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THE IMPROVEMENT OF THE KNESER THEOREM AND ITS APPLICATIONS

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Abstract

In this paper, first it is proved that on the Möbius strip M there is a unique periodic orbit of the continuous flow f which is the generator of the fundamental group $\pi_1(M)$, where f is tangent (or transversal) to the boundary and has no fixed point on M. Then the results of the Kneser theorem are augmented. On the base of these two results, the classification theorems for M and the Klein bottle are given, which are some more profound than those given in [1]. At last, applying the improved Kneser theorem to some continuous flows on torus, the author gets the results that there exist periodic orbits the number of which is even (at least 2), and describes some qualitative behaviors of the orbits. Moreover, some simple applications to the general nonorientable 2-manifold, particularly to the projective plane, are also mentioned.

§1. Periodic Orbits of Continuous Flow on Möbius Strip

Denote $I=[0, 1]$, $\Gamma=\{(x, y)\,|\,x\in[0, 1],\ y\in[0, 2]\}$ and action α: $(x, y)\mapsto(1-x, y+1)$.

Usually, the quotient space $M=\Gamma/\alpha$ is called a Möbius strip.

In this section, if no specification, we always denote by f the continuous flow on M tangent to the boundary and without fixed points.

Definition 1. *The following seven domains are named normal regions and denoted by* I, II, III, I′, II′, III′, IV′ *respectively.*

I *is homeomorphic to a closed plane annular domain, the inner and outer boundaries are the only two periodic orbits and they have the same positive direction, the ω-limit set and α-limit set of other orbit in the domain are the boundary periodic orbits;*

II *is the same as* I *except the difference that the boundary periodic orbits have the opposite positive directions;*

III *is homeomorphic to a closed plane annular domain filled with periodic orbits;*

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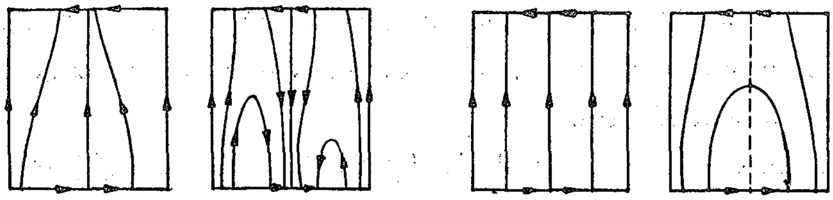

(a) (b) (c) (d)

Fig. 1

I′ *is homeomorphic to a Möbius strip M, the flow f is tangent to the boundary and possesses a unique periodic orbit which is not situated on the boundary, this orbit represents the generator of fundamental group $\pi_1(M)$, it has the same positive direction as the boundary periodic orbit (see Figure 1a);*

II′ *is the same as I′, but the two periodic orbits have the opposite positive direction (see Figure 1b);*

III′ *is homeomorphic to a Möbius strip filled with periodic orbits (see Figure 1c);*

IV′ *is a "half" domain of type II′, corresponding to a nonorientable direction field (see Figure 1d).*

Definition 2. *A periodic orbit L is called a 1-periodic orbit if it represents the generator of $\pi_1(M)$; 2-periodic if it is the square of the generator of $\pi_1(M)$.*

If notice that any continuous flow is orientable, then, by the classification of [1] Ch. 3.6.10, we can get the following theorem.

Theorem 1. *M is divided by the orbits of f into several normal regions, each of them belongs to one of the six types I, II, III, I′, II′, III′.*

The next theorem plays an important role in the paper.

Theorem 2. *If the continuous flow is tangent to the boundary of M and has no fixed points, then there is exactly one periodic orbit representing the generator of $\pi_1(M)$, and the others (at least one on the boundary) must be 2-periodic.*

Proof Suppose there is no 1-periodic orbit, we show it will lead to a contradiction.

Consider the ω-limit set $(\omega(\gamma))$ and α-limit set $(\alpha(\gamma))$ of arbitrary one orbit γ which is not situated on the boundary. Since there is no fixed points and non-trivial recurrent orbit, both $\omega(\gamma)$ and $\alpha(\gamma)$ are periodic orbits. If $\omega(\gamma)\neq\alpha(\gamma)$, there is at least one periodic orbit which is different from the boundary; if $\omega(\gamma)=\alpha(\gamma)$, then $\omega(\gamma)$ is a periodic orbit different to the boundary.

Bacause there are only two kinds of non-null-homotopic Jordan curve on M: either the generator of $\pi_1(M)$ or the square of the generator, it follows that, from the above assumption, there is a 2-periodic orbit which is different from the boundary and divides M into a cylinder and a Möbius strip M_1.

Repeating the above discussion on M_1, we get another 2-periodic orbit which splits up M_1 into a cylinder and a Möbius strip $M_2\cdots$.

Repeatedly, we obtain a sequence of Möbius strips: $M\supset M_1\supset M_2\supset\cdots\supset M_n\supset\cdots$. Denote their intersection by A_ω. The set A_ω is non-empty, compact, closed and invariant, since each M_i does.

Because there exists a minimal set of f on A_ω and, from [3] or [6] (p. 316), it must be a periodic orbit, we see there is a periodic orbit $L_\omega\subset A_\omega$ and, by the assumption of no 1-periodic orbit, it is 2-periodic.

For any positive integer i, the closure of M_i-L_ω consists of a cylinder and a Möbius strip M_ω. It followsa that $M_\omega\subset A_\omega$.

Proceeding again as above, we can get another Möbius strip $M_{\omega+1}\subset M_\omega$.

Let β be a transfinite number. Suppose we have got a Möbius strip M_α for each $\alpha<\beta$. Denote $A_\beta=\bigcap_{\alpha<\beta} M_\alpha$. For the same reason as above, we have another Möbius strip $M_\beta\subset A_\beta$.

Thus, we get a transfinite sequence of Möbius strips contained one by one: $M\supset M_1\supset\cdots\supset M_n\supset\cdots\supset M_\omega\supset M_{\omega+1}\supset\cdots\supset M_\beta\supset\cdots$.

It follows that, from the Cantor-Baire theorem ([4] p. 458), there is a transfinite number μ of second class, such that $M_\mu=M_{\mu+1}=M_{\mu+2}=\cdots$.

But the preceding proof says that there should be a Möbius strip $M^*\subset M_\mu$ and $M^*\neq M_\mu$.

This contradiction implies that there exists at least one 1-periodic orbit on M. On the other hand, by [1], there is at most one 1-periodic orbit on M, so the theorem follows immediately.

Corollary 1. *Every continuous flow transversal to the boundary ∂M has exactly one 1-periodic orbit, if without fixed points.*

Proof It is easy to see that M is invariant in either positive or negative direction. So there must exist a periodic orbit L. Since L is non-null-homotopic, we may as well assume L is 2-periobic, and it cuts down a Möbius strip M_1 from M. Since the restriction of f on M_1 is well defined, we see that, by Theorem 2, there is exactly one 1-periodic orbit of f on M_1. The uniqueness of 1-periodic orbit on M follows from the proof of Theorem 2.

Remark 1. Corollary 1 strengthens the results of the Proposition 6.9 of [1] (p. 126). From the above proof, moreover, we can see that Corollary 1 still holds if there exists a point $x\in\partial M$, such that $\omega(x)\subset M$ or $\alpha(x)\subset M$.

Similarly, we can apply Theorem 2 to strengthen the conclusion of [1] Ch. 3.6.10, IV): "there is at most one component which belongs to one of the four types I′, II′, III′ and IV′ (corresponding to nonorientable flow)".

Corollary 2. *A direction field tangent to boundary and without fixed point divides M into some normal regions, exactly one of them belongs to one of the types I′, II′, III′ and IV′. The necessary and sufficient condition under which normal regions of* *types* I, II *and* III *exist is the existence of* 2–*periodic orbit different to the boundary*.

Protf Since only regions of types I′, II′, III′ can contain 1–periodic orbit, the corollaoy follows directly from Theorem 2 and its proof if f is orientable. Now suppose that f is nonorientable. It is easy to see that there is exactly one normal region of the type IV′, and the others are of types I, II, III. Moreover, there exist regions of types I, II and III if and only if the boundary of region IV′ is different from ∂M.

In the section 2, we will apply Theorem 2 to improve the Kneser theorem and sharpen the classification of [1] Ch. 4.2.8.

§ 2. The Improvement of the Kneser Theorem and the Classification on K^2

For $u, v\in\mathbb{R}$, define transformations on $\mathbb{R}^2$:

$$h: (u, v)\mapsto(u+1, v),\ k: (u, v)\mapsto(-u, v+1).$$

Denote by G the transformation group on $\mathbb{R}^2$ generated by h and k. The quotient space $K^2=\mathbb{R}^2/G$ (the orbit space of group G) is called Klein bottle.

For arbitrary integers m, n, p, q, there exists the following relationship:

$$(k^m h^n)(k^p h^q)=k^{m+p}h^{(-1)^p n+q},$$
$$(k^m h^n)^{-1}=k^{-m}h^{(-1)^{m+1}n}. \tag{1}$$

G is isomorphic to the fundamental group $\pi_1(K^2)$ (cf. [1] Ch. IV, p. 137).

Let T^2 be the two–fold covering space of K^2 corresponding to the normal subgroup of $\pi_1(K^2)$ generated by h, k^2. Then T^2 is a torus obtained by identifying the opposite sides of the rectangle $\left[-\frac{1}{2}, \frac{1}{2}\right]\times[-1, 1]$.

Suppose f is a continuous flow without fixed points defined on K^2. Denote by $\tilde{f}$ the flow on T^2 induced by the two–fold covering map: $T^2\to K^2$ from f.

By the Kneser Theorem and the Lemma 3 of [1] Ch. 4.2.3, we have the following lemma.

Lemma 1. *Every continuous flow without fixed points on* K^2 *has periodic orbits, as a non–null–homotopic Jordan curve, each of them represents one of the following elements of* $\pi_1(K^2)$: h, h^{-1}, k^2, k^{-2}, $k^{\pm1}h^n$ (cf. Figure 2).

From [1] Ch. 4.2.8 we have the following lemma.

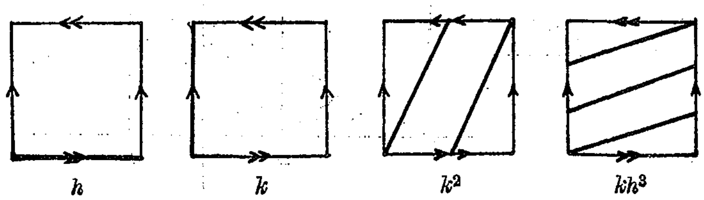

Fig. 2

Lemma 2. *If f is a continuous flow without fixed points on K^2, then K^2 is divided into several normal regions by the periodic orbits (one is denoted by L) of f. If L represents the element $h^{\pm1}$ of $\pi_1(K^2)$, then these regions are of types* I, II, III; *if L is of class $k^{\pm2}$, then they may be of types* I, II, III, I′, II′ *and* III′. *When there exist exactly two periodic orbits and both belong to the class $k^{\pm1}h^n$, the normal region has type* I′ *or* II′.

In this section, we always assume that we have defined a continuous flow without fixed points on K^2. Now we apply Theorem 2 to prove the main result of the paper.

Theorem 3. *If there is no periodic orbits of class $h^{\pm1}$, then there exists a unique integer s, such that there are exactly two periodic orbits of class $k^{\pm1}h^s$, moreover, the other periodic orbits (if exist) must represent class $k^{\pm2}$.*

Proof By Lemma 1 and the hypothesis of the theorem, we may assume that there exists a periodic orbit L of class either $k^{\pm2}$ or $k^{\pm1}h^n$.

If L belongs to class $k^{\pm2}$, then K^2 is cut up by L into two Möbius strips M_1 and M_2. From Theorem 2, it follows that there are two periodic orbits L_1 and L_2, and L_i is the generator of $\pi_1(M_i)$ for $i=1, 2$. By Lemma 1, L_i represents the class $k^{\pm2}h^{n_i}$. Thus we may as well assume that L is of class $k^{\pm1}h^n$.

Denote by T the regular covering space of K^2 corresponding to subgroup G_1 generated by k^{+1}, i.e., T is the quotient space of $\mathbb{R}^2$ under the action of group G_1. As described in Figure 3, T is the quotient space of the strip region $\{(x, y)\,|\,0\leqslant y\leqslant 1\}$ under the action k, and K^2 is the quotient space of the unit square

$$\left\{(x,y)\,\middle|\,x\in\left[-\frac{1}{2},\frac{1}{2}\right],\ y\in[0,1]\right\}$$

under the action k. L_0 and L_1 ($L_1=hL_0$) are two preimages of L under the covering map π: $T\to K^2$. The shadowy region M in Figure 3 is a Möbius strip. It is easy to see from the figure that π: $M-L_0\cup L_1\to K^2-L$ is a homeomorphism. Therefore, to research the periodic orbits on K^2 except L we need only to consider the periodic orbits different from the boundary of M. Now Theorem 2 says that there exists exactly one 1-periodic orbit L_3 on M, and the other periodic orbits (denoted by L_λ), if exist, must be 2-periodic. Since the abscissa increases by n along L_3, the image

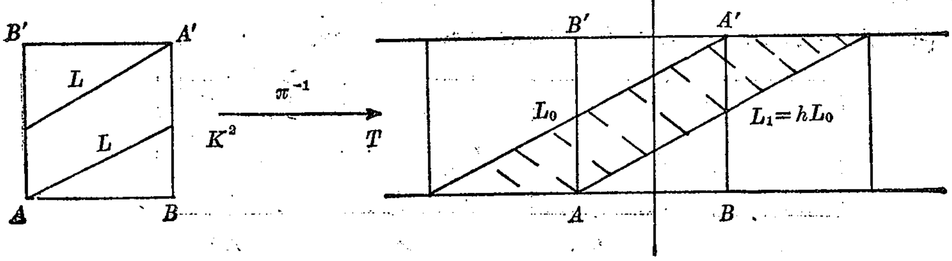

Fig. 3

$\pi(L_3)$ of L_3 on K^2 belongs to class $k^{\pm 1}h^n$. And since L_λ is homotopic to $L_0 \cup L_1$ on M, the image $\pi(L_\lambda)$ is of class $(k^{\pm 1}h^n)^2 = k^{\pm 2}$ (cf. formula (1)). Thus we have completed the proof of the theorem.

Now we have obtained a strengthened Kneser Theorem:

Theorem 4. *If f is a continuous flow without fixed points on the Klein bottle K^2, then there must exist periodic orbits and each of them represents one of the elements of $\pi_1(K^2)$: $h^{\pm 1}$, $k^{\pm 2}$, $k^{\pm 1}h^n$ $(n \in Z)$. Moreover, if f has no periodic orbits of class $h^{\pm 1}$, then there exists a unique integer s such that f has exactly two periodic orbits of class $k^{\pm 1}h^s$, and if there exists any other periodic orbits else, it must be of class $k^{\pm 2}$.*

Remark 2. The following two examples show respectively that a continuous flow f without fixed points on K^2 may have a unique periodic orbit of class $h^{\pm 1}$ (must be two-sided and half stable) or arbitrary number of periodic orbits of class $k^{\pm 2}$ besides two $k^{\pm 1}h^s$ periodic orbits when f has no $h^{\pm 1}$ periodic orbits.

Example 1

$$\dot{\theta} = t(1-t), \quad \dot{t} = -\left(t - \frac{1}{2}\right)^2, \tag{2}$$

θ, $t \in [0, 1]$. If we identify $(0, t)$ and $(\theta, 0)$ with $(1, t)$ and $(1-\theta, 1)$ respectively, then system (2) defines a continuous flow on Klein bottle which has a unique and half stable periodic orbit.

Example 2 Construct an oppositely directed homeomorphism f on $S^1 = \mathbb{R}/Z$: $f(x) = 1 - x$, for any $x \in [0, 1]$, where we identify 0 with 1.

Obviously, $f^2(x) = f(f(x)) = f(1-x) = x$, $\forall x \in [0, 1]$, and only points 0 and 1/2 satisfy $f(x) = x$. Thus all the points on S^1 but the fixed points 0 and 1/2 are 2-periodic points of f.

Taking a fundamental square $\{(x, y) \mid 0 \leqslant x, y \leqslant 1\}$ on $x-y$ plane, and identifying the points $(0, y)$ with $(1, y)$, we get a cylinder H. Let S_1^1, S_2^1 be the upper and lower boundary circle respectively, and x the coordinate of S_1^1 and S_2^1. Now we first twist the generatrix of H n cycles counter clockwise and keep H itself invariant, then identify the points of x on S_1^1 with $f(x)$ on S_2^1, such that H becomes a Klein bottle K^2. Clearly, the generatrix of H defines a continuous flow on K^2, all orbits are closed and belong to class $k^{\pm 2}$ as well as two $k^{\pm 1}h^n$ periodic orbits.

By a similar way, for any given integers $N \geqslant 0$ and s, we can construct a continuous flow on K^2, such that there exist exactly $N+2$ periodic orbits, where two are of class $k^{\pm 1}h^s$ and N pieces belong to class $k^{\pm 2}$.

Using Theorem 4 and Lemma 2, we have the following classification theorem for continuous flows without fixed points on K^2.

Theorem 5. *Suppose f is a continuous flow without fixed points on K^2. Then K^2 is divided by the orbits of f into several normal regions, and the set F of all these normal regions has one of the following four types:*

1) *F only consists of the regions of types* I, II *and* III, *and at least one is of type* II *(corresponding to the case that f only has $h^{\pm 1}$ periodic orbits)*;

2) *F only contains one element and it belongs to type* I′ *or* II′ *or* III′ *(the first two corresponding to the case that f has no $h^{\pm 1}$ and $k^{\pm 2}$ periodic orbits, the last one corresponding to K^2 filled with periodic orbits)*;

3) *F has exactly two elements and they are either of types* I′ *and* II′ *or of type* III′ *(corresponding to the case that $k^{\pm 2}$ periodic orbits are either unique or filled with a Möbius strip)*;

4) *F has exactly two elements of types* I′, II′ *and* III′, *and at least one element of types* I, II *and* III *(corresponding to other cases)*.

Proof Case 1. Because each region of types I′, II′, III′ contains a $k^{\pm 1}h^n$ periodic orbit, F only can consist of elements of types I—III when f has only $h^{\pm 1}$ periodic orbits (denote one by L). Evidently, K^2-L is a cylinder, and both periodic orbits situated at the upper and lower boundaries have opposite positive directions. Hence F possesses at least one normal region II.

The Case 2 is intuitive.

Case 3. If we cut up K^2 along the unique (or the boundary of the Möbius strip filled with $k^{\pm 2}$ periodic orbits) $k^{\pm 2}$ periodic orbit into two Möbius strips, then the assertion follows directly.

Case 4. Now K^2 has at least two $k^{\pm 2}$ periodic orbits L_1, L_2, which are not contained in the same one normal region III′. K^2 is divided by L_1 into two Möbius strips M_1 and M_2, and one of them is cut up by L_2 into a Möbius strip and a cylinder. Thus we see F has at least one element belonging to one of the types I—III. Since a region of types I, II, III does not contain $k^{\pm 1}h^n$ periodic orbit, F has exactly two elements of types I′—III′.

Corollary 3. *If there is no $h^{\pm 1}$ periodic orbits, then K^2 contains at least one and at most two normal regions of types* I′—III′.

§3. Application

In this section we apply Theorem 4 to a class of differential equations defined on the torus, and give some simple applications of Theorem 2, Corollaries 1 and 2 to nonorientable surfaces, particularly, to the projective plane.

We still take the notations K^2, f, T^2, $\tilde{f}$ used in section 2, where f is a continuous flow without fixed points on K^2. Denote by L the periodic orbit of f and $\tilde{L}$ the lifting of L on T^2.

Lemma 3. *Let $k^s h^t$ and (m, n) represent the elements of $\pi_1(K^2)$ and $\pi_1(T^2)$ respectively, where $s, t, m, n \in Z$. If L represents $h^{\pm 1}$, then L has two preimages $\tilde{L}_1$, $\tilde{L}_2$ on T^2 and they represent the same element $(1, 0)$; if L represents $k^{\pm 2}$, then L has also* *two preimages* $\tilde{L}_1$, $\tilde{L}_2$ *on* T^2 *representing* $(0, 1)$; *if* L *represents* $k^{\pm 1}h^n$, *then* L *has only one preimage* $\tilde{L}$ *on* T^2 *and it is of class* $(0, 1)$.

Proof The first two assertions are obvious, we only prove the last one. Since T^2 is a regular two-fold cover of K^2, the lifting of L on T^2 is equal to the square of L. By formula (1), we have $(k^{\pm 1}h^n)^2 = k^{\pm 2}$. If we notice that the transformation $k^{\pm 2}$ acting on T^2 is equivalent to the transformation defined by the Jordan curve of class $(0, 1)$, the lemma follows immediately.

Lemma 4. $\tilde{f}$ *has non-zero even number of periodic orbits on* T^2 *and they represent one of the generators of* $\pi_1(T^2)$. *Moreover,*

(i) *If* $\tilde{f}$ *has* $(1, 0)$ *periodic orbits, then* T^2 *contains at least two normal regions of type* II;

(ii) *If* $\tilde{f}$ *has exactly two* $(1, 0)$ *periodic orbits, then* T^2 *consists of exactly two regions of type* II;

(iii) *The necessary condition under which* T^2 *consists of finite number of normal regions and each of them belongs to type* II *is that* $\tilde{f}$ *has exactly* $2k$ *periodic orbits, where* k *is odd;*

(iv) *If* $\tilde{f}$ *has* $(0, 1)$ *periodic orbits, then there is a nonnegative integer* s *such that every periodic orbit winds around the torus first* s *cycles in positive direction, then again* s *cycles in negative direction, or in the reverse order. The* $(1, 0)$ *periodic orbit of* $\tilde{f}$ *has not any spiral phenomenon.*

Proof The first part of the conclusion is a direct consequence of Theorem 4 and Lemma 3. The (i) and (ii) in the second part of the conclusion are intuitive, we only need to notice that $\tilde{L}_1$, $\tilde{L}_2$ (the same meaning as in Lemma 3) have opposite directions and the boundary periodic orbits of regions I and II have the same positive direction.

The assertion (iii) follows from the fact that if k is even, then there must exist region either of type I or of type III. For simplicity, we may assume $k=2$. Now there are periodic orbits $\tilde{L}_{ij}$ on T^2, $i, j=1, 2$, such that $\pi(\tilde{L}_{ij}) = L_i$, where L_1 and L_2 are $h^{\pm 1}$ periodic orbits on K^2. Since the positive direction of $\tilde{L}_{i1}$ is opposite to that of $\tilde{L}_{i2}$, there exist at least two (exactly two as $k=2$) regions of types I and III, no matter what positive directions $\tilde{L}_{11}$ and $\tilde{L}_{21}$ take.

The assertion (iv) can be deduced from Lemma 3, Theorem 3 and its proof.

Let X be a $C^r (r \geqslant 0)$ vector field on T^2 defined as following.

$X = (X_1, X_2)$, where X_1 and X_2 have no common zero point, and satisfy the condition H_1:

$$X_1(x, y) = -X_1(-x, y+1), \quad X_2(x, y) = X_2(-x, y+1),$$

$$X\left(-\frac{1}{2}, y\right) = X\left(\frac{1}{2}, y\right), \quad X(x, -1) = X(x, 1),$$

for any $x\in\left[-\frac{1}{2}, \frac{1}{2}\right]$, $y\in[-1, 1]$.

By Lemma 4, we get the following theorem.

Theorem 6. *Suppose Y is a $C^r(r\geqslant 0)$ vector field on T^2 without singular points, and topologically conjugate to X, then Y has non-zero even number of homotopic periodic orbits, and they represent the generator of $\pi_1(T^2)$. Moreover, the flow $\tilde{f}$ defined by Y has the same behavior and structure as described in Lemma 4.*

Example 3

$$\begin{aligned}\dot{x}&=\beta\sin\pi x\pm\gamma\sin\pi y,\\ \dot{y}&=1+\alpha\sin\pi x\cos\pi y,\end{aligned}\tag{3}$$

where

$$|\alpha|\leqslant 1 \text{ and } \beta\neq 0 \text{ as } |\alpha|=1.$$

$$\begin{aligned}\dot{x}&=\beta\sin\pi x-\gamma\cos\pi y,\\ \dot{y}&=1+\alpha\sin\pi x\cos\pi y,\end{aligned}\tag{4}$$

where $|\alpha|\leqslant 1$ and $\beta+\gamma\neq 0$ as $\alpha=1$; $\beta-\gamma\neq 0$ as $\alpha=-1$.

It is easy to see that under the given conditions the systems (3) and (4) are smooth, have no singular points, and satisfy condition H_1 (change x into $2x$). Hence, for each system there exists at least a couple of periodic orbits.

Since $\dot{y}\geqslant 0$ and it takes zero value only at several points, the flows defined by (3) and (4) have not any (1, 0) periodic orbits. And from $\dot{y}=1$ as $|y|=\frac{1}{2}$, we see that every periodic orbit has the same positive direction. It follows that there are not any regions of type II on T^2.

At last, we give an application of Theorem 2 and Corollaries 1 and 2. Each nonorientable closed surface with genus g may be regarded as a sphere with g holes to each of them glued one Möbius strip[5]. The next theorem follows directly from Theorem 2 and its proof.

Theorem 7. *Suppose f is a continuous flow defined on a nonorientable surface M^2. If there exists a closed region $D\subset M^2$ homeomorphic to a Möbius strip, such that f has no fixed points situated in D and region D contains a half-orbit of f, then there is a one-sided periodic orbit of f on M^2.*

Particularly, when the genus of M^2 is 1, we have the following corollary.

Corollary 4. *Suppose f is a continuous flow defined on a projective plane P^2. If there exists a two-sided Jordan curve L such that the fixed points of f are situated at the same side of L, then the condition that L is either a periodic orbit or a closed transverse implies that there exists a unique non-null-homotopic periodic orbit on P^2.*

Proof By [6] Lemma 4(iii), L is a null-homotopic Jordan curve and bounds a simple connected region D. Hence, P^2-D is a Möbius strip. Since there must exist fixed points in D whether L is a closed orbit or transverse, f has no fixed points in

P^2-D. Thus the existence of non-null-homotopic (equivalent to one-sided) periodic orbit L^* is a direct consequence of Theorem 7. The uniqueness follows from the fact that P^2-L^* is homeomorphic to a disk.

Remark 3. In [8], we have shown that every continuous flow defined on a 2-manifold (projective plane) has either (one-sided) periodic orbits or singular closed orbits if the set of cluster points of the fixed points is countable.

Theorem 8. *If a continuous flow f defined on projective plane P^2 has unique and elementary fixed point, then only one of the following two cases can occur:*

i) *f has unique and non-null-homotopic periodic orbit, the whole P^2 is a simple connected spiral region*[7];

ii) *f has at least one null-homotopic periodic orbit besides a non-null-homotopic one. P^2 is cut up by the orbits of f into several normal regions, one is a simple connected spiral region, one is of the type either* I′ *or* II′ *or* III′, *and the others are of types* I, II, III. *There exist regions of types* I, II, III *on P^2 if and only if there exist at least two null-homotopic periodic orbits.*

Proof Let S be the unique, elementary fixed point of f. We may assume S is a source, since the Euler characteristic number of P^2 is 1. We can easily construct a closed transverse T around S, and see that P^2-T consists of a disk and a Möbius strip. Then Corollary 1 implies that f has exactly one one-sided, non-null-homotopic periodic orbit L.

If L is the unique periodic orbit of f, then P^2 wholely is a normal region—a simple connected spiral region. In fact, if we denote by Σ the single point compactization of P^2-L, $\chi(\Sigma)$ the Euler characteristic number, then from the formula

$$\chi(\Sigma)=\chi(P^2-L)+1=\chi(P^2)+1=2,$$

we see Σ is a sphere, so P^2-L is homeomorphic to a disk.

Now suppose that there exists at least one two-sided periodic orbit L_1 as well as L. It follows that, from the proof of Corollary 4, L_1 is null-homotopic and bounds a unique simple connected closed region D. We may assume there is no periodic orbits at the inner of D. Then D is a spiral region, and $M=(P^2-D)\cup L_1$ is a Möbius strip. The flow f is tangent to $\partial M=L_1$. Then the theorem is a consequence of Corollary 2, if we notice that every two-sided Jordan curve on P^2 is null-homotopic and every 2-periodic orbit on M is two-sided.

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