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BIFURCATIONS OF LIMIT CYCLES FORMING COMPOUND EYES IN THE CUBIC SYSTEM

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Abstract

Let $H(n)$ be the maximal number of limit cycle of planar real polynomial differential system with the degree n and C^k_m denote the nest of k limit cycles enclosing m singular points. By computing detection functions, tne authors study bifurcation and phase diagrams in the class of a planar cubic disturbed Hamiltonian system. In particular, the following conclusion is reached: The planar cubic system (E_3) has 11 limit cycles, which form the pattern of compound eyes of $C^1_9 \supset 2[C'_3 \supset (2C^2_1)]$ and have the symmetrical structure; so the Hilbert number $H(3) \geqslant 11$.

§1. Introduction

In 1974, F. Takens[1] listed "all bifurcation" of two parameter nodegenerate Hamiltonian disturbed systems on plane. But, as P. J. Holmes and J. E. Marsden[2] said: "It is not strictly correct to speak of a "list" of two-parameter bifurcation, since the various analyses have not been conveniently gathered in one article". In this paper we will give some bifurcations that have not been listed by Takens.

Let $H(n)$ be the maximal number of limit cycles for planar ploynomial differential systems with the degree n. In the past thirty years, many results have been obtained if $n=2$ (See [3]). C. S. Coleman in "Hilbert 16th problem: How many cycles?"[4] said: "For $n>2$ the maximal number of eyes is not known, nor is it known just which complex patterns of eyes within eyes or eyes enclosing more than a single critical point can exist." It is important that for $n=3$ these problems are considered. As W. A. Coppel[5], to use vague analogy, we have something corresponding to odd functions. Can we understand something corresponding to even functions?

Let C^k_m denote a nest of k limit cycles which encloses m singular points. The sign "$\supset$" is used to show enclosing relations between limit cycles. And the sign "$+$" is

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used to divide limit cycles enclosing different critical points. Denote simply that $C_m^k + C_m^k = 2C_m^k$, etc.

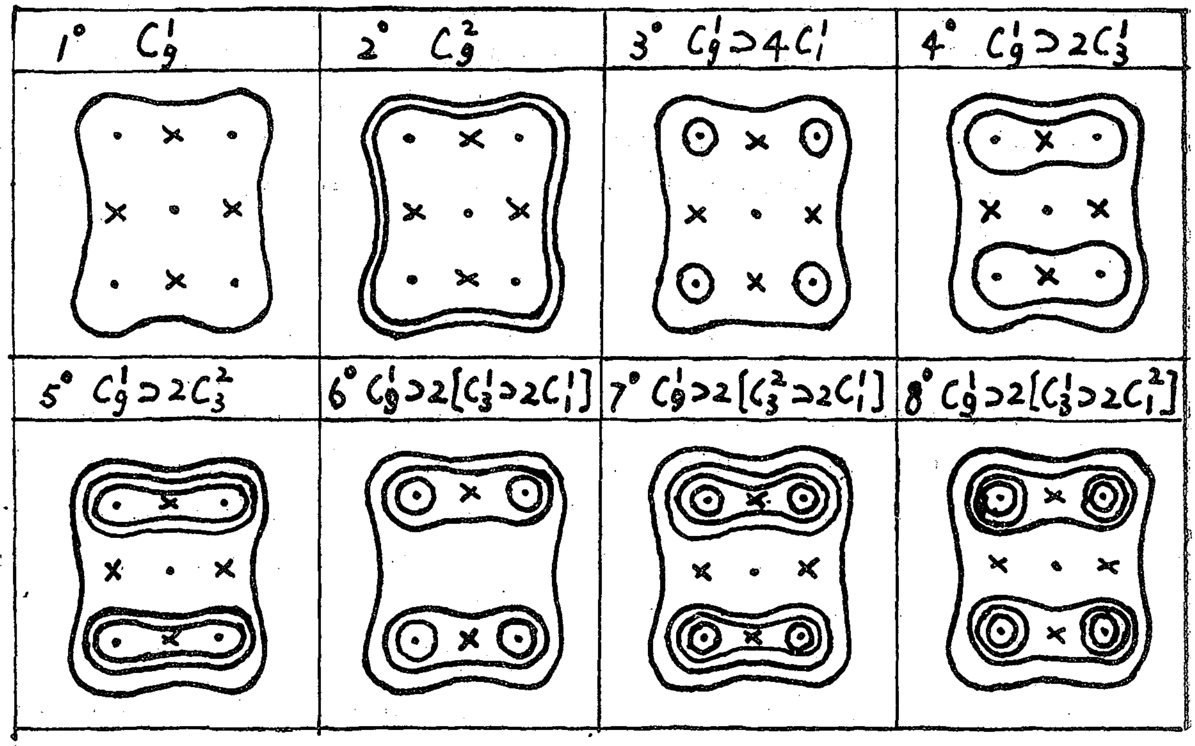
We discuss the two-parameter family of disturbed cubic Hamiltonion system depending on λ, μ:

$$\frac{dx}{dt} = y(1-cy^2) + \mu x(mx^2+ny^2-\lambda),$$

$$\frac{dy}{dt} = -x(1-ax^2) + \mu y(mx^2+ny^2-\lambda), \qquad (1.1)_\mu$$

where $a>c>0$, $0<\mu\ll 1$. Using the theorem of Pontryagin and Zhang Zhifen and Melnikov's Method, and studying detection curvs[6] of $(1, 1)_\mu$, we obtain bifurcations and distributions of limit cycles of $(1.1)_\mu$ listed in Table 1.

Table 1



§ 2. Qualitative Analysis of $(1.1)_\mu$

System $(1.1)_{\mu=0}$ has 9 finite singular points. S_1^0 $(0, 1/\sqrt{c})$ S_2^0 $(0, -1/\sqrt{c})$, S_3^0 $(-1/\sqrt{a}, 0)$ and $S_4^0(1/\sqrt{a}, 0)$ are saddle points; A_1^0 $(1/\sqrt{a}, 1/\sqrt{c})$, A_2^0 $(-1/\sqrt{a}, 1/\sqrt{c})$, A_3^0 $(-1/\sqrt{a}, -1/\sqrt{c})$, A_4^0 $(1/\sqrt{a}, -1/\sqrt{c})$ and the origin $O(0,0)$ are centers. For $0<\mu\ll 1$, $(1.1)_\mu$ also has 9 critical points S_i, $A_i (i=1-4)$ and $O(0, 0)$. Except $O(0, 0)$, all the 8 critical points lie on the curve

$$ax^4+cy^4-(x^2+y^2)=0,$$

i. e.

$$r^2=\frac{1}{a\cos^4\theta+c\sin^4\theta}. \tag{2.1}$$

S_i and A_i take respectively slight displacements from S_i^0 and A_i^0, as μ varies slightly.

Write (ξ_1, η_1) and (ξ_2, η_2) as the coordinates of critical points A_1 and A_2. From the right side of $(1.1)_\mu$, we have

$$\begin{cases}\xi_1=1/\sqrt{a}+[\mu/2\sqrt{c}+O(\mu^2)][\lambda-(m/a+n/c)],\\ \eta_1=1/\sqrt{c}-[\mu/2\sqrt{a}+O(\mu^2)][\lambda-(m/a+n/c)];\end{cases} \tag{2.2}$$

$$\begin{cases}\xi_2=-1/\sqrt{a}+[\mu/2\sqrt{c}+O(\mu^2)][\lambda-(m/a+n/c)],\\ \eta_2=1/\sqrt{c}+[\mu/2\sqrt{a}+O(\mu^2)][\lambda-(m/a+n/c)].\end{cases} \tag{2.3}$$

Since the vector field defined by $(1.1)_\mu$ is invariant under a rotation overt π, critical points A_3 and A_4 have similar formulas. It is easily seen that if $\mu\to 0$ or $\lambda\to\left(\frac{m}{a}+\frac{n}{c}\right)$, then $A_i\to A_i^0$, $S_i\to S_i^0$ $(i=1-4)$.

System $(1.1)_\mu$ has a first integral

$$H(x, y)=-ax^4-cy^4+2(x^2+y^2)=h. \tag{2.4}$$

By use of polar coordinates, (2, 4) becomes

$$r^2=r^2(\theta, h)=\frac{1\pm\sqrt{1-h(a\cos^4\theta+c\sin^4\theta)}}{a\cos^4\theta+c\sin^4\theta}\overset{\text{def.}}{=\!=}\frac{1\pm\sqrt{V(\theta, h)}}{u(\theta)}. \tag{2.5}$$

Denote $r_+^2=\frac{1+\sqrt{V}}{u}$, $r_-^2=\frac{1-\sqrt{V}}{u}$. With h varying, the quartic algebraic curves defined by (2.4) can be divided to 4 types (See Fig 1).

(i) $\{\Gamma_1^h\}$: $-\infty<h<1/a$. It is a family of global closed curves enclosing all the 9 singular points.

(ii) $\{\Gamma_2^h\}$: $0<h<1/a$. I is a closed family surrouding the origin $O(0, 0)$.

(iii) $\{\Gamma_3^h\}$: $1/a<b<1/c$. It is two closed families surrouding respectively three singular points A_1^0, A_2^0 and S_1^0 or A_3^0, A_4^0 and S_2^0

(iv) $\{\Gamma_4^h\}$: $1/c<h<1/a+1/c$. It is four closed families surrouding respectively one singular point A_i^0 $(i=1-4)$.

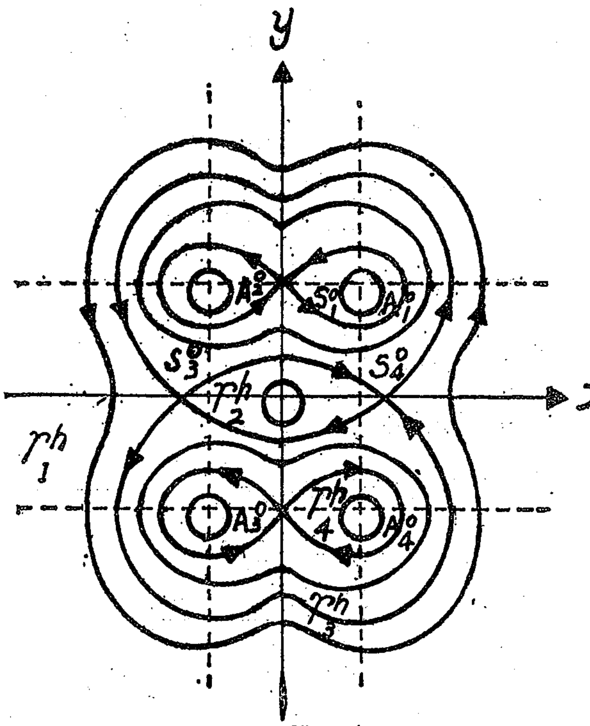


Fig. 1

For $h=1/a$, (2.4) is four branches of heteroclinic orbits connecting two critical points S_3^0 and S_4^0. For $h=1/c$, (2.4) is two homoclinic orbits with a figure of eight, connecting respectively in S_1^0 and S_2^0.

As h increasing, the curve Γ_2^h extends outside, but the other curves constrict inside.

By means of the definition of detection functions, we consider four detection functions $\lambda_i(h=\psi_i(h)/\varphi_i(h)$ $(i=1-4)$, corresponding to $\{\Gamma_i^h\}$. Let $g(\theta)=m\cos^2\theta+n\sin^2\theta$. From Fig. 1, for $i=1, 2$, we have

$$\lambda_i(h)=\int_0^{\pi/2} r_i^4(\theta, h)g(\theta)d\theta\Big/\int_0^{\pi/2} r_i^2(\theta, h)d\theta, \tag{2.6}$$

where $i=1$, $r_1=r_+$, $-\infty<h<1/a$; $i=2$, $r_2=r_-$, $0<h<1/a$.

For $i=3, 4$, we have

$$\lambda_i(h)=\int_{\theta_1(h)}^{\theta_i(h)} (r_+^4-r_-^4)\, g(\theta)d\theta\Big/\int_{\theta_1(h)}^{\theta_i(h)} (r_+^2-r_-^2)d\theta$$
$$=\int_{\theta_1(h)}^{\theta_i(h)} \frac{2\sqrt{V}g}{u^2}\,d\theta\Big/\int_{\theta_1(h)}^{\theta_i(h)} \frac{\sqrt{V}}{u}\,d\theta, \tag{2.7}$$

where $\theta_1(h)=\frac{1}{2}\arccos\left\{\left[2\left(\frac{a+c}{h}-ac\right)^{1/2}-(a-c)\right]/(a+c)\right\}$, and if $i=3$, $\theta_3(h)=\pi/2$, $1/a<h<1/c$; if $i=4$, $1/c<h<1/c+1/a$,

$$\theta_4(h)=\frac{1}{2}\arccos\left\{-\left[2\left(\frac{a+c}{h}-ac\right)^{1/2}+(a-c)\right]/(a+c)\right\}.$$

From (2.6) and (2.7), we easily see that every $\lambda_i(h)$ is a one-valued and differentiable function when h varies in its domain. Using the theorems of bifurcations for closed orbits and homoclinic or heteroclinic orbits[6], we have the values of parameters related to global and local bifurcations as follows:

$$\tilde{\delta}_1=\lambda_1(1/a)=\int_0^{\pi/2} r_+^4\left(\theta, \frac{1}{a}\right)g(\theta)d\theta\Big/\int_0^{\pi/2} r_+^2(\theta, 1/a)d\theta$$
$$=\left[m\left(2I_3+2I_7-\frac{1}{a}I_1\right)+(n-m)\left(2I_4+2I_9-\frac{1}{a}I_2\right)\right]\Big/(I_1+I_5), \tag{2.8}$$

$$\tilde{\delta}_2=\lambda_2(1/a)=\int_0^{\pi/2} r_-^4\left(\theta, \frac{1}{a}\right)g(\theta)d\theta\Big/\int_0^{\pi/2} r_-^2(\theta, 1/a)d\theta$$
$$=\left[m\left(2I_3-2I_7-\frac{1}{a}I_1\right)+(n-m)\left(2I_4-2I_9-\frac{1}{a}I_2\right)\right]\Big/(I_1-I_5), \tag{2.9}$$

$$\tilde{\delta}_3=\lambda_3(1/a)=\left[\int_0^{\pi/2} 2\sqrt{V(\theta, 1/a)}\,(g(\theta)/u^2(\theta))d\theta\right]\Big/\int_0^{\pi/2} (\sqrt{V(\theta, 1/a)}/u(\theta))\,d\theta$$
$$=2[mI_7+(m-n)I_9]/I_5, \tag{2.10}$$

$$\tilde{\delta}_4=\lambda_3(1/c)=\lambda_4(1/c)=\int_{\theta_1(1/c)}^{\pi/2} \frac{2\sqrt{V(\theta, 1/c)}g(\theta)}{u^2(\theta)}\,d\theta\Big/\int_{\theta_1(1/c)}^{\pi/2} \frac{\sqrt{V(\theta, 1/c)}}{u(\theta)}\,d\theta$$
$$=2[mI_8+(n-m)I_{10}]/I_6, \tag{2.11}$$

$$\tilde{\delta}_5=\lambda_4\left(\frac{1}{a}+\frac{1}{c}\right)=2\left(\frac{m}{a}+\frac{n}{c}\right), \tag{2.12}$$

where I_1-I_{10} are following integrals:

$$I_1=\int_0^{\pi/2} \frac{d\theta}{u(\theta)}, \qquad I_2=\int_0^{\pi/2} \frac{\sin^2\theta}{u(\theta)}\,d\theta,$$
$$I_3=\int_0^{\pi/2} \frac{d\theta}{u^2(\theta)}, \qquad I_4=\int_0^{\pi/2} \frac{\sin^2\theta}{u^2(\theta)}\,d\theta,$$
$$I_5=\int_0^{\pi/2} \frac{\sqrt{V(\theta, 1/a)}}{u(\theta)}\,d\theta, \qquad I_6=\int_{\theta_1(1/c)}^{\pi/2} \frac{\sqrt{V(\theta, 1/c)}}{u(\theta)}\,d\theta,$$

$$I_7=\int_0^{\pi/2}\frac{\sqrt{V(\theta,1/c)}}{u^2(\theta)}d\theta,\qquad I_8=\int_{\theta_1(1/c)}^{\pi/2}\frac{\sqrt{V(\theta,1/c)}}{u^2(\theta)}d\theta,$$

$$I_9=\int_0^{\pi/2}\frac{\sqrt{V(\theta,1/a)}\sin^2\theta}{u^2(\theta)}d\theta,\qquad I_{10}=\int_0^{\pi/2}\frac{\sqrt{V(\theta,1/c)}\sin^2\theta}{u^2(\theta)}d\theta.$$

These definite integral can be computed exactly by calculus or computer.

§ 3. The Case of $m=-1, n=1$

Consider the system with numerical coefficients:

$$\frac{dx}{dt}=y(1-y^2)+\mu x(y^2-x^2-\lambda),\qquad\qquad (3.1)_\mu$$

$$\frac{dy}{dt}=-x(1-2x^2)+\mu y(y^2-x^2-\lambda).$$

For $(3.1)_\mu$, $g(\theta)=\sin^2\theta-\cos^2\theta=-\cos 2\theta$, $u(\theta)=2\cos^4\theta+\sin^4\theta$,

$$\theta_1(1/c)=\frac{1}{2}\arccos\frac{1}{3}.$$

Using (2.8) – (2.12), we have

$$\tilde{b}_1=\lambda_1(1/2)=1.7632/2.69237=0.654885,$$
$$\tilde{b}_2=\lambda_2(1/2)=-0.0319787/0.496503=-0.0644079,$$
$$\tilde{b}_3==\lambda_3(1/2)=2\times 0.448794/1.09794=0.817523,$$
$$\tilde{b}_4=\lambda_3(1)=\lambda_4(1)=2\times 0.26425/0.496482=1.06452,$$
$$\tilde{b}_5=\lambda_4(3/2)=1.$$

Clearly, $\tilde{b}_4>\tilde{b}_5>\tilde{b}_3>\tilde{b}_1>b_2$. Write $\tilde{u}(\theta)=2\sin^4\theta+\cos^4\theta$, $\tilde{V}(\theta)=(1-\tilde{u}(\theta)h)^{\frac{1}{2}}$.

Lemma 3.1. $\lambda_1(h)>0$, $\lim\limits_{h\to-\infty}\lambda_1(h)=+\infty$, $\lim\limits_{h\to1/2-0}\lambda_1'(h)=+\infty$.

Proof Since $\varphi_1(h)$ is the area inside Γ_1^h, we have $\varphi_1(h)>0$. To prove $\lambda_1(h)>0$, it suffices to discuss $\psi_1(h)>0$. Note that

$$\psi_1(h)=\int_0^{\pi/2}r_+^4(-\cos 2\theta)d\theta=\int_0^{\pi/4}r_+^4(-\cos 2\theta)d\theta+\int_{\pi/4}^{\pi/2}r_+^4(-\cos 2\theta)d\theta.$$

By using the transformation $\tilde{\theta}=\frac{\pi}{2}-\theta$ to the second integral, we have

$$\psi_1(h)=\int_0^{\pi/4}\frac{\cos 2\theta}{u^2\tilde{u}^2}[u(1+\sqrt{\tilde{v}})+\tilde{u}(1+\sqrt{\tilde{v}})][\cos 2\theta+u\sqrt{\tilde{v}}-\tilde{u}\sqrt{\tilde{v}}]d\theta.\qquad (3.2)$$

When $\theta\in\left(0,\frac{\pi}{4}\right)$, $\cos 2\theta>0$, $u(\theta)>\tilde{u}(\theta)$ and $1-hu>0$, $1-h\tilde{u}>0$ for all $h\in(-\infty, 1/2)$. It follows that $u\sqrt{\tilde{v}}-\tilde{u}\sqrt{v}>0$. Hence, the integrand of (3.2) is positive. So $\psi_1(h)>0$.

Similarly, we have

$$\varphi_1(h)=\int_0^{\pi/2}r_+^2(\theta)d\theta=\int_0^{\pi/4}\frac{u+(1+\sqrt{\tilde{v}})+\tilde{u}(1+\sqrt{v})}{u\tilde{u}}d\theta.\qquad (3.3)$$

It is evident that the ratio of the integrand of (3.2) to that of (3.3) approaches

$+\infty$, as $h\to-\infty$. Thus $\lim\limits_{h\to-\infty}\lambda_1(h)=+\infty$. We see that

$$\varphi_1'(h)=-\int_0^{\pi/2}\frac{d\theta}{2\sqrt{v}},\quad \psi_1'(h)=\int_0^{\pi/2}\frac{(1+\sqrt{v})\cos 2\theta}{u\sqrt{v}}\,d\theta,$$

$$\lambda_1'(h)=[\psi_1'(h)-\lambda_1(h)\varphi_1'(h)]/\varphi_1(h). \tag{3.4}$$

If $h=1/2$, $\theta=0$, then $v(\theta,h)=1-h(2\cos^4\theta+\sin^4\theta)=0$. It follows that integrals $\int_0^{\pi/2} d\theta/\sqrt{v}$ and $\int_0^{\pi/2} d\theta/u\sqrt{v}$ are divergent when $h\to1/2$ and $\theta\to0$.

$$\psi_1'(h)-\lambda_1(h)\varphi_1'(h)=\int_0^{\pi/2}\frac{\cos 2\theta}{u(\theta)}\,d\theta+\int_0^{\pi/2}\frac{\cos 2\theta}{u\sqrt{v}}\,d\theta+\lambda_1(h)\int_0^{\pi/2}\frac{d\theta}{2\sqrt{v}}$$

$$=\Big[\int_0^{\pi/2}\frac{\cos 2\theta}{u(\theta)}\,d\theta+\int_{\pi/4}^{\pi/2}\frac{\cos 2\theta}{u\sqrt{v}}\,d\theta+\lambda_1(h)\int_{\pi/4}^{\pi/2}\frac{d\theta}{2\sqrt{v}}\Big]$$

$$+\Big[\int_0^{\pi/4}\frac{\cos 2\theta}{u\sqrt{v}}\,d\theta+\lambda_1(h)\int_0^{\pi/2}\frac{d\theta}{2\sqrt{v}}\Big]. \tag{3.5}$$

In (3.5), three integrals in the first square brackets are convergent, but two integrals in the second square brackets are divergent if $h\to1/2-0$. So $\lim\limits_{h\to1/2-0}\lambda_1'(h)=+\infty$.

In the same way as the proof of Lemma 3.1, we have the following lemma.

Lemma 3.2. $\lambda_2(h)<0$, $\lim\limits_{h\to1/2-0}\lambda_2'(h)=-\infty$.

Lemma 3.3. $\lambda_3(h)>0$, $\lim\limits_{h\to1/2+0}\lambda_3'(h)=+\infty$, $\lim\limits_{h\to1}\lambda_3'(h)=-\infty$.

Proof For $h\in(1/2,1)$, $\varphi_3(h)>0$, we prove $\psi_3(h)>0$. Since $\theta_1(h)<\frac{1}{2}\arccos\frac{1}{3}<\pi/4$, we have

$$\psi_3(h)=\int_{\theta_1(h)}^{\pi/2}\frac{-2\cos 2\theta\sqrt{v}}{u^2}\,d\theta$$

$$=\int_{\theta_1(h)}^{\pi/4}\frac{-2\cos 2\theta}{u^2}\sqrt{v}\,d\theta+\int_{\pi/4}^{\pi/2}\frac{-2\cos 2\theta}{u^2}\sqrt{v}\,d\theta$$

$$=2\int_0^{\pi/4}\cos 2\theta\Big[\frac{\sqrt{\tilde v}}{\tilde u^2}-\frac{\sqrt{v}}{u^2}\Big]d\theta+\int_0^{\theta_1(h)}\frac{\cos 2\theta}{u^2}\sqrt{v}\,d\theta$$

$$=2\int_0^{\pi/4}\frac{u^2\sqrt{\tilde v}-\tilde u^2\sqrt{v}}{u^2\tilde u^2}\cos 2\theta\,d\theta+\int_0^{\theta_1(h)}\frac{\cos 2\theta}{u^2}\sqrt{v}\,d\theta. \tag{3.6}$$

It is easily seen that $u^2\sqrt{\tilde v}-\tilde u^2\sqrt{v}>0$ if $\theta\in(0,\pi/4)$ and $h\in(1/2,1)$.

From (3.6), it follows that $\psi_3(h)>0$. Thus, $\lambda_3(h)>0$.

$$\varphi_3'(h)=-\int_{\theta_1(h)}^{\pi/2}\frac{d\theta}{2\sqrt{v}},\quad \psi_3'(h)=\int_{\theta_1(h)}^{\pi/2}\frac{\cos 2\theta}{u\sqrt{v}}\,d\theta. \tag{3.7}$$

If $\theta=\pi/2$ and $h=1$, $v(\theta,h)=0$, $\lim\limits_{h\to1/2+0}\theta_1(h)=0$. Two integrals of (3.7) have singularity in $\theta=\theta_1(h)$ and $\theta=\pi/2$ as $h\to1/2+0$ and $h\to1-0$. We have

$$\lambda_3'(h)=[\psi_3'(h)-\lambda_3(h)\varphi_3'(h)]/\varphi_3(h). \tag{3.8}$$

Take θ_0: $\pi/4<\theta_0<\frac{\pi}{2}$ such that $|\cos 2\theta_0|\geqslant0.8$. Thus,

$$\begin{aligned}\psi_3'(h)-\lambda_3(h)\varphi_3'(h)&=\int_{\theta_1(h)}^{\pi/2}\frac{\cos 2\theta}{u\sqrt{v}}\,d\theta+\lambda_3(h)\int_{\theta_1(h)}^{\pi/2}\frac{d\theta}{2\sqrt{v}}\\&=\left[\int_{\theta_1(h)}^{\theta_0}\frac{\cos 2\theta}{u\sqrt{v}}\,d\theta+\lambda_3(h)\int_{\theta_1(h)}^{\theta_0}\frac{d\theta}{2\sqrt{v}}\right]\\&\quad+\left[\int_{\theta_0}^{\pi/2}\frac{\cos 2\theta}{u\sqrt{v}}\,d\theta+\lambda_3(h)\int_{\theta_0}^{\pi/2}\frac{d\theta}{2\sqrt{v}}\right].\end{aligned}$$

If $h\to 1-0$, integrals of the first square brackets are convergent. Consider the second brackets. For $\theta\in(\theta_0,\pi/2)$, $\left(\frac{1}{u(\theta)}\right)>1$, $-\cos 2\theta>0.8$. Hence

$$\lambda_3(h)\int_{\theta_0}^{\pi/2}\frac{d\theta}{2\sqrt{v}}-\int_{\theta_0}^{\pi/2}\frac{(-\cos 2\theta)}{u\sqrt{v}}\,d\theta\leqslant\lambda_3(h)\int_{\theta_0}^{\pi/2}\frac{d\theta}{2\sqrt{v}}-\int_{\theta_0}^{\pi/2}\frac{0.8}{\sqrt{v}}\,d\theta. \tag{3.9}$$

Because $\lim\limits_{h\to 1-0}\frac{1}{2}\lambda_3(h)=\frac{1}{2}\times 1.06452<0.8$, the right side of (3.9) approaches $-\infty$ as $h\to 1-0$. Since $\lim\limits_{h\to 1-0}\varphi_3(h)=\text{constant}$, from (3.8) $\lim\limits_{h\to 1-0}\lambda_3'(h)=-\infty$.

Similarly, we have $\lim\limits_{h\to 1/2+0}\lambda_3'(h)=-\infty$.

By using the analogous method, we can prove the following Lemma.

Lemma 3.4. $\lambda_4(h)>0$, $\lim\limits_{h\to 1+0}\lambda_4'(h)=-\infty$.

Note that in saddle points S_1, S_2 and S_3, S_4 the saddle values $\sigma_{1,2}>0$ and $\sigma_{3,4}<0$. The above conclusions of $\lambda_i'(h)\,(i=1\sim 4)$ conform to information obtained from these saddle values.

By virtue of preceding discuss and differentiability of detection functions, we can obation some local knowledge of detection curves. Using computer to determine global detection curves, we have the result in Fig 2. By using the theorems of [6], it can indicate the number and positions of limit cycles of $(3.1)_\mu$. Hence, we have the following theorem.

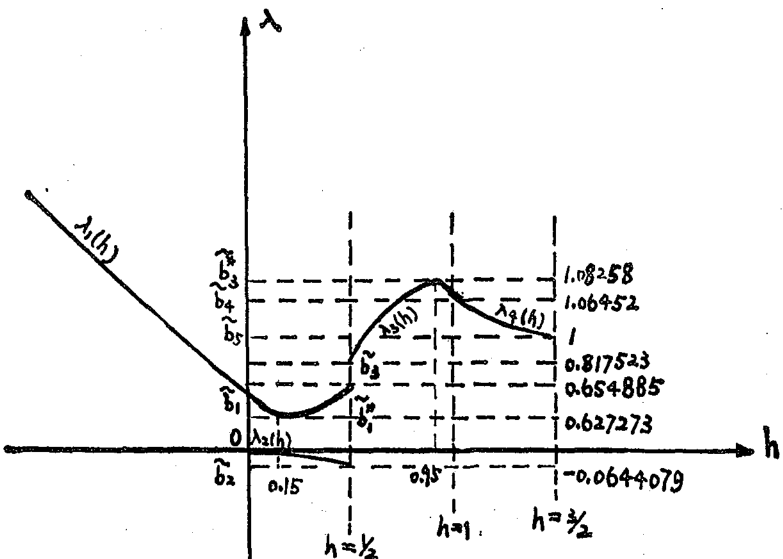


Fig. 2 Ditection curves of $(3.1)_\mu(m=-1, n=1)$

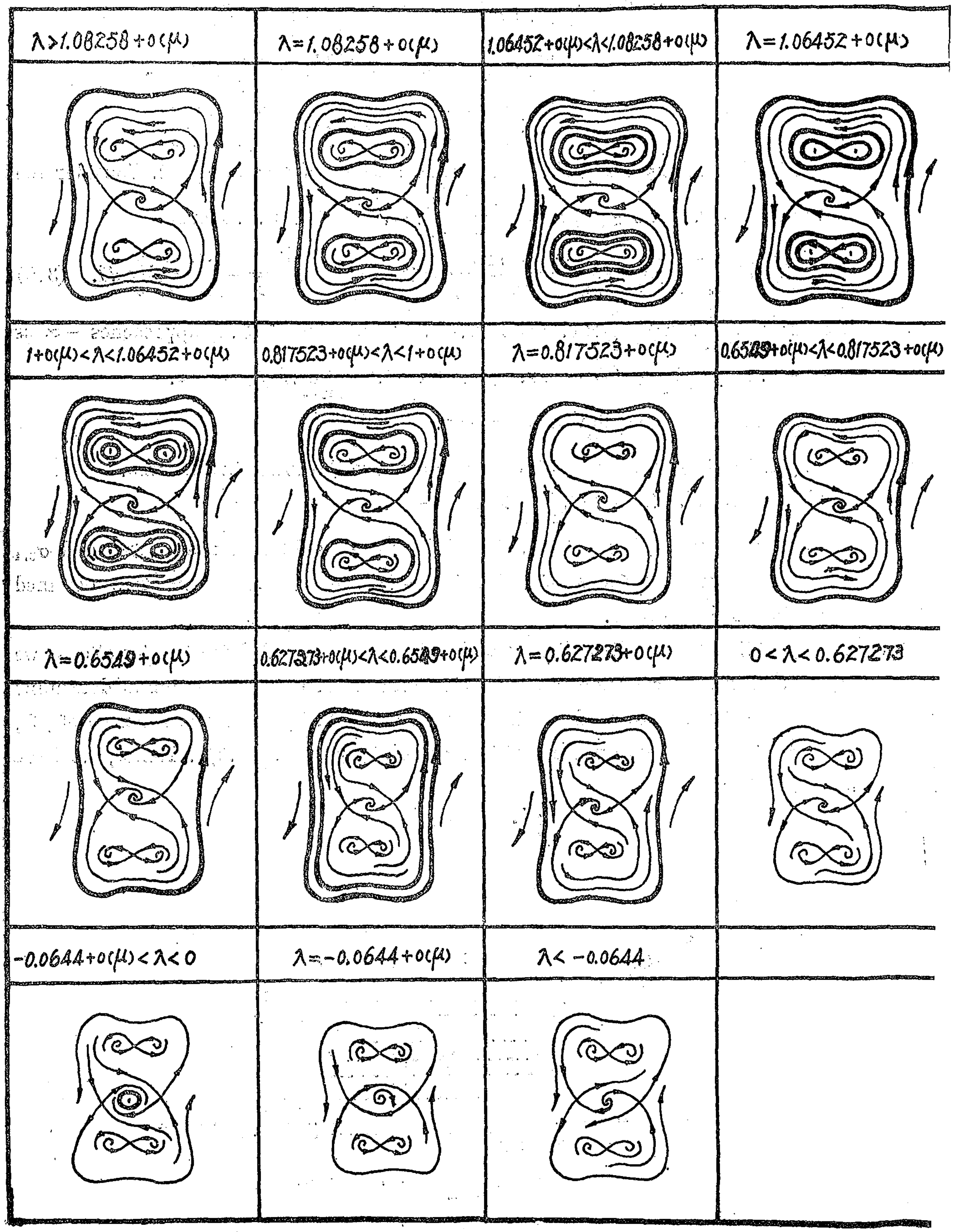

λ>1.08258+o(μ)
λ=1.08258+o(μ)
1.06452+o(μ)<λ<1.08258+o(μ)
λ=1.06452+o(μ)
1+o(μ)<λ<1.06452+o(μ)
0.817523+o(μ)<λ<1+o(μ)
λ=0.817523+o(μ)
0.6549+o(μ)<λ<0.817523+o(μ)
λ=0.6549+o(μ)
0.627273+o(μ)<λ<0.6549+o(μ)
λ=0.627273+o(μ)
0<λ<0.627273
-0.0644+o(μ)<λ<0
λ=-0.0644+o(μ)
λ< -0.0644

Fig. 3

Theorem 1. *For fixed μ, $0<\mu\ll 1$, as the parameter λ varies, System $(3.1)_\mu$ has the bifurcations as follows.*

(i) *If $\lambda>b_3^*=\tilde{b}_3^*+O(\mu)\approx 1.08258+O(\mu)$, then there exists one unstable limit cycle of $(3.1)_\mu$ with the distribution of C_9^1;*

(ii) *If $\lambda=b_3^*$, then there exist 3 limit cycles of $(3.1)_\mu$ with the distribution $C_9^1\supset 2C_3^1$;*

(iii) *If $b_4=\tilde{b}_4+O(\mu)<\lambda<b_3^*$, then there exist 5 limit cycles of $(3.1)_\mu$ with the distribution $C_9^1\supset 2C_3^2$;*

(iv) *If $b_5=\tilde{b}_5+O(\mu)<\lambda<b_4$, then there exist 7 limit cycles of $(3.1)_\mu$ with the distribution $C_9^1\supset 2[C_3^1\supset 2C_1^1]$;*

(v) *If $b_3=\tilde{b}_3+O(\mu)<\lambda<b_5$, then there exist 3 limit cycles of $(3.1)_\mu$ with the distribution $C_9^1\supset 2C_3^1$;*

(vi) *If $b_1=\tilde{b}_1+O(\mu)<\lambda<b_3$, then there exists one unstable limit cycle of $(3.1)_\mu$ with the distribution C_9^1;*

(vii) *If $b_{1*}=\tilde{b}_{1*}+O(\mu)<\lambda<b$, then there exist two limit cycles of $(3.1)_\mu$ with the distribution C_9^2;*

(viii) *If $\lambda=b_{1*}$, then there exists one semistable limit cycle of $(3.1)_\mu$ with the distribution C_9^1;*

(ix) *If $0<\lambda<b_{1*}$ or $\lambda<b_2$, then System $(3.1)_\mu$ has no limit cycle.*

(x) *If $b_2=\tilde{b}_2+O(\mu)<\lambda<0$, then there exists one limit cycle of $(3.1)_\mu$ with the distribution $C_{\bar{1}}^{\bar{1}}$.*

Bifurcations and phase portraits of $(3.1)_\mu$ are shown as Fig 3.

§ 4. The Case of $m=1, n=-3$

Consider the system

$$\begin{aligned}\frac{dx}{dy}&=y(1-y^2)+\mu x(x^2-3y^2-\lambda),\\ \frac{dy}{dx}&=-x(1-2x^2+\mu y(x^2-3y-\lambda).\end{aligned}\tag{4.1$_\mu$}$$

Note that for $(4.1)_\mu$, $g(\theta)=4\cos^2\theta-3$. Computing (2.8)—(2.12), we have

$$\begin{aligned}\tilde{b}_1&=\lambda_1(1/2)=-8.94528/2.69237=-3.32245,\\ \tilde{b}_2&=\lambda_2(1/2)=-0.0978258/0.4965503=-0.19703,\\ \tilde{b}_3&=\lambda_3(1/2)=-2.21184/1.09794=-4.0291,\\ \tilde{b}_4&=\lambda_3(1)=\lambda_4(1)=-1.19231/0.496482=-4.80305,\\ \tilde{b}_5&=\lambda_4(3/2)=-5.\end{aligned}$$

We see that $\tilde{b}_2>\tilde{b}_1>\tilde{b}_3>\tilde{b}_4>\tilde{b}_5$. For the detection functions $\lambda_i(h)$ $(i=1-4)$ of $(4.1)_\mu$ we also have following lemmas.

Lemma 4.1. $\lambda_1(h)<0$, $\lim\limits_{h\to-\infty}\lambda_1(h)=-\infty$, $\lim\limits_{h\to 1/2-0}\lambda_1'(h)=-\infty$.

Proof Since

$$g(\theta)=4\cos^2\theta-3\begin{cases}>0 & \text{for } 0<\theta<\pi/6,\\ <0 & \text{for } \pi/6<\theta<\pi/2,\end{cases}$$

letting $\tilde{g}(\theta)=4\sin^2\theta-3$, we have

$$\begin{aligned}\psi_1(h)&=\int_0^{\pi/2}\frac{g(\theta)(1+\sqrt{v})^2}{u^2}d\theta\\&=\int_0^{\pi/6}\frac{g(\theta)(1+\sqrt{v})^2}{u^2}d\theta+\int_{\pi/6}^{\pi/3}\frac{g(\theta)(1+\sqrt{v})^2}{u^2}d\theta\\&\quad+\int_{\pi/3}^{\pi/2}\frac{g(\theta)(1+\sqrt{v})^2}{u^2}d\theta.\end{aligned}\tag{4.2}$$

For the third integralof (4.2), by changing the variable $\tilde{\theta}=\frac{\pi}{2}-\theta$, (4.2) converts into

$$\begin{aligned}\psi_1(h)&=\int_0^{\pi/6}\left[\frac{g(\theta)(1+\sqrt{v})^2}{u^2}+\frac{\tilde{g}(\theta)(1+\sqrt{\tilde{v}})}{\tilde{u}^2}\right]d\theta\\&\quad+\int_{\pi/6}^{\pi/3}\frac{g(\theta)(1+\sqrt{v})^2}{u^2}d\theta.\end{aligned}\tag{4.3}$$

If $\theta\in(0,\pi/6)$, then $u(\theta)>\tilde{u}(\theta)$. For $h>0$, $\sqrt{v}<\sqrt{\tilde{v}}$, it follows that

$$\begin{aligned}\tilde{u}^2g(\theta)(1+\sqrt{v})^2+u^2\tilde{g}(\theta)(1+\sqrt{\tilde{v}})^2&\leqslant u^2(\theta)(1+\sqrt{\tilde{v}})^2[g(\theta)+\tilde{g}(\theta)]\\&=-2u^2(\theta)(1+\sqrt{\tilde{v}})^2<0.\end{aligned}$$

For $h>0$, $\sqrt{v}>\sqrt{\tilde{v}}$, we have

$$\begin{aligned}&\tilde{u}^2g(\theta)(1+\sqrt{v})^2+u^2\tilde{g}(\theta)(1+\sqrt{\tilde{v}})^2\\&\leqslant|\tilde{g}(\theta)|[\tilde{u}^2(1+\sqrt{v})^2-u^2(1+\sqrt{\tilde{v}})^2]\\&=|\tilde{g}(\theta)|[\tilde{u}(1+\sqrt{v})+u(1+\sqrt{\tilde{v}})][-(u-\tilde{u})-(u\sqrt{\tilde{v}}-\tilde{u}\sqrt{v})]<0.\end{aligned}$$

So the integrands of (4.3) are negative for $h\in(-\infty,1/2)$, namely, $\psi_1(h)<0$. Since $\varphi_1(h)>0$, $\lambda_1(h)<0$ is proved.

Imitating the proof in Lemma 3.1, we have $\lim\limits_{h\to-\infty}\lambda_1(h)=-\infty$. To prove $\lim\limits_{h\to1/2-0}\lambda_1'(h)=-\infty$, we see that

$$\begin{aligned}\psi_1'(h)-\lambda_1(h)\varphi_1'(h)&=\left[-\int_0^{\pi/2}\frac{g(\theta)}{u(\theta)}d\theta-\int_{\pi/6}^{\pi/2}\frac{g(\theta)d\theta}{u\sqrt{v}}+\lambda_1(h)\int_{\pi/6}^{\pi/2}\frac{d\theta}{2\sqrt{v}}\right]\\&\quad+\left[-\int_0^{\pi/6}\frac{g(\theta)d\theta}{u\sqrt{v}}+\lambda_1(h)\int_0^{\pi/2}\frac{d\theta}{2\sqrt{v}}\right].\end{aligned}\tag{4.4}$$

For all $h\leqslant1/2$, the part in the first brackets of (4.4) is bounded. As $h\to1/2-0$, in $\theta=0$ the integrands of the second brackets have singularity. Because $g(\theta)>0$, $\theta\in(0,\pi/6)$ and $\lim\limits_{h\to1/2-0}\lambda_1(h)=-3.32245$, $\lim\limits_{h\to1/2-0}\varphi_1(h)=2.69237$, it follows that $\lim\limits_{h\to1/2-0}\lambda_1'(h)=-\infty$.

Similarly, we have the following Lemmas.

Lemma 4.2. $\lambda_2(h)<0$, $\lim\limits_{h\to1/2-0}\lambda_2'(h)=+\infty$.

Lemma 4.3. $\lambda_3(h)<0$, $\lim\limits_{h\to1/2+0}\lambda_3'(h)=-\infty$, $\lim\limits_{h\to1-0}\lambda_3'(h)=+\infty$.

Proof The first and second conclusions of the proof are similar to that of Lemma

4.1. Let us prove the third part. Because

$$\psi_3'(h)-\lambda_3(h)\varphi_3'(h)=-\int_{\theta_1(h)}^{\pi/2}\frac{g(\theta)\,d\theta}{u\sqrt{v}}+\lambda_3(h)\int_{\theta_1(h)}^{\pi/2}\frac{d\theta}{2\sqrt{v}},\tag{4.5}$$

taking θ_0 with $\pi/6<\theta_0<\pi/2$, such that $|g(\theta)|=|4\cos^2\theta-3|>2.5$, from (4.5) we have

$$\psi_3'(h)-\lambda_3(h)\varphi_3'(h)=\left[-\int_{\theta_1(h)}^{\theta_0}\frac{g(\theta)\,d\theta}{u\sqrt{v}}+\lambda_3(h)\int_{\theta_1(h)}^{\theta_0}\frac{d\theta}{2\sqrt{v}}\right]$$
$$+\left[-\int_{\theta_0}^{\pi/2}\frac{g(\theta)\,d\theta}{u\sqrt{v}}+\lambda_3(h)\int_{\theta_0}^{\pi/2}\frac{d\theta}{2\sqrt{v}}\right].\tag{4.6}$$

If $h\to1-0$, then the part in the first brackets of (4.6) is bounded. Consider the part in the second brackets, we have

$$\int_{\theta_0}^{\pi/2}\frac{-g(\theta)\,d\theta}{u\sqrt{v}}+\lambda_3(h)\int_{\theta_0}^{\pi/2}\frac{d\theta}{2\sqrt{v}}\geqslant\int_{\theta_0}^{\pi/2}\frac{2.5\,d\theta}{\sqrt{v}}+\lambda_3\int_{\theta_0}^{\pi/2}\frac{d\theta}{2\sqrt{v}}.\tag{4.7}$$

Since $\lim\limits_{h\to1-0}\frac{-\lambda_3(h)}{2}=\frac{1}{2}\times4.80305<2.5$ and, as $h\to1-0$, the integrands of (4.7) have singularity in $\theta=\pi/2$, the right side of (4.7) approaches $+\infty$ as $h\to1-0$. From $\lim\limits_{h\to1-0}\varphi_3(h)=$ positive constant, it follows that $\lim\limits_{h\to1-0}\lambda_3'(h)=+\infty$.

Similarly, we also have the following Lemma.

Lemmam 4.4. $\lambda_4(h)<0$, $\lim\limits_{h\to1+0}\lambda_4'(h)=+\infty$.

It is easy to see that in saddle points S_1, S_2 and S_3, S_4 the saddle values $\sigma_{1,2}>0$ and $\sigma_{3,4}<0$. The conclusions drawn from Lemmas 4.1—4.4 also conform to the information given from saddle values.

To sum up, it is similar to the discussion in Section 3. We obtain detection curves of (4.1) shown as Fig. 4. On the basis of the invariance of vector fields under a rotation over π, by the behaviour of the detection curves and the theorems of [6] we have the following theorem.

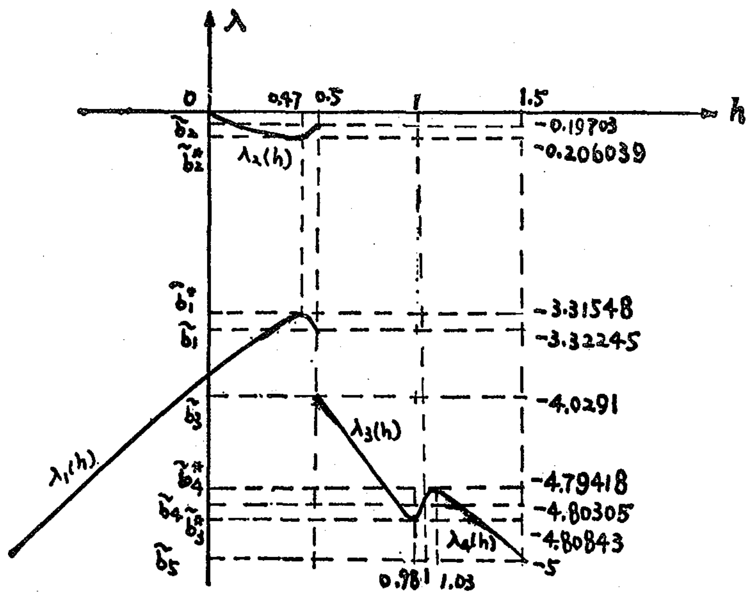


Fig 4. Ditection curves of $(4.1)_\mu$ $(m=1,\ n=-3)$

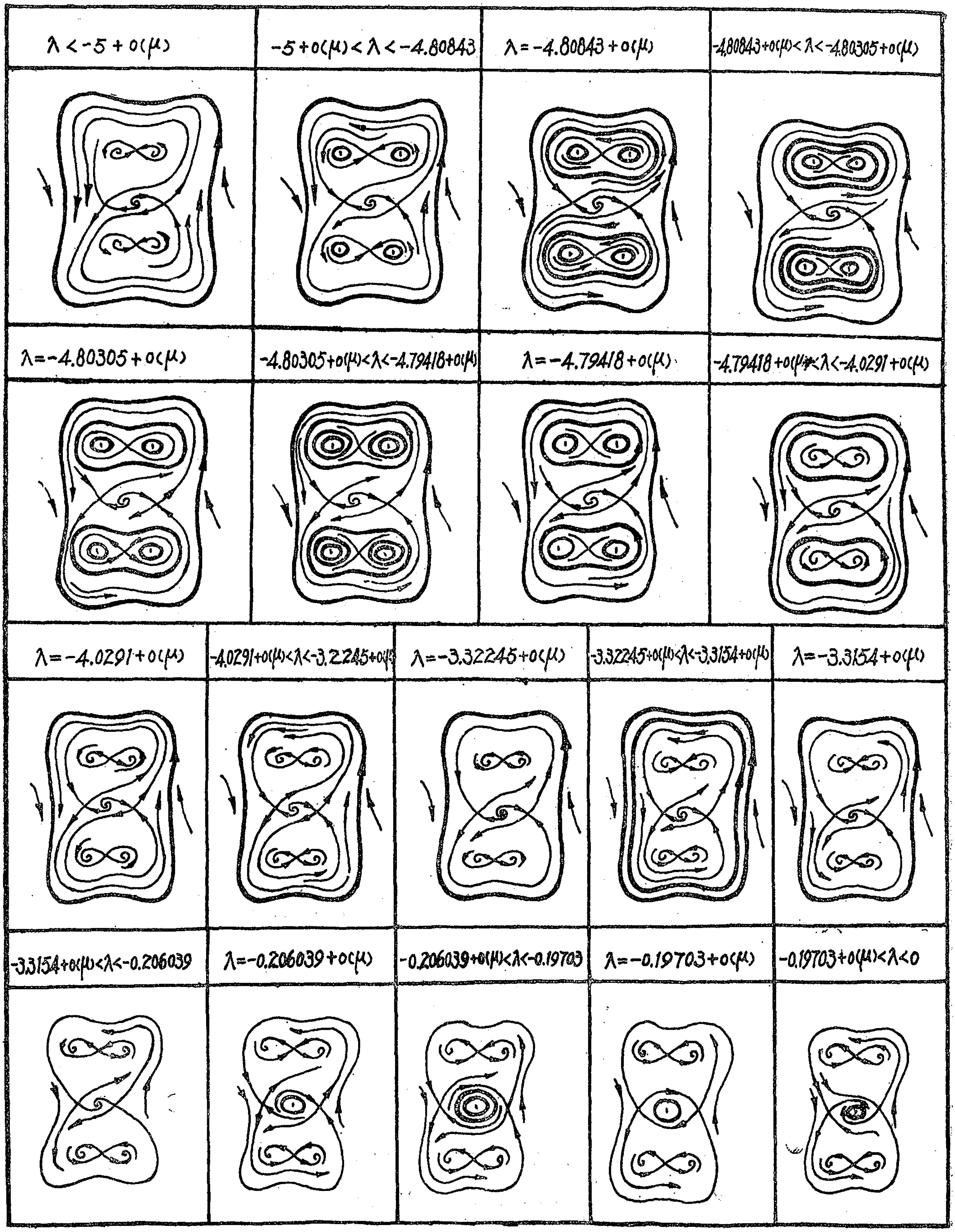

λ < -5 + o(μ)
-5 + o(μ) < λ < -4.80843
λ = -4.80843 + o(μ)
-4.80843 + o(μ) < λ < -4.80305 + o(μ)
λ = -4.80305 + o(μ)
-4.80305 + o(μ) < λ < -4.79418 + o(μ)
λ = -4.79418 + o(μ)
-4.79418 + o(μ) < λ < -4.0291 + o(μ)
λ = -4.0291 + o(μ)
-4.0291 + o(μ) < λ < -3.2245 + o(μ)
λ = -3.32245 + o(μ)
-3.32245 + o(μ) < λ < -3.3154 + o(μ)
λ = -3.3154 + o(μ)
-3.3154 + o(μ) < λ < -0.206039
λ = -0.206039 + o(μ)
-0.206039 + o(μ) < λ < -0.19703
λ = -0.19703 + o(μ)
-0.19703 + o(μ) < λ < 0

Fig. 5

Theorem 2. *For fixed μ, $0<\mu\ll 1$, as λ varies, $(4.1)_\mu$ has distributions of limit cycles as follows.*

(i) *If $\lambda<b_5=\tilde{b}_0+O(\mu)$, $(4.1)_\mu$ has one stable limit cycle with the distribution of C_9'.*

(ii) *If $b_5<\lambda<b_{3*}=\tilde{b}_{3*}+O(\mu)$, $(4.1)_\mu$ has 5 limit cycles with the distribution of $C_9^1\supset 4C_1^1$.*

(iii) *If $\lambda=b_{3*}=-4.80843+O(\mu)$, $(4.1)_\mu$ has 7 limit cycles with the distribution of $C_9^1\supset 2[C_3^1\supset 2C_1^1]$.*

(iv) *If $b_4=\tilde{b}_4+O(\mu)>\lambda>b_{3*}$, $(4.1)_\mu$ has 9 limit cycles with the distribution of $C_9^1\supset 2[C_3^2\supset 2C_1^1]$,*

(v) *If $b_4^*=-4.79418+O(\mu)>\lambda>b_4$, $(4.1)_\mu$ has 11 limit cycles with the distribution of $C_9^1\supset 2[C_3^1\supset C_2^2]$.*

(vi) *If $\lambda=b_4^*$, $(4.1)_\mu$ has 7 limit cycles with the distribution $C_9^1\supset 2[C_3^1\supset 2C_1^1]$.*

(vii) *If $b_3=\tilde{b}_3+O(\mu)>\lambda>b_4^*$, $(4.1)_\mu$ has 3 limit cycles with the distribution $C_9^1\supset 2C_3^1$.*

(viii) *If $b_1=\tilde{b}_1+O(\mu)>\lambda>b_3$, $(4.1)_\mu$ has one stable limit cycle with the distribution C_9'.*

(ix) *If $b_1^*=-3.31548+O(\mu)>\lambda>b_1$, $(4.1)_\mu$ has two limit cycles with the distribution C_9^2.*

(x) *If $\lambda=b_1^*$, $(4.1)_\mu$ has one semistable limit cycle with the distribution C_9^1.*

(xi) *If $\lambda=b_{2*}=\tilde{b}_{2*}+O(\mu)$ or $0>\lambda>b_2=\tilde{b}_2+O(\mu)$, $(4.1)_\mu$ has one limit cycle with the distribution C_1^1.*

(xii) *If $b_2>\lambda>b_{2*}$, $(4.1)_\mu$ has two limit cycles with the distribution C_1^2.*

(xiii) If $b_{2*}>\lambda>b_1^*$ or $\lambda\geqslant 0$, $(4.1)_\mu$ has no limit cycle.

In Fig. 5, we give the bifurcations and phase portraits of $(4.1)_\mu$. From Theorem 2 and Fig. 5, we may come to the following conclusion.

Theorem 3. *For the planar cubic system (E_3), the Hilbert number $H(3)\geqslant 11$ and limit cycles of (E_3) can form the patterns of compoundeyes and have the symmetric structure.*

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