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THE EXPECTATION OF STOPPED SEMIAMARTS AND REGULARITY AND CLOSENESS OF AMARTS

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Abstract

In this paper the author proves that the conditions (D): $\sup_n E_n|x_n|<\infty$ and (C): $E|x_\tau|I_{(\tau<\infty)}<\infty$, $\forall\tau\in\overline{T}$ are equivalent, if $(x_n, \mathscr{F}_n)$ is a semiamart, where $\overline{T}$ is the set of all stopping times. Therewith the regularity and closeness of amarts are discussed, some known results concerning the martingales are shown to remain valid for amarts.

§ 1. Introduction

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $(\mathscr{F}_n)_{n\geq1}$ an increasing sequence of sub-σ-algebras of $\mathscr{F}$. We denote respectively by $\overline{T}$, T_f and T the sets of all stopping times, a.s. finite stopping times and bounded stopping times. An integrable sequence $X=(x_n)_{n\geq1}$ of (real-valued) r.v.'s adapted to $(\mathscr{F}_n)_{n\geq1}$ is denoted by $(x_n, \mathscr{F}_n)_{n\geq1}$ or $(x_n, \mathscr{F}_n)$. Yamasaki[5] indicated that if $(x_n, \mathscr{F}_n)$ is a martingale then the conditions (D): $\sup_n E|x_n|<\infty$ and (C): $E|x_\tau|I_{(\tau<\infty)}<\infty$, $\forall\tau\in T$ are equivalent, but not equivalent for mils or sub (super) martingales. This naturally raises the question whether the condition of martingale can be weakened? An integrable adapted sequence $(x_n, \mathscr{F}_n)$ is said to be a semiamart if $\sup|Ex_\tau|<\infty$[4]. The following discussion shows that if $(x_n, \mathscr{F}_n)$ is a semiamart then the conditions (D) and (C) are also equivalent. Therewith the regularity and closeness of amarts are discussed.

§ 2. The Expectation of Stopped Semiamarts

Put

$$T(\sigma)=\{\tau\colon \tau\in T,\ \tau\geq\sigma\},\quad \forall\sigma\in T,$$
$$T_f(\sigma)=\{\tau\colon \tau\in T_f,\ \tau\geq\sigma\},\quad \forall\sigma\in T_f.$$

Definition 1. *An integrable adapted sequence $(x_n, \mathscr{F}_n)$ is a sub(super)*

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semiamart if $\sup_T (Ex_\tau)^{-(+)} < \infty$.

Lemma 1. *Suppose* $(x_n, \mathscr{F}_n)$ *is a subsemiamart. Then* $x_n = y_n + z_n$, $n \geqslant 1$, *where* $(y_n, \mathscr{F}_n)$ *is a submartingale*, $z_n \geqslant 0$ *a.s.. If* $\underline{\lim}_T Ex_\tau < \infty$, *then* $\underline{\lim}_T Ez_\tau = 0$ *and* $\underline{\lim}_n z_n = 0$ *a.s..*

Proof. For any n, set

$$y_n = \operatorname*{ess\,inf}_{\tau \in T(n)} E(x_\tau | \mathscr{F}_n), \quad z_n = x_n - y_n.$$

Then $(y_n, \mathscr{F}_n)$ is a submartingale, $z_n \geqslant 0$ a.s. and $\underline{\lim}_T Ex_\tau = 0$ if $\underline{\lim}_T Ex_\tau < \infty$. Therefore $\underline{\lim}_n z_n = 0$ a.s. by Proposition 4.2 of [3] and the proof is finished.

Lemma 2. *Suppose* $(x_n, \mathscr{F}_n)$ *is a subsemiamart. If* $\sup_n Ex_n^- = \infty$, *then*

$$\underline{\lim}_n Ex_n^+ = \infty.$$

Proof It follows from Lemma 1 and Theorem 3.6 of [7] that $x_n = y_n + z_n$, $n \geqslant 1$ and $\sup_n Ex_n^+ = \infty$, where $(y_n, \mathscr{F}_n)$ is a submartingale and $z_n \geqslant 0$ a.s.. Since $x_n \geqslant y_n$, we have $\sup_n Ey_n^- = \infty$. But $(y_n, \mathscr{F}_n)$ is a submartingale. Hence $\sup_T (Ey_\tau)^- < \infty$. Thus $\sup_n Ey_n^+ = \infty$ by Theorem 3.6 of [7]. Since $(y_n^+, \mathscr{F}_n)$ is also a submartingale, it follows that $Ey_n^+ \uparrow (n\uparrow)$, and $\underline{\lim}_n Ey_n^+ = \infty$. Therefore $\underline{\lim}_n Ex_n^+ = \infty$ and the assertion holds.

Lemma 3. *Suppose* $(x_n, \mathscr{F}_n)$ *is a subsemiamart. Then for any* k *and any* $B \in \mathscr{F}_k$, $\sup_{\tau \in T(k)} (Ex_\tau I_B)^- < \infty$.

Proof Fix $B \in \mathscr{F}_k$. It follows from Lemma 1 that $x_n I_B = y_n I_B + z_n I_B$, $n \geqslant 1$ and $(y_n I_B, \mathscr{F}_n, n \geqslant k)$ is a submartingale. Hence

$$\sup_{T(k)} (Ex_\tau I_B)^- \leqslant \sup_{T(k)} (Ey_\tau I_B)^- < \infty$$

and the proof is completed.

Theorem 4. *Suppose* $(x_n, \mathscr{F}_n)$ *is a subsemiamart. If* $\sup_n Ex_n^- = \infty$, *there is* $\tau \in \overline{T}$ *such that* $\int_{(\tau < \infty)} |x_\tau| = \infty$.

Proof Put $C_0 = \Omega$. Since $\underline{\lim}_n Ex_n^+ = \infty$ by Lemma 2, we can choose n_1 such that

$$\int_{C_0} x_{n_1}^- \geqslant 1, \int_{C_0} x_{n_1}^+ \geqslant 1.$$

Hence A_1, B_1, C_1, D_1 can be chosen as the proof of Theorem of [1]. If we define n_k, A_k, B_k, C_k and D_k for integer $k > 1$ by induction, then choose n_{k+1}, A_{k+1}, B_{k+1}, C_{k+1} and D_{k+1} as follows. Since $\sup_n \int_{C_k} x_n^- = \infty$, it follows that $\underline{\lim}_n (Ex_n^+ I_{C_k}) = \infty$ by Lemmas 3 and 2. Hence there is $n_{k+1} > n_k$ such that

$$\int_{C_k} x_{n_{k+1}}^- \geqslant 1, \quad \int_{C_k} x_{n_{k+1}}^+ \geqslant 1.$$

Thus we can choose A_{k+1}, B_{k+1}, C_{k+1}, D_{k+1} and the theorem can be proved in the same way as the proof of Theorem [1].

Lemma 5. *Suppose* $(x_n, \mathscr{F}_n)$ *is an integrable adapted sequence. If* $\sup_T Ex_\tau^+ < \infty$, *then* $\int_{(\tau<\infty)} x_\tau^+ < \infty$ *for any* $\tau \in \overline{T}$.

Proof Assume the contrary, that is, there exists $\tau \in \overline{T}$ such that

$$\int_{(\tau<\infty)} x_\tau^+ = \sum_{k=1}^{\infty} \int x_k^+ I_{(\tau=k)} = \infty.$$

Then for any n there exists k_n such that

$$\sum_{k=1}^{k_n} \int x_k^+ I_{(\tau=k)} \geqslant n.$$

Let $\tau_n = \tau \wedge k_n$. Then $\tau_n \in T$ and $Ex_{\tau_n}^+ \geqslant n$. We get $\sup_n Ex_{\tau_n}^+ = \infty$. This yields a contradiction and the proof is finished.

Now we can apply Theorem 4 and Lemma 5 to semiamarts.

Theorem 6. *Suppose* $(x_n, \mathscr{F}_n)$ *is a semiamart. Then conditions* (D): $\sup_n E|x_n| < \infty$ *and* (C): $\int_{(\tau<\infty)} |x_\tau| < \infty$, $\forall \tau \in \overline{T}$, *are equivalent.*

Proof (D)$\Rightarrow$(C). If $(x_n, \mathscr{F}_n)$ is an L^1-bounded semiamart, then $\sup_T E|x_\tau| < \infty$ by Proposition 1.3 of [2]. Therefore (C) holds by Lemma 5.

(C)$\Rightarrow$(D). Assume that (D) fails, that is, $\sup_n E|x_n| = \infty$. If $\sup_n Ex_n^- = \infty$, by Theorem 4 there exists $\tau \in \overline{T}$ such that $\int_{(\tau<\infty)} |x_\tau| = \infty$. This contradicts (C), and $\sup_n Ex_n^- < \infty$ holds. $\sup_n Ex_n^+ < \infty$ can be showed by the above assertion for $(-x_n, \mathscr{F}_n)$ and the proof ends.

Suppose $X = (x_n, \mathscr{F}_n)$ is an integrable adapted sequence. Let

$$T_f(X) = \{\tau\colon \tau \in T_f,\ Ex_\tau \text{ exists}\}.$$

The sets $\overline{T}$, T_f and T are directed sets to the right. But in general the ordered set $T_f(X)$ does not have this property[3]. Obviously the case of $T_f(X) = T_f$ is interesting.

Corollary 6.1. *If* $X = (x_n, \mathscr{F}_n)$ *is an* L^1*-bounded semiamart, then* $T_f(X) = T_f$.

Corollary 6.2. *If* $X = (x_n, \mathscr{F}_n)$ *is an* L^1*-bounded amart, then* $\lim_{T_f} Ex_\tau$ *exists and is finite.*

The proof is obvious from Corollary 6.1 and Theorem 4.4 of [3].

Definition 2. *An integrable adapted sequence* $X = (x_n, \mathscr{F}_n)$ *is said to be a* $T_f(X)$*-amart, if* $T_f(X)$ *is a directed set and* $\lim_{T_f(X)} Ex_\tau$ *exists* (finite).

Krengel and Sucheston[4] indicated that there are L^1-bounded semiamarts converging to zero in L^1 and a.s. which are not amarts. This naturally raises the question: what are these sequences? The assertion given below answers this question.

Theorem 7. *If* $X=(x_n, \mathscr{F}_n)$ *is an* L^1*-bounded semiamart, then that* (1) $\lim\limits_n x_n$(a.s.) *exists and* (2) $(x_n, \mathscr{F}_n)$ *is a* T_f*-amart are equivalent.*

Proof (1)$\Rightarrow$(2). Since $T_f(X)=T_f$ by Corollary 6.1, it follows from Proposition 4.3 and Theorem 2.1 of [3] that $\lim\limits_{T_f} Ex_\tau = E(\lim\limits_n x_n)$. So $(x_n, \mathscr{F}_n)$ is a T_f-amart.

(2)$\Rightarrow$(1). Since $\sup\limits_n E|x_n|<\infty$, $E(\underline{\lim}_n\, x_n)$ and $E(\overline{\lim}_n\, x_n)$ exist. Therefore $\lim\limits_n x_n$ exists and $\lim\limits_n x_n \in L^1$ by Theorem 3.1 of [3].

§ 3. The Closeness and Regularity of Amarts

Amarts are an important generalization of martingales. The great majority of fundamental properties possessed by martingales remain true for amarts. Now the closeness and regularity of amarts are discussed.

Define $x_\infty = \overline{\lim}_n\, x_n$ for a sequence (x_n) of r.v.'s.

Definition 3. *An amart* $(x_n, \mathscr{F}_n)$ *is called a right-closed amart, if*

$$\lim_n \int |E(x_\infty|\mathscr{F}_n)-x_n|=0.$$

Definition 4. *An amart* $(x_n, \mathscr{F}_n)$ *is called a* T_f*-regular amart, if*

$$\overline{\lim_{T_f}}\, E|x_\tau| \geqslant \overline{\lim_{T}}\, E|x_\sigma|.$$

Obviously, the regularity and closeness of amarts are weaker than the regularity and closeness of martingale.

Theorem 8. *Suppose* $(x_n, \mathscr{F}_n)$ *is an* L^1*-bounded amart. Then the following assertions are equivalent.*

(1) $(x_n, \mathscr{F}_n)$ *is a* T_f*-regular amart.*

(2) $\lim\limits_T E|x_\tau| = E|x_\infty|$.

(3) $(x_\tau)_{\tau\in T}$ *is uniformly integrable.*

(4) $(x_{\tau\wedge n}, \mathscr{F}_{\tau\wedge n})$ *is a right-closed amart for any* $\tau\in T_f$.

(5) (x_n) *is uniformly integrable.*

(6) $(x_n, \mathscr{F}_n)$ *is a right-closed amart.*

(7) *There exists* $y\in L^1$ *such that* $\lim\limits_n \int |E(y|\mathscr{F}_n)-x_n|=0$.

Proof (3)$\Rightarrow$(5)$\Rightarrow$(6)$\Rightarrow$(7)$\Rightarrow$(5) are obvious. We only have to prove (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4)$\Rightarrow$(1) and (5)$\Rightarrow$(1). Now $x_\infty=\lim\limits_n x_n$ and $x_\infty\in L^1$ by the convergence theorem of amarts[2].

(1)$\Rightarrow$(2). Since $(|x_n|, \mathscr{F}_n)$ is also an L^1-bounded amart[2], $(|x_n|, \mathscr{F}_n)$ is a T_f-amart by Corollary 6.2. It follows from Theorem 3.1 of [3] that

$$E|x_\infty| = \lim_{T_f} E|x_\tau| = \lim_{T} E|x_\tau|.$$

(2)$\Rightarrow$(3). Since $|x_\sigma| \to |x_\infty|$ (pr.) $(\sigma \in T)$, from $[|x_\infty| - |x_\sigma|]^+ \leqslant |x_\infty| \in L^1$, we see that $\lim\limits_T \int [|x_\infty| - |x_\sigma|]^+ = 0$ and $\lim\limits_T \int [|x_\infty| - |x_\sigma|]^- \leqslant \lim\limits_T \int [|x_\infty| - |x_\sigma|]^+ + \lim\limits_T \left| \int |x_\infty| - \int |x_\sigma| \right| = 0$. Therefore $(x_\sigma)_{\sigma \in T}$ is uniformly integrable.

(3)$\Rightarrow$(4). It is evident that $x_{\tau\wedge n} \to x_\tau (n\uparrow)$ and $(x_{\tau\wedge n})_{n\geqslant 1} \subset (x_{\tau\wedge\sigma})_{\sigma\in T} \subset (x_\sigma)_{\sigma \in T}$, so $(x_{\tau\wedge n}, \mathscr{F}_{\tau\wedge n})$ is an amart by Theorem 4.2 of [3]. Since $x_\tau \in L^1$ by Theorem 6, we have

$$\lim_n \int |E(x_\tau | \mathscr{F}_{\tau\wedge n}) - x_{\tau\wedge n}| \leqslant \lim_n \int |x_\tau - x_{\tau\wedge n}| = 0,$$

by the uniform integrability of $(x_{\tau\wedge n})$. Therefore $(x_{\tau\wedge n}, \mathscr{F}_{\tau\wedge n})$ is a right-closed amart.

(4)$\Rightarrow$(1). Since $(|x_n|, \mathscr{F}_n)$ is an amart, we see now that $\lim\limits_T \int |x_\sigma|$ exists. Let $a = \lim\limits_T \int |x_\sigma|$ and $\varepsilon > 0$. Then there exists n_1 such that

$$\left| \int |x_\sigma| - a \right| < \frac{\varepsilon}{2}, \quad \sigma \in T(n_1), \tag{8.1}$$

for any fixed $\tau \in T_f(n_1)$. Since $(x_{\tau\wedge n}, \mathscr{F}_{\tau\wedge n})$ is a right-closed amart and so $(x_{\sigma\wedge n})$ is uniformly integrable, we see that $(|x_{\tau\wedge n}|, \mathscr{F}_{\tau\wedge n})$ is also uniformly integrable. Then there exists n_2 such that

$$\int \left| E(|x_\tau| \,|\, \mathscr{F}_{\tau\wedge n}) - |x_{\tau\wedge n}| \right| < \frac{\varepsilon}{2}, \quad n \geqslant n_2. \tag{8.2}$$

Now we choose $n \geqslant \max(n_1, n_2)$. Thus by (8.1) and (8.2)

$$\left| \int |x_\tau| - a \right| < \varepsilon, \quad \tau \in T_f(n).$$

Therefore $\lim\limits_{T_f} \int |x_\tau| = a$ and (1) holds.

(5)$\Rightarrow$(1). Let $x_n = y_n + z_n$ be the Riesz decomposition of (x_n)[2], where $(y_n, \mathscr{F}_n)$ is a martingale, $z_n \to O(L^1$ and a.s.), $\lim\limits_T E|z_\sigma| = 0$. Hence $\lim\limits_{T_f} E|z_\tau| = 0$ by Proposition 4.1 of [3] and Corollary 6.1. Since (x_n) is uniformly integrable, it follows that (y_n) is also uniformly integrable and $(y_n, \mathscr{F}_n)$ is a T_f-regular martingale. Thus

$$\begin{aligned} \overline{\lim_T} \int |x_\sigma| &\leqslant \overline{\lim_T} \int |y_\sigma| + \overline{\lim_T} \int |z_\sigma| \leqslant \overline{\lim_{T_f}} \int |y_\tau| \\ &\leqslant \overline{\lim_{T_f}} \int |x_\tau| + \overline{\lim_{T_f}} \int |z_\tau| = \overline{\lim_{T_f}} \int |x_\tau|. \end{aligned}$$

The proof of the theorem is completed.

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