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THE HIGHER ORDER APPROXIMATION OF SOLUTION OF QUASILINEAR SECOND ORDER SYSTEMS FOR SINGULAR PERTURBATION

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Abstract

In this paper, using the theory of invariant region, the author considers the existence and the asymptotic behavior of solution of vector second order quasi-linear boundary value problem:

$$\varepsilon y''=f(x, y, \varepsilon)y'+g(x, y, \varepsilon),$$
$$y(0, \varepsilon)=A(\varepsilon), \quad y(1, \varepsilon)=B(\varepsilon)$$

as the positive perturbation parameter ε tends to zero, where y, g, A and B are vector-valued and f is a matrix function. Under the appropriate assumptions the author obtains, involving the boundary layer, uniformly valid asymptotic solution of higher order approximation.

§1. Introduction

Many physical and chemical problems can be studied as singularly perturbed two-point vector boundary value problems of the form:

$$\varepsilon y''=f(x, y, \varepsilon)y'+g(x, y, \varepsilon), \quad 0\leqslant x\leqslant 1, \tag{1.1}$$
$$y(0, \varepsilon)=A(\varepsilon), \ y(1, \varepsilon)=B(\varepsilon), \tag{1.2}$$

where ε is a positive perturbed small parameter, y, g, A and B are vector valued and f is a matrix function (cf. e.g., Amundson[1] and Cohen[2]). Scalar problems of this form are analyzed quite thoroughly (cf. Howes[3], Chang and Howes[4]). An enlightening case history of such analyses was given by Erdélyi[5], and important early work includes that of Coddington and Levinson[6] and Wasow[7]. Vector problems of this form, when boundary is unperturbed, were considered by K. W. Chang[8] and W. G. Kelley[9], but they did not actually obtain higher order terms or complete boundary layes behavior. By applying the theory of invariant region to the vector boundary value problem (1.1), (1.2), however, the higher order approximations of solution are obtained.

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§ 2. Some Preliminary Results

In this section we collect for the convenience of the reader the results to be used in the proofs of our main theorems. Let us consider the two-point boundary value problem

$$x''=g(t, x, x'), \tag{2.1}$$

$$x(0)=A, \quad x(1)=B, \tag{2.2}$$

where g: $[0, 1]\times R^n\times R^n\to R^n$ is continuous and $A, B\in R^n$.

For $i=1, 2, \cdots, N$, let $r_i(t, x)$ be of class C^2 on $[0, 1]\times R^n$, $W_i(t, x)$ the gradient vector of r_i, $V_i(t, x)$ the gradient vector of $\partial r_i/\partial t$, where these gradients are taken with respect to x, and $P_i(t, x)$ the Hessian of r_i with respect to x. Let the first and the second derivatives of r_i with respect to (2.1) be denoted by

$$r_i'=\partial r_i/\partial t+W_i\cdot x', \tag{2.3}$$

$$r_i''=\partial^2 r_i/\partial t^2+2V_i\cdot x'+x'P_i\cdot x'+W_i\cdot g \tag{2.4}$$

for $i=1, 2, \cdots, N$, where the dot indicates the usual scalar product in R^n. Define $D=\{(t, x, y)\colon 0\leqslant t\leqslant 1, r_i(t, x)<0 \text{ for } i=1, 2, \cdots, N, y\in R^n\}$.

We give two types of Nagumo conditions for g:

N_1: There exists a sequence $\{\varphi_i\}_{i=1}^n$ of positive, nondecreasing continuous functions on $(0, \infty)$ such that

$$\int^\infty \frac{s\,ds}{\varphi_i(s)}=\infty$$

and

$$|g^i(t, x, y)|\leqslant\varphi_i(|y^i|), \quad \text{for } (t, x, y)\in D,\ i=1, 2, \cdots, n.$$

N_2: There is a positive, nondecreasing, continuous function φ on $(0, \infty)$ such that

$$s^2/\varphi(s)\to\infty, \quad \text{as } s\to\infty$$

and

$$\|f(t, x, y)\|\leqslant\varphi(\|y\|) \quad \text{for } (t, x, y)\in D.$$

The following theorem is a special case of Theorem 4 in [10].

Theorem 1. *Assume* $\{(t, x)\colon 0\leqslant t\leqslant 1, r_i(t, x)\leqslant 0\}$ *is a bounded set and*

(a) *the functions* r_i *described above satisfy for* $i=1, 2, \cdots, N$

$$r_i''>0 \text{ when } r_i=0 \text{ and } r_i'=0; \tag{2.5}$$

(b) *there is a function of class* C^2 *on* $[0, 1]$ *which satisfies* (2.2) *and whose trajectory is contained in* D;

(c) *initial value problems for* (2.1) *have unique solutions;*

(d) g *satisfies either* N_1 *or* N_2 *on* D.

Then the boundary value problem (2.1), (2.2) *has a solution* $x(t)$ *with* $r_i(t, x(t))<0$ *for* $0\leqslant t\leqslant 1$ *and* $i=1, 2, \cdots, N$.

§ 3. Singular Perturbation Problems

We begin by considering the system (1.1) on the interval [0, 1] with boundary perturbation conditions (1.2). For simplicity, let us assume that f and g are infinitely differentiable in y and x and that f, g, $A(\varepsilon)$ and $B(\varepsilon)$ possess asymptotic power series expansions in ε as $\varepsilon\to 0$. We shall first consider the vector problem under the assumption that the reduced problem

$$f(x, u_R, 0)u_R'+g(x, u_R, 0)=0, \quad u_R(1)=y(1, 0)=B(0) \tag{3.1}$$

is stable throughout $0\leqslant x\leqslant 1$ in the sense that u_R exists and

$$f(x, u_R, 0)<0 \tag{3.2}$$

there (i.e., f is a strictly stable matrix having eigenvalues with negative real parts). We first realize that u_R cannot generally represent the solution to (1.1) near $x=0$ because we cannot expect to have $u_R(0)=y(0, 0)=A(0)$. Instead, we must expect boundary layer behavior to occur near $x=0$, providing the required nonuniform convergence form $y(0, 0)$ to $u_R(0)$ as $\varepsilon\to 0$, the (very) small "boundary layer jump" $\|y(0, 0)-u_R(0)\|$ is needed. We must require an additional "boundary layer stability", namely the inner product

$$\xi^T\int_0^\xi f(0, u_R(0)+s, 0)ds<0 \tag{3.3}$$

remains negative for $\xi+u_R(0)$ along all paths connecting $u_R(0)$ and $y(0, 0)$ with $0<\|\xi\|\leqslant\|y(0, 0)-u_R(0)\|$ (Here, T represents the transpose and $\|z\|=(z^Tz)^{\frac{1}{2}}$). Indeed, (3.3) directly generalizes the (minimal) hypotheses used by Howes for the scalar problem and it is weaker than the common assumption that $f(0, y, 0)<0$ for all y.

The results of Howes and O'Malley[11] and others suggest that under such hypotheses, (1.1), (1.2) will have a solution $y(x, \varepsilon)$ of the form

$$y(x, \varepsilon)=U(x, \varepsilon)+V(\tau, \varepsilon), \tag{3.4}$$

where the outer solution U has an asymptotic expansion

$$U(x, \varepsilon)\sim\sum_{j=0}^{\infty}U_j(x)\varepsilon^j \tag{3.5}$$

providing the asymptotic solution for $x>0$, while the boundary layer correction V has an expansion

$$V(\tau, \varepsilon)\sim\sum_{j=0}^{\infty}V_j(\tau)\varepsilon^j \tag{3.6}$$

whose terms all tend to zero as the stretched variable

$$\tau=x/\varepsilon \tag{3.7}$$

ternds to infinity.

The terms U_0, U_j ($j=1, 2, \cdots$) of outer expansion U must satisfy the reduced problem and linear problems, respectively,

$$f(x, U_0(x), 0)U_0'(x)+g(x, U_0'(x), 0)=0, \ U_0(1)=y(1, 0)=B(0), \tag{3.8}$$

$$f(x, U_0(x), 0)U_j'(x)+f_y(x, U_0(x), 0)U_0'(x)U_j(x)+g_y(x, U_0(x), 0)U_j(x) \\ =h_{j-1}(x), \ U_j(1)=B_j, \tag{3.9}$$

where h_{j-1} is known in terms of x, $U_0(x)$, $\cdots$, $U_{j-1}(x)$; $B_j=\frac{1}{j!}\frac{\partial^j B(\varepsilon)}{\partial \varepsilon^j}\Big|_{\varepsilon=0}$.

The stability assumption (3.2) implies that (3.8) and (3.9) are nonsingular initial value problem, so they have a unique solution thoughout $0\leqslant x\leqslant 1$. Thus, there is no difficulty in generating the outer expansion $U(x, \varepsilon)$ with

$$U(x, 0)=U_R(x)=U_0(x).$$

The terms V_0, V_j $(j=1, 2, \cdots)$ of boundary layer correction V must necessarily be a decaying solution of the nonlinear problem and linear problem, respectively

$$\frac{d^2V_0}{d\tau^2}=f(0, U_0(0)+V_0(\tau), 0)\frac{dV_0}{d\tau}=0, \ V_0(0)=y(0, 0)-U_0(0)=A(0)-U_0(0), \tag{3.10}$$

$$\frac{d^2V_j}{d\tau^2}-f(0, U_0(0)+V_0(\tau), 0)\frac{dV_j}{d\tau}-f_y(0, U_0(0)+V_0(\tau), 0)V_j(\tau)\frac{dV_0}{d\tau} \\ =k_{j-1}(\tau), \ V_j(0)=A_j-U_j(0), \tag{3.11}$$

where k_{j-1} is a linear combination of preceding terms V_l and their derivatives $dV_l/d\tau$, $l<j$, with coefficients that are functions of τ and $V_0(\tau)$, $A_j=\frac{1}{j!}\frac{\partial^j A(\varepsilon)}{\partial \varepsilon^j}\Big|_{\varepsilon=0}$.

The decaying solution of (3.10) must satisfy

$$\frac{dV_0}{d\tau}-\int_\infty^\tau f(0, U_0(0)+V_0, 0)\frac{dV_0}{d\tau}d\tau=0$$

and, thereby, the initial value problem

$$\frac{dV_0}{d\tau}=\int_0^{V_0(\tau)} f(0, U_0(0)+W, 0)dW, \ \tau>0, \ V_0(0)=y(0, 0)-U_R(0). \tag{3.12}$$

Multiplying by V_0^T, the boundary layer stability condition (3.3) implies that

$$\frac{1}{2}\frac{d}{d\tau}\|V_0(\tau)\|^2=V_0^T\int_0^{V_0(\tau)} f(0, U_0(0)+z, 0)dz<0 \tag{3.13}$$

for nonzero values of $V_0(\tau)$ satisfying $\|V_0(\tau)\|\leqslant\|y(0, 0)-U_R(0)\|=\|V_0(0)\|$. Thus, our boundary layer stability implies that $\|V_0(\tau)\|$ will decrease monotonically as τ increases until we reach the rest point $V_0(\tau)=0$ of (3.12) at $\tau=\infty$. Ultimately, $V_0(\tau)$ will become so small that (3.2) (for $x=0$) implies that the eigenvalues of $f(0, U_0(0)+V_0(0), 0)$ will thereafter have real parts smaller than some $-k<0$ and (3.12) then implies that

$$V_0(\tau)=O(e^{-k\tau}), \tag{3.14}$$

i.e. V_0 is exponentially decaying as $\tau\to\infty$. Although we can seldom explicitly integrate the nonlinear system (3.12), we can approximate its solution arbitrarily closely by using a successive approximations procedure on (3.12) (cf. Erde'lyi[12]).

Knowing V_0, we next integrate (3.11) for $j=1$ and then proceed termwise. Rearranging (3.11) and integrating, we obtain

$$\frac{dV_j}{d\tau}-f(0,\ U_0(0)+V_0(\tau),\ 0)V_j+l_j=0,$$

where

$$l_j(\tau)=-\int_\infty^\tau\left\{f_y(0,\ U_0(0)+V_0(r),\ 0)\left[V_j(r)\frac{dV_0}{dr}(r)-\frac{dV_0}{dr}(r)V_j(r)\right]+k_{j-1}(r)\right\}dr$$

is known whenever V_j and $dV_0/d\tau$ commute. Thus, V_j satisfies the integral equation

$$V_j(\tau)=P(\tau)U_j(0)+\int_0^\tau P(\tau)P^{-1}(r)l_j(r)dr, \tag{3.15}$$

where $P(\tau)$ is the exponentially decaying fundamental matrix for the linear system

$$dV/d\tau-f(U_0(0)+V,\ 0,\ 0)V=0,\quad \tau\geqslant 0,\ V(0)=I.$$

In general, (3.15) must also be solved via successive approximations, though it directly provides the solution of (3.11) when the commutator $[V_j,\ dV_0/d\tau]\equiv 0$.

We would have to limit the expansions to finite order approximations, under weaker smoothness assumptions on f and g, and we would have the following result.

Theorem 2. *Assume:*

(a) *the reduced problem* (3.1) *has a* $C^{(m+1)}[0,\ 1]$ *solution* $u_R(x)$ *so that* $f(x,\ u_R(x),\ 0)<0$, *moreover, the inner product*

$$\xi^T\int_0^\xi f(0,\ u_R(0)+s,\ 0)ds<0$$

remains negative for $\xi+u_R(0)$ *along all paths connecting* $u_R(0)$ and $y(0,\ 0)$ *with* $0<\|\xi\|\leqslant\|y(0,\ 0)-u_R(0)\|$;

(b) $f,\ g,\ A$ *and* B *are sufficiently differentiable in*

$$E=\{(x,\ y,\ y',\ \varepsilon):\ 0\leqslant x\leqslant 1,\ \|y-Y_m\|^2\leqslant(C\varepsilon^{m+1})^2,\ y'\in R^n,\ 0<\varepsilon\leqslant\varepsilon_1\},$$

where C *is some positive constant,* $Y_m=\sum_{j=0}^m[U_j(x)+V_j(\tau)]\varepsilon^j$;

(c) *there exists a constant* $\delta>0$ *so that*

$$\frac{\partial f}{\partial y}Y'_m+\frac{\partial g}{\partial y}-\frac{1}{4\varepsilon}(f)(f^*)-\delta I\geqslant 0$$

for all $(x,\ y,\ y',\ \varepsilon)\in E$, *where* f^* *denotes the adjoint of* f;

(d) *the "boundary layer jump"* $\|y(0,\ 0)-u_R(0)\|$ *is sufficient small.*

Then for each ε, $0<\varepsilon\leqslant\varepsilon_1$, *problem* (1.1), (1.2) *has a solution* $y(x,\ \varepsilon)$, *which satisfies*

$$\|y(x,\ \varepsilon)-\sum_{j=0}^m U_j(x)\varepsilon^j-\sum_{j=0}^m V_j(\tau)\varepsilon^j\|^2\leqslant(C\varepsilon^{m+1})^2$$

for $0\leqslant x\leqslant 1$.

Proof Let $z_m(x,\ \varepsilon)=y(x,\ \varepsilon)-Y_m(x,\ \varepsilon)$. From the preceding derivation we can see that the following forms hold:

$$z_m''=f(x,\ z_m+Y_m,\ \varepsilon)(z_m'+Y_m')+g(x,\ z_m+Y_m,\ \varepsilon)$$
$$-f(x,\ Y_m,\ \varepsilon)Y_m'-g(x,\ Y_m,\ \varepsilon) \tag{3.16}$$
$$z_m(0,\ \varepsilon)=O(\varepsilon^{m+1}),\ z_m(1,\ \varepsilon)=O(\varepsilon^{m+1}). \tag{3.17}$$

Fix ε so that $0<\varepsilon\leqslant\varepsilon_1$ and define

$$r(x,\ z_m)=\|z_m\|^2-(C\varepsilon^{m+1})^2$$

for $0\leqslant x\leqslant 1$ and $z_m\in R^n$. Among the hypotheses of Theorem 1, only (a) is not immediate.

For this choice of r, we have $W(z_m)=2z_m$, $V(z_m)=0$ and $P(z_m)=2I$ for all $z_m\in R^n$. Thus, if $r(x,\ z_m)=0$, then we have $\|z_m\|^2=(C\varepsilon^{m+1})^2$, therefore, $\frac{\partial^2 r}{\partial x^2}(x,\ z_m)=0$. Thus' in E we obtain

$$r''=2z_m'\cdot z_m'+2z_m\cdot\frac{1}{\varepsilon}\,[f(x,\ z_m+Y_m,\ \varepsilon)(z_m'+Y_m')$$
$$+g(x,\ z_m+Y_m,\ \varepsilon)-f(x,\ Y_m,\ \varepsilon)Y_m'-g(x,\ Y_m,\ \varepsilon)].$$
$$=2z_m'\cdot z_m'+2z_m\cdot\frac{1}{\varepsilon}\,\{f(x,\ z_m+Y_m,\ \varepsilon)z_m'+[f(x,\ z_m+Y_m,\ \varepsilon)$$
$$-f(x,\ Y_m,\ \varepsilon)](Y_m')+[g(x,\ z_m+Y_m,\ \varepsilon)-g(x,\ Y_m,\ \varepsilon)]\}.$$

By applying the mean value theorem, we have

$$r''=2z_m'\cdot z_m'+2z_m\cdot\frac{1}{\varepsilon}\Big[f(x,\ z_m+Y_m,\ \varepsilon)(z_m')+\frac{\partial f(x,\ Y_m,\ \varepsilon)}{\partial y}(z_m)(Y_m')$$
$$+\frac{\partial g(x,\ Y_m,\ \varepsilon)}{\partial y}(z_m)\Big]=2\left|z_m'+\frac{1}{2\varepsilon}f(x,\ z_m+Y_m,\ \varepsilon)(z_m)\right|^2$$
$$+\frac{2z_m}{\varepsilon}\cdot\Big[\Big(\frac{\partial f}{\partial y}\Big)(Y_m')+\frac{\partial g}{\partial y}-\frac{1}{4\varepsilon}(f)(f^*)\Big](z_m)$$
$$\geqslant\frac{2z_m}{\varepsilon}\cdot\Big[\Big(\frac{\partial f}{\partial y}\Big)(Y_m')+\frac{\partial g}{\partial y}-\frac{1}{4\varepsilon}(f)(f^*)\Big](z_m)\geqslant\frac{2}{\varepsilon}\,\delta\|z_m\|^2\geqslant 0,\quad \delta>0.$$

Theorem 1 applies, and the problem (3.16), (3.17) has a solution $z_m(x,\ \varepsilon)$ which satisfies $r(x,\ z_m(x,\ \varepsilon))<0$ for $0\leqslant x\leqslant 1$, that is, $\|z_m\|^2\leqslant(C\varepsilon^{m+1})^2$, or

$$\left\|y(x,\ \varepsilon)-\sum_{j=0}^{m}U_j\varepsilon^j-\sum_{j=0}^{m}V_j\varepsilon^j\right\|^2\leqslant(C\varepsilon^{m+1})^2.$$

We could also consider the reduced problem

$$f(x,\ u_L,\ 0)u_L'+g(x,\ u_L,\ 0)=0,$$
$$u_L(0)=y(0,\ 0)=A(0).$$

Then the stability condition (3.2) and the boundary layer stability condition (3.3) would be replaced by

$$f(x,\ u_L(x),\ 0)>0$$

for $0\leqslant x\leqslant 1$ and the assumption that

$$\theta^T\int_0^\theta f(1,\ u_L(1)+z,\ 0)\,dz>0$$

for all $\theta+u_L(1)$ on paths between $u_L(1)$ and $y(1,\ 0)$ satisfying $0<\|\theta\|\leqslant\|y(1,\ 0)-u_L(1)\|$. Nonuniform convergence of the solution to (1.1) would then take place near $x=1$, depending on the stretched variable

$$\sigma=(1-x)/\varepsilon,$$

and the limiting solution on $0\leqslant x\leqslant 1$ would be $u_L(x)$.

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