)}{N+2}}}{R^2}\iint_{Q_\mu(R_l)}\phi_{k_l}^2(v)dx\,dt. \tag{3.15}$$

Now define

$$J_l = \iint_{Q_\mu(R_l)} \phi_{k_l}^2(v)\,dx\,dt.$$

In view of Hölder inequality it follows that

$$J_{l+1} = \iint_{Q_\mu(R_{l+1})} \phi_{k_{l+1}}^2(v)\,dx\,dt$$

$$\leqslant \Big[\iint_{Q_\mu(R_{l+1})} [\phi_{k_{l+1}}(v)]^{\frac{2(N+2)}{N}} dx\,dt\Big]^{\frac{N}{N+2}} [\operatorname{meas} Q_\mu(R_{l+1}) \cap \{\phi_{k_{l+1}}(v) > 0\}]^{\frac{2}{N+2}}. \quad (3.16)$$

Notice that

$$J_l \geqslant \frac{1}{C} \iint_{Q_\mu(R_l)} [(v-k_l)^-]^2\,dx\,dt \geqslant \frac{1}{C}(k_l - k_{l+1})^2 \operatorname{meas} Q_\mu(R_l) \cap \{v < k_{l+1}\}$$

$$\geqslant \frac{M_1^{2\alpha}}{C4^l} \operatorname{meas} Q_\mu(R_{l+1}) \cap \{\phi_{k_{l+1}}(v) > 0\}. \quad (3.17)$$

The inequality (3.15), (3.16) and (3.17) results in

$$J_{l+1} \leqslant \frac{C\,16^l \mu^{\frac{2(p-2)}{N+2}}}{M_1^{\frac{4\alpha}{N+2}} R^2} J_l^{1+\frac{2}{N+2}}.$$

Set

$$Y_l = J_l \mu^{p-2}/M_1^{2\alpha} R^{N+2}.$$

The previous inequality can be written in the following

$$Y_{l+1} \leqslant C16^l Y_l^{1+\frac{2}{N+2}}.$$

According to [4, Lemma II. 5.6, p. 95] therefore

$$Y_l \to 0 \quad \text{if } l \to \infty, \quad (3.18)$$

provided

$$Y_0 \leqslant \hat{\varepsilon}_0,$$

which means (3.3) for some ε_0. The fact (3.18) implies

$$\operatorname*{ess\,inf}_{Q_\mu(\frac{R}{2})} v \geqslant \frac{M_1^\alpha}{2},$$

and (3.4) follows at once.

This proves the lemma under the additional assumption that u is a smooth solution of (2.1). In the general case we recall Theorem 2.2 and prove by routine arguments that v is a weak solution of (3.9): note $|\nabla u|^{p-2}|v|^{\frac{1-\alpha}{2\alpha}}|\nabla u|^2 \in L(Q_\mu(R))$. The rest of the proof is the same.

Lemma 3.2. *Suppose that μ satisfies*

$$2M_{1,\mu}^+(R) \geqslant \mu \geqslant M_\mu(R).$$

Then for any $\varepsilon_0 > 0$ there exist constants $0 < \lambda$, $\beta < 1$, depending only on p, N and ε_0, such that we have

$$\operatorname{meas}\{(x, t) \in Q_\mu(R) \,|\, u_{x_1} \leqslant (1-\beta) M_{1,\mu}^+(R)\} > \lambda \operatorname{meas} Q_\mu(R), \quad (3.19)$$

if the inequality

$$\iint_{Q_\mu(R)} [(M_{1,\mu}^+(R))^\alpha - |u_{x_1}|^{\alpha-1}u_{x_1}]^2\,dx\,dt \leqslant \varepsilon_0 (M_{1\mu}^+(R))^\alpha \tag{3.20}$$

fails, where $\alpha=\min\{p-1, 1\}$.

Proof Suppose that (3.19) fails (for β, λ as selected below) and set as before $M_1=M_{1,\mu}^+(R)$. Then

$$\iint_{Q_\mu(R)} (M_1^\alpha - |u_{x_1}|^{\alpha-1}u_{x_1})^2\,dx\,dt$$

$$= \iint_{Q_\mu(R)\cap\{u_{x_1}\leqslant(1-\beta)M_1\}} + \iint_{Q_\mu(R)\cap\{u_{x_1}>(1-\beta)M_1\}}$$

$$\leqslant [M_1^\alpha + M_\mu^\alpha(R)]^2 \lambda \operatorname{meas} Q_\mu(R) + (\beta M_1)^{2\alpha} \operatorname{meas} Q_\mu(R) \quad \left(\beta \leqslant \frac{1}{2}\right)$$

$$\leqslant [\lambda(1+2^\alpha)^2 + \beta^{2\alpha}] M_1^{2\alpha} \operatorname{meas} Q_\mu(R).$$

Select λ and β such that

$$(1+2^\alpha)^2\lambda + \beta^{2\alpha} \leqslant \varepsilon_0, \quad \beta \leqslant \frac{1}{2}.$$

The lemma has been proved.

Lemma 3.3. *Suppose that*

$$2M_{1,\mu}^+(R) \geqslant \mu \geqslant M_\mu(R),$$

and there exist constants $0<\lambda$, $\beta<1$ *such that*

$$\operatorname{meas}\{(x, t) \in Q_\mu(R)\,|\,u_{x_1}(x, t) \leqslant (1-\beta)M_{1,\mu}^+(R)\} \geqslant \lambda \operatorname{meas} Q_\mu(R). \tag{3.21}$$

Then there exist constants $0<\delta$, $\gamma<1$, *depending only on* N, p, λ *and* β, *such that*

$$M_{1,\mu}^+(\delta R) \leqslant \gamma M_{1,\mu}^+(R). \tag{3.22}$$

Proof For any $\varepsilon>0$, define

$$\psi_\varepsilon(s) = \begin{cases} (s^2+\varepsilon^2)^{\frac{1}{2}} - \varepsilon, & \text{if } s \geqslant 0, \\ 0, & \text{if } s<0, \end{cases}$$

and set

$$k = (1-\beta)M_{1,\mu}^+(R), \quad w = (u_{x_1} - k)^+,$$

$$w_\varepsilon(x, t) = \psi_\varepsilon(u_{x_1} - k).$$

It is easy to get, for any $\phi \in C_0^\infty(Q_\mu(R))$ and $\phi \geqslant 0$,

$$\iint_{Q_\mu(R)} [w_\varepsilon \phi_t - a_{ij}|\nabla u|^{p-2} w_{\varepsilon x_j} \phi_{x_i}]\,dx\,dt \geqslant 0.$$

Let $\varepsilon \to 0$, we can get, for any $\phi \in C_0^\infty(Q_\mu(R))$ and $\phi \geqslant 0$,

$$\iint_{Q_\mu(R)} [w\phi_t - \tilde{a}_{ij} w_{x_j} \varphi_{x_i}]\,dx\,dt \geqslant 0,$$

where

$$\tilde{a}_{ij} = \begin{cases} a_{ij}|\nabla u|^{p-2}, & \text{if } u_{x_1} \geqslant k, \\ u^{p-2}\delta_{ij}, & \text{if } u_{x_1} < k. \end{cases}$$

This means that $w(x, t)$ satisfies in the weak sense

$$w_t - (\tilde{a}_{ij} w_{x_j})_{x_i} \leqslant 0 \quad \text{in } Q_\mu(R).$$

Let $\tau = t\mu^{p-2}$. Then $w(x, \tau)$ satisfies in the weak sense

$$w_\tau - \left(\frac{\tilde{a}_{ij}}{\mu^{p-2}} w_{x_j}\right)_{x_i} \leqslant 0 \quad \text{in } \tilde{Q}(R),$$

where $\tilde{Q}(R) = \{(x, \tau) \mid |x| < R, \ -R^2 < \tau < 0\}$. It is easy to observe

$$\frac{1}{C}|\xi|^2 \leqslant \frac{\tilde{a}_{ij}}{\mu^{p-2}} \xi_i \xi_j \leqslant C|\xi|^2, \ \forall \xi \in \mathbb{R}^N \text{ and } (x, \tau) \in \tilde{Q}(R),$$

where C depends only on N, p and β.

Define

$$\hat{a}_{ij} = \tilde{a}_{ij}/\mu^{p-2}, \quad v = \frac{\operatorname{ess\,sup} w - w}{\operatorname{ess\,sup} w}.$$

The assumption (3.21) implies

$$\operatorname{meas}\{(x, \tau) \in \tilde{Q}(R) \mid v(x, \tau) \geqslant 1\} \geqslant \lambda \operatorname{meas} \tilde{Q}(R).$$

The function $v(x, \tau)$ satisfies in the weak sense

$$v_\tau - (\hat{a}_{ij} v_{x_j})_{x_i} \geqslant 0.$$

By an estimate of Kružkov (see the appendix in [1]) there exist constants $0 < \delta, \gamma < 1$ such that

$$\operatorname*{ess\,inf}_{\tilde{Q}(\delta R)} v \geqslant 1 - \gamma > 0,$$

that is

$$\operatorname*{ess\,sup}_{\tilde{Q}(\delta R)} w \leqslant \gamma \operatorname*{ess\,sup}_{\tilde{Q}(R)} w.$$

Back to the original time variable t, we have

$$M_{1,\mu}^{+}(\delta R) \leqslant \gamma M_{1,\mu}^{+}(R)$$

as claimed.

§ 4. Interior Hölder Continuity of ∇u

Now we determine ε_0 by Lemma 3.1, and then constants λ, β by ε_0 and Lemma 3.2, and finally δ, γ by λ, β and Lemma 3.3. In this case, δ and λ depend only on N and p.

Select a constant s: $1 < s < 2$ if $p > 2$ and $2 < s < 3$ if $1 < p < 2$, such that

$$\delta^{\frac{2(2-s)}{s(p-2)}} > \max\left\{\frac{1}{2}, \gamma\right\}. \tag{4.1}$$

It suffices to choose s close to 2. Let $\Omega_T' \subset\subset \Omega_T$ and suppose

$$\|\nabla u\|_{\infty, \Omega_T'} \leqslant \overline{M}_0, \tag{4.2}$$

and define

$$M_0 = \overline{M}_0 \delta^{-\frac{2(2-s)}{s(p-2)}}.$$

Fix the point $P_0(x^0, t^0) \in \Omega_T'$ and set

$$Q_\mu(P_0, R) = \left\{(x, t) \,\middle|\, |x - x^0| < R, \ t^0 - \frac{R^2}{\mu^{p-2}} < t < t^0\right\}. \tag{4.3}$$

The constant $0 < R_0 \leqslant 1$ is selected such that

$$Q_{2M_0}(P_0, R_0) \subset \Omega_T'. \tag{4.4}$$

Set

$$\hat{Q}(P_0, R)=\left\{(x, t)\,\big|\,|x-x^0|<R,\ t^0-\frac{R^s R_0^{2-s}}{(2M_0)^{p-2}}<t<t^0\right\} \tag{4.5}$$

and

$$M_i^{\pm}(R)=\operatorname*{ess\,sup}_{\hat{Q}(P_0,R)}(\pm u_{x_i}) \quad (i=1, 2, \cdots, N),$$
$$M(R)=\max_{1\le i\le N}\ \operatorname*{ess\,sup}_{\hat{Q}(P_0,R)}|u_{x_i}|, \tag{4.6}$$
$$\operatorname*{osc}_{\hat{Q}(P_0,R)} u_{x_i}=\operatorname*{ess\,sup}_{\hat{Q}(P_0,R)} u_{x_i}-\operatorname*{ess\,inf}_{\hat{Q}(P_0,R)} u_{x_i}=M_i^{+}(R)+M_i^{-}(R).$$

Theorem 4.1. *Let u be a solution of the equation* (2.1). *Then there exist constants $0<\rho<1$ and $C>0$, depending only on N and p, such that*

$$\operatorname*{osc}_{\hat{Q}(P_0,R)} u_{x_i}\le CM_0\left(\frac{R}{R_0}\right)^{\rho} \quad \text{for } 0<R\le R_0,\ i=1, 2, \cdots, N, \tag{4.7}$$

if $\frac{3}{2}<p<2$ *and*

$$\operatorname*{osc}_{Q_{2M_0}(P_0,R)} u_{x_i}\le CM_0\left(\frac{R}{R_0}\right)^{\rho} \quad \text{for } 0<R\le R_0,\quad i=1, 2, \cdots, N. \tag{4.8}$$

if $p>2$.

Proof Define

$$R_1=\sup\left\{R\in[0, R_0]\,\big|\,\exists 1\le j\le N,\ \theta\in\{+, -\}\ni |M_j^{\theta}(R)|>2M_0\left(\frac{R}{R_0}\right)^{\frac{2-s}{p-1}}\right\}. \tag{4.9}$$

Then we can suppose $R_1>0$, since otherwise the theorem has been proved. In addition we have $R_1\le\delta^{\frac{2}{s}}R_0$ by the definition of M_0 and $\overline{M}_0$.

Thus, there must exist R_2:

$$\delta^{\frac{2}{s}}R_2<R_1<R_2\le R_0 \tag{4.10}$$

such that we have

$$|M_j^{\pm}(R_2)|\le 2M_0\left(\frac{R}{R_0}\right)^{\frac{2-s}{p-2}} \quad \text{for } j=1, 2, \cdots, N, \tag{4.11}$$

and there exist i_0 and θ, say $i_0=1$, $\theta=+$, such that

$$M_1^{+}(\delta^{\frac{2}{s}}R_2)>2M_0\left(\frac{\delta^{\frac{2}{s}}R_2}{R_0}\right)^{\frac{2-s}{p-2}}. \tag{4.12}$$

Set

$$\mu=2M_0\left(\frac{R_2}{R_0}\right)^{\frac{2-s}{p-2}}. \tag{4.13}$$

At first we prove that

$$\iint_{Q_\mu(P_0,R_2)}[(M_1^{+}(R_2))^{\alpha}-|u_{x_1}|^{\alpha-1}u_{x_1}]^2\,dx\,dt\le\varepsilon_0((M_1^{+}R_2))^{2\alpha}, \tag{4.14}$$

where $Q_\mu(P_0, R_2)$ is defined in (4.3).

For $R\le R_2$, define

$$M_{1,\mu}^{+}(R)=\operatorname*{ess\,sup}_{Q_\mu(P_0,R)} u_{x_1},$$
$$M_\mu(R)=\max_{1\le i\le N}\ \operatorname*{ess\,sup}_{Q_\mu(P_0,R)}|u_{x_i}|. \tag{4.15}$$

Notice that $Q_\mu(P_0, R_2)=\hat{Q}(P_0, R_2)$ and $M_{1,\mu}^{+}(R_2)=M_1^{+}(R_2)$.

By (4.1) and (4.13), (4.12) implies

$$M_{1,\mu}^{+}(R_2) > M_1^{+}(\delta^{\frac{2}{s}} R_2) > \max\left\{\gamma, \frac{1}{2}\right\}\mu. \tag{4.16}$$

Then we have, noting (4.11), (4.13) and (4.16),

$$2M_{1,\mu}^{+}(R_2) > \mu \geqslant M_\mu(R_2). \tag{4.17}$$

If (4.14) fails, it follows by Lemma 3.2 that

$$\text{meas}\{(x, t) \in Q_\mu(R_2) \,|\, u_{x_1} \leqslant (1-\beta) M_{1,\mu}^{+}(R_2)\} > \lambda \,\text{meas}\, Q_\mu(R_2). \tag{4.18}$$

By Lemma 3.3, (4.17) and (4.18) result in

$$M_{1,\mu}^{+}(\delta R_2) \leqslant \gamma M_{1,\mu}^{+}(R_2).$$

Noting that $Q_\mu(P_0, \delta R_2) = \hat{Q}(P_0, \delta^{\frac{2}{s}} R_2)$, we get

$$M_1^{+}(\delta^{\frac{2}{s}} R_2) = M_{1,\mu}^{+} \delta R_2) \leqslant \gamma M_{1,\mu}^{+}(R_2) \leqslant \gamma\mu,$$

which contradicts (4.16). (4.14) has been proved.

By Lemma 3.1, (4.14) and (4.17) implies

$$\operatorname*{ess\,inf}_{Q_\mu\left(P^0, \frac{R_2}{2}\right)} u_{x_1} \geqslant \frac{M_1^{+}(R_2)}{2^{\frac{1}{\alpha}}} \left(\geqslant \frac{1}{2 \cdot 2^{\frac{1}{\alpha}}}\, \mu\right). \tag{4.19}$$

We know that u_{x_k} satisfies in the weak sense

$$\frac{\partial u_{x_k}}{\partial t} - (a_{ij} |\nabla u|^{p-2} u_{x_k x_j})_{x_i} = 0,$$

where a_{ij} are defined in (3.6).

Set

$$\xi = x - x^0, \quad \tau = \mu^{p-2}(t - t^0),$$

$$Q'(R) = \{(\xi, \tau) \,|\, |\xi| < R^2, \ -R^2 < \tau < 0\},$$

$$v(\xi, \tau) = u_{x_k}(x, t).$$

Then v satisfies in the weak sense

$$\frac{\partial v}{\partial \tau} - \left(a_{ij} \frac{|\nabla u|^{p-2}}{\mu^{p-2}} v_{\xi_j}\right)_{\xi_i} = 0.$$

In view of (4.19), it follows that

$$\frac{1}{C}|\eta|^2 \leqslant a_{ij} \frac{|\nabla u|^{p-2}}{\mu^{p-2}} \eta_i \eta_j \leqslant C|\eta|^2, \quad \forall \eta \in \mathbb{R}^N, \ (\xi, \tau) \in Q'\left(\frac{R_2}{2}\right).$$

Appling Hölder interior estimates for non-degenerate equations, we have

$$\operatorname*{osc}_{Q'(R)} v \leqslant C\left(\frac{R}{R_2}\right)^{\bar{\beta}} \operatorname*{osc}_{Q'\left(\frac{R_2}{4}\right)} v \quad \text{for } 0 < R < \frac{R_2}{4},$$

where $C > 0$ and $0 < \bar{\beta} < 1$ depend only on N and p.

Back to the variables (x, t), we obtain

$$\operatorname*{osc}_{Q_\mu(P_0, R)} u_{x_k} \leqslant C\left(\frac{R}{R_2}\right)^{\bar{\beta}} \operatorname*{osc}_{Q_\mu\left(P_0, \frac{R_2}{4}\right)} u_{x_k} \quad \text{for } 0 < R < \frac{R_2}{4}, \ k = 1, 2, \cdots, N. \tag{4.20}$$

If $R \geqslant R_2$, then by the definition of R_1

$$\operatorname*{osc}_{\hat{Q}(P_0, R)} u_{x_k} \leqslant |M_k^{+}(R)| + |M_k^{-}(R)| \leqslant 4M_0\left(\frac{R}{R_0}\right)^{\frac{2-s}{p-2}}. \tag{4.21}$$

If $\frac{R_2}{4} \leqslant R \leqslant R_2$, then

$$\operatorname*{osc}_{\hat{Q}(P_0, R)} u_{x_k} \leqslant \operatorname*{osc}_{\hat{Q}(P_0, 4R)} u_{x_k} \leqslant 4M_0 \left(\frac{4R}{R_0}\right)^{\frac{2-s}{p-2}}. \tag{4.22}$$

If $0 < R < \frac{R_2}{4}$, then it follows from (4.20) and (4.22) that

$$\operatorname*{osc}_{Q_\mu(P_0, R)} u_{x_k} \leqslant C\left(\frac{R}{R_2}\right)^{\hat{\beta}} \cdot 4M_0 \left(\frac{4R_2}{R_0}\right)^{\frac{2-s}{p-2}}.$$

Set $\rho = \min\left\{\bar{\beta}, \frac{2-s}{p-2}\right\}$, we obtain

$$\operatorname*{osc}_{Q_\mu(P_0, R)} u_{x_k} \leqslant CM_0 \left(\frac{R}{R_2}\right)^{\rho} \left(\frac{R_2}{R_0}\right)^{\rho} = CM_0 \left(\frac{R}{R_0}\right)^{\rho}. \tag{4.23}$$

If $1 < p < 2$, then $Q_\mu(P_0, R) \supset \hat{Q}(P_0, R)$ for $0 < R < \frac{R_2}{4}$, and if $p > 2$ then $\hat{Q}(P_0, R) \supset Q_{2M_0}(P_0, R)$ for $R \geqslant \frac{R_2}{4}$ and $Q_\mu(P_0, R) \supset Q_{2M_0}(P_0, R)$ for $0 < R \leqslant \frac{R_2}{4}$. Thus (4.21), (4.22) and (4.23) imply what we want.

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