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THE PERIODIC HARRY-DYM EQUATION

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Abstract

The development of the inverse scattering transform (I. S. T) has made it possible to solve certain physically significant nonlinear evolution equations with periodic boundary conditions. Date and Tanaka[2] have considered kdv equation; Ma and Ablowitz[3] have discussed the cubic Schrodinger equation. In this paper, following closely the analysis in [2, 3] the author considers Harry-Dym eqution

$$(q^2)_t = -2r_{xxx}, \tag{I}$$

where $q(x, t)$is periodic in x with period π for all time $q(x, t) = q(x+\pi, t)$, $q(x, t) = r^{-1}(x, t) > 0$

§ 1. The Direct Scattering Problem

We consider the eigenvalue problem

$$\phi_{1,xx} = -\xi q^2(x)\phi_1, \quad \xi = k^2. \tag{1.1}$$

We denote the solutions of (1.1) by $C(x, x_0, \xi)$, $S(xx, x_0, \xi)$ which satisfy

$$C(x_0, x_0, \xi) = S_x(x_0, x_0, \xi) = 1,$$
$$C_x(x_0, x_0, \xi) = S(x_0, x_0, \xi) = 0. \tag{1.2}$$

From $q(x+\pi) = q(x)$, it is easy to show that

$$\Phi(x+\pi, x_0, \xi) = \Phi(x, x_0, \xi)\hat{T}(x_0, \xi), \tag{1.3}$$

where
$$\Phi(x, x_0, \xi) = \begin{pmatrix} C(x, x_0, \xi) & S(x, x_0, \xi) \\ C_x(x, x_0, \xi) & S_x(x, x_0, \xi) \end{pmatrix},$$
$$\hat{T}(x_0, \xi) = \Phi(x_0+\pi, x_0, \xi).$$

Using (1.2), we obtain $\det \hat{T} = 1$.

Let m be an eigenvalue of the matrix $\hat{T}$. Then

$$m^2 - (\operatorname{tr} \hat{T})m + \det \hat{T} = 0,$$

i.e.

$$m^2 - \Delta(\xi)m + 1 = 0, \tag{1.4}$$

where $\Delta(\xi) = C(x_0+\pi, x_0, \xi) + S_x(x_0+\pi, x_0, \xi)$. Let $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be an eigenvector belonging to m. Then solution

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$$u(x, \xi)=v_1C(x, x_0, \xi)+v_2S_x(x, x_0, \xi)$$

of (1.1) satisfies

$$u(x+\pi, \xi)=mu(x, \xi). \tag{1.5}$$

By (1.4) and (1.5), we have

$$u(x+\pi, \xi)=\pm u(x, \xi)\Leftrightarrow\Delta(\xi)=\pm 2.$$

Next we describe a series of results[1].

(i) $C(x_0+\pi, x_0, \xi)$ and $S(x_0+\pi, x_0, \xi)$ are entire functions of ξ, therefore so is $\Delta(\xi)$.

(ii) Zeros $\{\xi_0, \xi_1, \cdots|$ordered from left to right$\}$ of $\Delta^2(\xi)-4$ and zeros $\{r_1, r_2, \cdots$orderd from left to right$\}$ of $S(x_0+\pi, x_0, \xi)$ are real.

$$\{\xi: \Delta^2(\xi)-4<0\}=\bigcup_{j=1}^{\infty} J_j,$$

$$\{\xi: \Delta^2(\xi)-4\geqslant 0\}=\bigcup_{j=0}^{\infty} I_i,$$

where $I_0=(-\infty, \xi_0]$, $I_i=[\xi_{2j-1}, \xi_{2j}]$, $J_j=(\xi_{2j-2}, \xi_{2j-1})$, $j=1, 2, \cdots$. All intervals are finite with the exception of I_0. In the interval J_i, all solutions of (1.1) are bounded; in the open interval I_j^0, (1.1) has no solution that is bounded; when $\xi=\xi_j$, at least there is a bounded solution of (1.1). Therefore the intervals J_j are called the zones of stability, the intervals I_j the zones of instability.

(iii) $\xi_0, \xi_{4j-1}, \xi_{4j}$ are zeros of $\Delta(\xi)-2$ and ξ_{4j-3}, ξ_{4j-2} are zeros of $\Delta(\xi)+2$. These zeros are simple except for the cases $\xi_{4j-1}=\xi_{4j}$ or $\xi_{4j-3}=\xi_{4j-2}$ when these zeros are double. $r_j\in I_j$, $j=1, 2, \cdots$.

$$\xi_0<\xi_1<r_1\leqslant\xi_2<\xi_3\leqslant r_2\leqslant\xi_4<\cdots<\xi_{2j-1}\leqslant r_j\leqslant\xi_{2j}<\cdots.$$

In the following we consider the case when finite zones of instability I_j degenerate to points except exactly N of them. We call such potential $q(x)$ N-band potential. Renumbering them, we order these I_j and r_j from left to right

$$I_0, I_j=[\xi_{2i-1}, \xi_{2j}], r_i\in I_j, j=1, 2, \cdots.$$

§2. The Inverse Scattering Problem

Introduce the function $\phi_2=\phi_{1,x}$ and rewrite equation (1.1) as follows

$$\phi_x=M\phi, M=\begin{pmatrix} 0 & 1 \\ -\xi q^2 & 0\end{pmatrix}, \quad \phi=\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}. \tag{2.1}$$

Making the transformation

$$\phi=q^{-\frac{1}{2}}(x)e^{ik\theta(x)}B\psi, \ \xi=k^2, \tag{2.2}$$

where $B=\begin{pmatrix} -i & i \\ kq(x) & kq(x)\end{pmatrix}$, $\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$, $\theta(x)=\int_{x_0}^{x} q(t)\,dt$. Following the transformation (2.2), (2.1) can be written as follows

$$\psi_x = A\psi,\ A = \begin{pmatrix} 0 & -q_x/2q \\ -q_x/2q & -2ikq \end{pmatrix}. \tag{2.3}$$

Solutions $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ of (2.3) which satisfy

$$f(x_0, x_0, k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ g(x_0, x_0, k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are equivalent to integral equations

$$\begin{cases} f_1(x, x_0, k) = 1 + \int_{x_0}^{x} \left(-\frac{q_t}{2q}\right) f_2 dt, \\ f_2(x, x_0, k) = e^{-2ik\theta(t)} \int_{x_0}^{x} \frac{-q_t}{2q} e^{2ik\theta(t)} f_1 dt \end{cases}$$

and

$$\begin{cases} g_1(x, x_0, k) = -\int_{x_0}^{x} (q_t/2q) g_2 dt, \\ g_2(x, x_0, k) = e^{-2ik\theta(x)} [1 - \int_{x_0}^{x} (q_t/2q) e^{2ik\theta(t)} g_1 dt]. \end{cases}$$

By use of the method of successive approximations, it is easy to derive the following lemma.

Lemma. *Let $q(x)$ be an enough smooth function, $k = k_1 + ik_2$, $k_2 \leqslant 0$, $x \in [x_0, x_0 + \pi]$. when $|k| \to \infty$, we have the following asymptotic series:*

$$f_1(x, x_0, k) = 1 + \frac{1}{8ik} \int_{x_0}^{x} \frac{(q_t)^2}{q^3} dt + \frac{1}{16k^2} \left[\frac{q_x^2(x_0)}{q^4(x_0)} - \frac{q_x(x) q_x(x_0)}{q^2(x) q^2(x_0)} e^{-2ik\theta(x)} \right]$$
$$- \frac{1}{32k^2} \left(\frac{q_x^2(x)}{q^4(x)} - \frac{q_x^2(x_0)}{q^4(x_0)} \right) - \frac{1}{128k^2} \left(\int_{x_0}^{x} \frac{q_t^2}{q^3} dt \right)^2 + O\left(\frac{1}{k^3}\right),$$

$$f_2(x, x_0, k) = \frac{-1}{4ik} \left(\frac{q_x(x)}{q^2(x)} - \frac{q_x(x_0)}{q^2(x_0)} e^{-2ik\theta(x)} \right)$$
$$- \frac{1}{8k^2} \left[\left(\frac{q_{xx}(x)}{q^3(x)} - \frac{2q_x^2(x)}{q^4(x)} \right) - e^{-2ik\theta(x)} \left(\frac{q_{xx}(x_0)}{q^3(x_0)} - \frac{2q_x^2(x_0)}{q^4(x_0)} \right) \right]$$
$$+ \frac{1}{32k^2} \frac{q_x(x)}{q^2(x)} \int_{x_0}^{x} \frac{q_t^2}{q^3} dt + \frac{1}{32k^2} \frac{q_x(x_0)}{q^2(x_0)} e^{-2ik\theta(x)} \int_{x_0}^{x} \frac{q_t^2}{q^3} dt + O\left(\frac{1}{k^3}\right),$$

$$g_1(x, x_0, k) = \frac{1}{4ik} \left(\frac{q_x(x)}{q^2(x)} e^{-2ik\theta(x)} - \frac{q_x(x_0)}{q^2(x_0)} \right)$$
$$+ \frac{1}{32k^2} \left(\frac{q_x(x_0)}{q^2(x_0)} + \frac{q_x(x)}{q^2(x)} e^{-2ik\theta(x)} \right) \int_{x_0}^{x} \frac{q_t^2}{q^3} dt$$
$$- \frac{1}{8k^2} \left[\left(\frac{q_{xx}(x)}{q^3(x)} - \frac{2q_x^2(x)}{q^4(x)} \right) e^{-2ik\theta(x)} - \left(\frac{q_{xx}(x_0)}{q^3(x_0)} - \frac{2q_x^2(x_0)}{q^4(x_0)} \right) \right] + O\left(\frac{1}{k^3}\right),$$

$$g_2(x, x_0, k) = e^{-2ik\theta(x)} - \frac{e^{-2ik\theta(x)}}{8ik} \int_{x_0}^{x} \frac{q_t^2}{q^3} dt - \frac{e^{-2ik\theta(x)}}{128k^2} \left(\int_{x_0}^{x} \frac{q_t^2}{q^2} dt \right)^2$$
$$+ \frac{1}{16k^2} \left(\frac{q_x^2(x_0)}{q^4(x_0)} e^{-2ik\theta(x)} - \frac{q_x(x_0) q_x(x)}{q^2(x_0) q^2(x)} \right)$$
$$- \frac{e^{-2ik\theta(x)}}{32k^2} \left(\frac{q_x^2(x)}{q^4(x)} - \frac{q_x^2(x_0)}{q^4(x_0)} \right) + O\left(\frac{1}{k^3}\right).$$

From (1.2) and (2.2), we have

$$C(x, x_0, \xi)=\frac{q^{\frac{1}{2}}(x_0)e^{ik_0\theta(x)}}{2q^{\frac{1}{2}}(x)}[f_1(x)-f_2(x)-g_1(x)+g_2(x)],$$

$$S(x, x_0, \xi)=-\frac{iq^{\frac{1}{2}}(x_0)e^{ik_0\theta(x)}}{2kq^{\frac{1}{2}}(x)q(x_0)}[f_1(x)-f_2(x)+g_1(x)-g_2(x)],$$

$$S_x(x, x_0, \xi)=\left(\frac{q(x)}{q(x)}\right)^{\frac{1}{2}}\frac{e^{ik_0\theta(x)}}{2}[f_1(x)+f_2(x)+g_1(x)+g_2(x)].$$

Using the lemma, we get

$$\Delta^2(\xi)=4\left[\cos^2 k\theta+\frac{c}{4k}\sin k\theta\cos k\theta-\frac{1}{64k^2}(\cos^2 k\theta-\sin^2 k\theta)c^2\right]+O\left(\frac{e^{2|k_2|\theta}}{k^3{}_2}\right),$$

$$S^2(x_0+\pi, x_0, \xi)=\frac{1}{k^2q(x_0)}\left[\sin^2 k\theta-\frac{C}{4k}\sin k\theta\cos k\theta+\frac{1}{4k^2}\frac{q_x^2(x_0)}{q^4(x_0)}\sin^2 k\theta\right.$$
$$\left.+\frac{1}{64k^2}(\cos^2 k\theta-\sin^2 k\theta)C^2+\frac{D}{2k^2}\sin^2 k\theta+O\left(\frac{e^{2|k_2|\theta}}{k^3}\right)\right],$$

where $\theta=\theta(x_0+\pi)$, $C=\int_{x_0}^{x_0+\pi}\frac{q_t^2}{q^3}dt$, $D=\frac{q_{xx}(x_0)}{q^3(x_0)}-\frac{2q_x^2(x_0)}{q^4(x_0)}$.

Using asympotic estimates of $\Delta^2(\xi)$ and $S^2(x_0+, x_0, \quad)$, we find that

$$\frac{4-\Delta^2(\xi)}{S^2(x_0+\pi, x_0, \xi)}\cdot\frac{\prod_{i=1}^{N}(\xi-r_i)^2}{\prod_{i=0}^{2N}(\xi-\xi_i)}\longrightarrow 4q^2(x_0),\ \text{as } |k|\to\infty.$$

Since $\frac{4-\Delta^2(\xi)}{S^2(x_0+\pi, x_0, \xi)}\frac{\prod_{i=1}^{N}(\xi-r_i)^2}{\prod_{i=0}^{2N}(\xi-\xi_i)}$ is an entire function of ξ, by Liouville's theorem, we can conclude that

$$\frac{4-\Delta^2(\xi)}{S^2(x_0+\pi, x_0, \xi)}=4q^2(x_0)\frac{\prod_{i=0}^{2N}(\xi-\xi_i)}{\prod_{i=1}^{N}(\xi-r_i)^2}\quad\text{for all } \xi. \tag{2.4}$$

Using the asymptotic forms of $\Delta^2(\xi)$ and $S^2(x_0+\pi, x_0, \xi)$, comparing the cofficient of the $(1/\xi)$ term in (2.4), we obtain

$$q_{xx}(x_0)r^3(x_0)=2\left(\sum_{i=0}^{2N}\xi_i-2\sum_{i=1}^{N}r_i\right). \tag{2.5}$$

Changing the point x_0 to x_0+dx_0, we get $\Phi(x, x_0+dx_0, \xi)$. Using the Taylor's theorem, we have

$$\Phi(x, x_0+dx_0, \xi)=\Phi(x, x_0, \xi)(I+Q(x_0)dx_0)+O((dx_0)^2), \tag{2.6}$$

where $Q(x_0)$ is an unknown 2×2 matrix. Since $\Phi(x, x_0+dx_0, \xi)$ is a solving matrix of (2.1), $Q(x_0)$ is independent of x.

Replacing x in (2.6) by $x+\pi$ and using (1.3), we get

$$\frac{\partial\hat{T}(x_0, \xi)}{\partial x_0}=\hat{T}Q-Q\hat{T}. \tag{2.7}$$

From (2.6), we obtain $Q(x_0)=\Phi_{x_0}(x, x_0, \xi)|_{x=x_0}$. It is easy to derive $\Phi_e(x, x_0, \xi)$

$=\Phi(x, x_0, \xi)\begin{pmatrix} 0 & -1 \\ \xi q^2(x) & 0 \end{pmatrix}$. Therefore $Q(x_0)=\begin{pmatrix} 0 & -1 \\ \xi q^2(x) & 0 \end{pmatrix}$.

By (2.7), we get $\frac{\partial \Delta(x_0, \xi)}{\partial x_0}=0$. Hence $\xi_j(j=0, 1, \cdots, 2N)$ is independent of x_0.

Taking the square root of equation (2.4), we see that

$$\frac{S(x_0+\pi, x_0, \xi)}{i\sqrt{\Delta^2(\xi)-4}}=\frac{\sigma'\prod_{i=1}^{N}(\xi-r_i)}{2q(x_0)\sqrt{\prod_{i=0}^{2N}(\xi-\xi_i)}}, \tag{2.8a}$$

where the square root is defined so that the real part of it is positive and $\sigma'=\pm 1$. Using (2.7), we get

$$\frac{\partial S(x_0+T, x_0, r_i)}{\partial x_0}=S_x(x_0+\pi, x_0, r_j)-C(x_0+\pi, x_0, r_j)=\sigma'_j\sqrt{\Delta^2(r_j)-4} \tag{2.8b}$$

with $\sigma'_j=\pm 1$. Combining a differentiation of (2.8a) with respect to x_0 and (2.8b), we show that

$$\frac{dr_j}{dx_0}=\frac{2iq(x_0)\sigma_j\sqrt{\prod_{i=0}^{2N}(r_j-\xi_i)}}{\prod_{i\neq j}(r_j-r_i)}, \tag{2.9a}$$

$j=1, 2, \cdots, N$; $\sigma_j=\pm 1$. Then

$$q(x_0)=\frac{\prod_{i\neq j}(r_j-q_i)}{2i\sigma_j\sqrt{\prod_{i=0}^{2N}(r_j-\xi_i)}}\frac{dr_j}{dx_0}. \tag{2.9b}$$

From (2.9b) and (2.5), we may get the differential equation of r_j with respect to x_0. This gives the motion of r_i with respect to x_0.

§ 3. The Time Depedence of Spectrum

Associated to equation (I) and the vector function ϕ of (2.1) the time dependence[4]

$$\phi_t=N\phi,\ N=\begin{pmatrix} 2\xi r_x(x) & -4\xi r(x) \\ 2\xi r_{xx}(x)+4\xi^2 q(x) & -2\xi r_x(x) \end{pmatrix}^{(*)}. \tag{3.1}$$

For the solving matrix of (2.1), we have

$$\Phi_t(x, t, \xi)-N\Phi(x, t, \xi)=\Phi(x, t, \xi)\Lambda, \tag{3.2}$$

where $\Lambda=\begin{pmatrix} \lambda & \alpha \\ \mu & \beta \end{pmatrix}$ is independent of x. To keep $\Phi(x_0, t, \xi)=I$ for all time we take

$$\Lambda=-N(x_0).$$

Replacing x in (3.2) by x_0 and using (1.3), we obtain

$$\frac{\partial \hat{T}}{\partial t}=\hat{T}\Lambda-\Lambda\hat{T}. \tag{3.3}$$

From (3.3) we have $\frac{\partial \Delta(\xi)}{\partial t}=0$. Therefore $\xi_j, j=0, 1, \cdots, 2N$, is time independent.

(*) The c of formula (45) in [4] should be $2k^2 r_{xx}+4k^2\rho$.

If $q(x)$ is an N-band potential initially, it will be an N-band potential for all time t.

Using (3.3), we get

$$\left.\frac{\partial S(x_0+\pi, t, \xi)}{\partial t}\right|_{\xi=r_j} = 4i r_j \sigma'_j \sqrt{\Delta^2(r_j)-4}\, r(x_0), \tag{3.4}$$

$\sigma'_j = \pm 1$. Combining a differentiation of (2.8a) with respect to t and (3.4), we conclude that

$$\frac{dr_j}{dt} = \frac{-8\sigma_j r_j \sqrt{\prod_{i=0}^{2N}(r_j-\xi_i)}}{\prod_{i\neq j}(r_j-r_i)}, \tag{3.5}$$

$j=1, 2, \cdots, N$; $\sigma_j = \pm 1$.

For an N-band potential periodic Harry-Dym equation (I) can be solved by following procedures. First, we solve the equatipn (1.1) to get spectra $\xi_0, \cdots, \xi_{2N}$ and $r_1, r_2, \cdots, r_N$ as $t=0$. Second, considering these $r_i(x)$ as initial conditions at $t=0$, solving equation (3.5), we get $r_i(x, t)$. By (2.9b), we can obtain the solution of the periodic Harry-Dym equation $q(x, t)$ with N-band.

On the other hand, because an arbitrary periodic potential can have infinite number of zones of instability I_j, here we give a method to find an N-band potential for the equation (I). At a certain point x_0, given real ξ_i, r_i, r_{ix} and r_{ixx} that satisfy properties in part 1 as boundary conditions, then solving the third order systerm of r_i, we can obta in $r_i(x)$. By (2.9b) wereconstruct the potential $q(x)$, this $q(x)$ is an N-band potential at $t=0$.

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