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WEAK CHEBYSHEV SPACES ON LOCALLY ORDERED TOPOLOGY SPACE AND THE RELATED CONTINUOUS METRIC SELECTIONS

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Abstract

Let $C(X)$ be the space of all continuous real-valued functions on a compact Hausdorff space X under the uniform norm:

$$\|f\|=\max\{|f(x)|: x\in X\}.$$

For $G\subset C(X)$, define

$$P_G(f)=\{g\in G: \|f-g\|=\inf\{\|f-p\|: p\in G\}\}.$$

If there exists a continuous mapping S from $C(X)$ to G such that $S(f)\in P_G(f)$ for every f in $C(X)$, then S is called a continuous selection of the metric projection P_G.

And G is called a Z-subspace of $C(X)$, if, for every nonzero g in G, g does not vanish on any open subset of X.

In this paper, the author gives several characterizations of Z-subspaces G whose metric projections P_G have continuous selections. The following results are obtained:

If X is locally connected and G is an n-dimensional Z-subspace of $C(X)$, then P_G has a continuous selection if and only if every nonzero g in G has at most n zeros and has at most $n-1$ zeros with sign changes.

§1. Introduction

Let $C(X)$ be the Banach space of all real-valued continuous functions on the compact Hausdorff space X, the norm on $C(X)$ is defined as

$$\|f\|=\max\{|f(x)|: x\in X\}.$$

Suppose that G is a subspace of $C(X)$, the metric projection is defined as

$$P_G(f)=\{g\in G: \|f-g\|=\inf\{\|f-p\|: p\in G\}\}.$$

If a mapping s from $C(X)$ to G satisfies that for every $f\in C(X)$, $s(f)\in P_G(f)$, then s is called a selection of $P_G(\cdot)$. Moreover, if s is continuous, then s is called a continuous selection of P_G; if $s(f+g)=s(f)+g$ holds for all $f\in C(X)$ and $g\in G$, then s is called a semi-additive selection of P_G; if for every $f\in C(X)$, there exists a constant $C(f)$ such that

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$$\|s(f)-s(h)\| \leqslant C(f)\|f-h\|, \quad h \in C(X),$$

then s is called a pointwise–Lipschitz–continuous selection of P_G. Let

$$CS_n(X) = \{G: G \text{ is an } n\text{–dimensional subspace of } C(X) \text{ and } P_G \text{ has a continuous selection}\}.$$

Lazar, Morris and Wulbert first studied the characterization of $CS_n(X)$ and characterized $CS_1(X)$[1]. Later, after a series of works, Nurnberger and Sommer gave a characterization of $CS_n(X)$ for all n and $X=[a, b]$[2]. If $X=\bigcup_{i=1}^{k}[a_i, b_i]$ is consisted of several closed intervals, Sommer could only give a sufficient condition of $G \in CS_n(X)$, his result is as follows:

Theorem A. *Suppose that $X=\bigcup_{i=1}^{k}[a_i, b_i]$ and G is an n–dimensional subspace of $C(X)$. If there exists $z \in X$ such that G, satisfies Haar condition on $X\backslash\{z\}$ and every g in G has at most $n-1$ zeros with sign changes, then P_G has a unique semi–additive and pointwise–Lipschitz–continuous selection.*

If for any non–zero g in G the set $Z(g)$ of all zeros of g does not contain any open subset of X, then G is called a Z–subspace of $C(X)$. Brown gave a necessary condition of G in $CS_n(X)$[4]:

Theorem B. *If $G \in CS_n(X)$ is a Z–subspace, then, for any non–zero g in G, g has at most n zeros and has at most $n-1$ z ros with sign changes.*

In this paper, we will delve further into the problems discussed in [3, 5]. We will give a characterization of Z–subspaces in $CS_n(X)$ for locally connected X, which improves Theorem A. For this purpose, we discuss weak Chebyshev subspaces on locally ordered topology space in section 2. Our main result is as follows:

If X is a locally connected, compact Hausdoaff space and G is an n–dimensional Z–subspace of $C(X)$, then P_G has a continuous selection if and only if, for every non–zero g in G, g has at most n zeros an l has at most $n-1$ zeros with sign changes

§2. Weak Chebyshev Space on Locally Ordered Topology Space

For a subset X of the set of all real numbers, Deutsch, Nurnberger and Singer defined weak Chebyshev subspaces of $C(X)$[6]. And in [7], Nuenberger discussed the existence of continuous metric selections about this kind of subspaces. In this section, we will introduce a new kind of weak Chebyshev subspaces defined on locally topology space. In section 3, we will see that the new weak Chebyshev subspaces play more important roles in dealing with the characterizations of $CS_n(X)$.

Definition 1. *Suppose that X is a compact Hausdorff space. If for every x there* *exist an open neighborhood* $U(x)$ *of* x *and an order* $\underset{(x)}{<}$ *such that* $(U(x), \underset{(x)}{<})$ *is a completely ordered set, then* $(X, \{U(x)\}, \{\underset{(x)}{<}\})$ *is called locally ordered topology space. And open neighborhoods of* x *in* $U(x)$ *are called completely ordered neighborhoods of* x.

Suppose that $(X, \{U(x)\}, \{\underset{(x)}{<}\})$ is a locally ordered topology space. For simplicity, we use $<$ as the order for a fixed completely ordered open neighborhood of x. Suppose that $G=\mathrm{span}\{g_1, \cdots, g_n\}$ is an n-dimensional subspace of $C(X)$. Let

$$D(x_1, \cdots, x_n)=\det\begin{pmatrix} g_1(x_1), \cdots, g_1(x_n) \\ \cdots\cdots \\ g_n(x_1), \cdots, g_n(x_n) \end{pmatrix}.$$

Definition 2. *If there exist open neighborhoods* $\{U(x)\}$ *and orders* $\{\underset{(x)}{<}\}$ *such that* $(X, \{U(x)\}, \{\underset{(x)}{<}\})$ *is a locally ordered topology space and* G *satisfies the following condition:*

For any r *distinct points* $x_1, \cdots, x_r$, *and* $0=m_0<m_1<\cdots<m_r=n$, *there exist completely ordered open neighborhoods* V_i *of* x_i *and* $\varepsilon=\pm 1$ *such that for any*

$$y_{m_i+1}<\cdots<y_{m_{i+1}}\in V_{i+1},\ 0\leqslant i\leqslant r-1,$$

there holds

$$\varepsilon D(y_1, \cdots, y_n)\geqslant 0,$$

then G *is called a weak Chebyshev subspace of* $C(X)$ *(about the orders* $\{\underset{(x)}{<}\}$*).*

This definition is different from that in [6]. In fact, it is a generilization of weak Chebyshev subspaces defined in [6].

Definition 3. *If for any* n *distinct points* $x_1, \cdots, x_n$ *there exist open neighborhoods* V_i *of* x_i, $1\leqslant i\leqslant n$, *and* $\varepsilon=\pm 1$ *such that*

$$\varepsilon D(y_1, \cdots, y_n)\geqslant 0 \text{ for } y_i\in V_i,\ 1\leqslant i\leqslant n, \tag{1}$$

then G *is called semi-definite.*

If G is a Z-subspace, then ε is uniquely determined by (1), and we call ε the sign of G at $(x_1, \cdots, x_n)$, denoted by $\Delta(x_1, \cdots, x_n)$.

Theorem 4. *Suppose that* X *is a locally path connected and compact Hausdorff space. If* n*-dimensional* Z*-subspace* G *is semidefinite, then* G *is a weak Chebyshev subspace of* $C(X)$.

Proof Suppose x in X and $U(x)$ is a path connected open neighborhood of x. Choose n-2 distinct points $x_3, \cdots, x_n$ in $X\backslash U(x)$. Define an order in $U(x)$ as follows:

For x_1, x_2 in $U(x)$, $x_1\neq x_2$, define

$x_1<x_2$, if $\Delta(x_1, \cdots, x_n)=1$;

$x_1>x_2$, if $\Delta(x_1, \cdots, x_n)=-1$.

From the path connectedness of $U(x)$ and the semi-definiteness of G, we can easily see that the relation "$<$" is transferable. Thus, $(X, \{U(x)\}, \{\underset{(x)}{<}\})$ is a locally ordered topology space. Now we will show that G is a weak Chebyshev subspace about the orders just defined.

Suppose that $x_1, \cdots, x_r$ in X are r distinct points and $0=m_0<m_1<\cdots<m_r=n$. Let V_i be completely ordered open neighborhoods of x_i such that $V_1, \cdots, V_r$ are mutually non-intersection. For

$$y_{m_i+1}<\cdots<y_{m_{i+1}} \text{ in } V_{i+1},$$
$$z_{m_i+1}<\cdots<z_{m_{i+1}} \text{ in } V_{i+1},$$

we can verify by induction that

$$\Delta(y_1,\cdots, y_n)=\Delta(z_1, \cdots, z_n). \tag{2}$$

In fact, suppose that (2) holds for $y_i=z_i$, $i\leqslant s+1$, then for $y_i=z_i$, $i\leqslant s$, $y_{s+1}\neq z_{s+1}$ we may assume that $y_{s+1}<z_{s+1}$. As y_{s+1}, z_{s+1} are in the same path connected set V_j for some j, there exists a continuous mapping $h(.)$ from $[0, 1]$ to V_j such that

$$h(0)=y_{s+1},\ h(1)=z_{s+1}.$$

Let
$$a=\sup\{t:\ 0\leqslant t\leqslant 1, \text{ and } h(t)=y_{s+1}\},$$
$$b=\inf\{t:\ a\leqslant t\leqslant 1, \text{ and } h(t)=z_{s+1}\}.$$

It is not hard to check that

$$h(a)=y_{s+1},\ h(b)=z_{s+1},$$

and
$$y_{s+1}<h(t)<z_{s+1}, \text{ for } a<t<b.$$

Thus

$$h([a, b])\cap\{y_1, \cdots, y_s, z_{s+2}, \cdots, z_n\}=\emptyset. \tag{3}$$

Define
$$u(t)=\Delta(y_1, \cdots, y_s, h(t), z_{s+2}, \cdots, z_n).$$

From (3) and the semi-definiteness of G, we see that $u(t)$ is continuous on $[a, b]$. But $u([a, b])$ in $\{-1, 1\}$ is connected, so $u(a)=u(b)$.

From the hypothesis of induction we get

$$\Delta(z_1, \cdots, z_n)=\Delta(y_1, \cdots, y_s, z_{s+1}, y_{s+2}, \cdots, y_n).$$

Thus
$$\Delta(y_1, \cdots, y_n)=u(a)=u(b)=\Delta(z_1, \cdots, z_n).$$

Lemma 5. *Suppose that an n-dimensional Z-space G is also a weak Chebyshev subspace. If f, f_k in $C(X)$ and real numbers r_k satisfy the following conditions:*

$$\lim\|f-f_k\|=0;$$
$$\lim r_k=r\neq 0;$$

and there exist $n+1$ distinct points $x_{k,0}, \cdots, x_{k,n}$ such that

$$f_k(x_{k,i})=r_k(-1)^i\cdot\Delta(x_{k,0}, \cdots, x_{k,i-1}, x_{k,i+1}, \cdots, x_{k,n}),\quad 0\leqslant i\leqslant n;$$

then there exist $n+1$ distinct points $x_0, \cdots, x_n$ and a subsequence $\{k_j\}$ such that

$$\lim x_{k_j,i}=x_i,\quad 0\leqslant i\leqslant n; \tag{4}$$

$$f(x_i)=r(-1)^i\cdot\Delta(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n),\quad 0\leqslant i\leqslant n. \tag{5}$$

Proof Obviously, there exists a subsequence $\{k_j\}$ such that (4) holds for some $x_0, \cdots, x_n$ in X.

If $x_0, \cdots, x_n$ are mutually distinct, then by the semi-definiteness of G we know that (5) is true. Otherwise, we may assume that

$$x_0=\cdots=x_{m_1}=y_1,\ \cdots,\ x_{m_{s-1}+1}=\cdots=x_n=y_s,$$

where $y_1, \cdots, y_s$ are mutually distinct and $m_1\geqslant 1$.

Because G is a weak Chebyshev subspace, there exist completely ordered open neighborhoods V_i of y_i and $\varepsilon=\pm 1$ such that

$$\varepsilon D(z_1, \cdots, z_n)\geqslant 0$$

for $$z_{m_i+1}<\cdots<z_{m_{i+1}}\in V_{i+1},\ 0\leqslant i\leqslant s-1,\ m_0=0,\ m_s=n.$$

Without loss of generality, we may assume that

$$x_{k_j, m_i+1}<\cdots<x_{k_j, m_{i+1}}\in V_{i+1},\quad 0\leqslant i\leqslant s-1.$$

Thus $$\Delta(x_{k_j,0}, x_{k_j,2}, \cdots, x_{k_j,n})=\Delta(x_{k_j,1}, \cdots, x_{k_j,n}),$$

which implies

$$f_{k_j}(x_{k_j,0})=-f_{k_j}(x_{k_j,1}).$$

Let $k_j\to\infty$. We obtain $r=-r$, i.e., $r=0$. This is impossible.

Definition 6. *Suppose that G is a semi-definite Z-subspace of $C(X)$, $f\in C(X)$ and $p\in G$. If there exist distinct points $x_0, \cdots, x_n$ and $\varepsilon=\pm 1$ such that*

$$f(x_i)-p(x_i)=\varepsilon(-1)^i\|f-p\|\Delta(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n),\quad 0\leqslant i\leqslant n,$$

then p is called an alternation element of f (about G).

One of the important properties of weak Chebyshev Z-subspace is that it keeps the limit of alternation elements being still an alternation element.

Theorem 7. *Suppose that G is a weak Chebyshev Z-subspace. If p_k is an alternation element of f_k and $\lim\limits_{k\to\infty}(\|f-f_k\|+\|p-p_k\|)=0$, then p is an alternation element of f.*

Proof If f in G, then $f=p$, the conclusion is obviously true. Now, we assume $r=\|f-p\|>0$, and write $r_k=\|f_k-p_k\|$. Because p_k is an alternation element of f_k, there exist $x_{k,0}, \cdots, x_{k,n}$ and $\varepsilon=\pm 1$ such that

$$f_k(x_{k,i})=(-1)^i\cdot\varepsilon\cdot r_k\cdot\Delta(x_{k,0}, \cdots, x_{k,i-1}, x_{k,i+1}, \cdots, x_{k,n}),\quad 0\leqslant i\leqslant n.$$

By selecting a subsequence, we may assume that $\varepsilon_k=\varepsilon$ for all k. Using Lemma 5, we see that there exist $n+1$ distinct points $x_0, \cdots, x_n$ such that

$$\begin{aligned} f(x_i)&=(-1)^i\varepsilon r\cdot\Delta(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)\\ &=(-1)^i\varepsilon\|f-p\|\cdot\Delta(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n),\quad 0\leqslant i\leqslant n. \end{aligned}$$

This means that p is an alternation element of f.

§ 3. Characterization of Z-Subspaces in $CS_n(X)$

By studying alternation elements, we obtain a sufficient condition for G in $CS_n(X)$. And it turns out that if X is locally connected and G is a Z-subspace, then the converse of Theorem B in section 1 is also true.

Lemma 8. *Suppose that G is a weak Chebyshev Z-subspace, If, for every f in $C(X)$, f has one and only one alternation element, then P_G has a unique pointwise-*

Lipschitz-continuous semi-additive selection.

Proof Combining the proof of Theorem 1.7 in [3] and Lemma 5, we get the proof of this lemma.

Lemma 9. *Suppose that G is an n-dimensional weak Chebyshev Z-subspace of $C(X)$ and B is a nowhere dense subset of X. If G satisfies Haar condition on $X\backslash B$, then every f in $C(X)$ has at least one alternation element.*

Proof Let

$$B_k=\{x\text{: there exists } y \text{ in } B \text{ such that } |f(x)-f(y)|<1/k \text{ and}$$
$$\max\{|g(x)-g(y)|: g \text{ in } G \text{ with } \|g\|=1\}<1/k\};$$
$$X_k=X\backslash B_k,\ k\geqslant 1.$$

Because G satisfies Haar condition on X_k, there exists g_k in G and $x_{k,0}, \cdots, x_{k,n}$ in X_k, $\varepsilon_k=\pm 1$, such that

$$f(x_{k,i})-g_k(x_{k,i})=(-1)^i\varepsilon_k\|f-g_k\|_{X_k}\cdot\Delta(x_{k,0}, \cdots, x_{k,i-1}, x_{k,i+1}, \cdots, x_{k,n}),$$
$$0\leqslant i\leqslant n.$$

By selecting a subsequence, we may assume that $\varepsilon_k\equiv\varepsilon$ and

$$\lim_{k\to\infty} g_k=g. \tag{6}$$

From (6) and the fact that $\bigcup_{k=1}^{\infty} X_k=X\backslash B$ is a dense subset of X, we obtain

$$\lim_{k\to\infty}\|f-g_k\|_{X_k}=\|f-g\|,$$

where $\|f-g_k\|_{X_k}=\max\{|f(x)-g_k(x)|: x \text{ in } X_k\}$.

Now, using Lemma 5, we know that g is an alternation element of f (about G).

This lemma is an improvement of Theorem 1.5 in [5].

Theorem 10. *If G is an n-dimensional semi-definite Z-subspace of $C(X)$ and every nonzero g in G has at most n zeros, then any f in $C(X)$ has at most one alternation element.*

Proof Suppose that f in $C(X)$ has two alternation elements p_1 and p_2, i. e., there exist $x_{i,0}, \cdots, x_{i,n}$ in X and $\varepsilon=\pm 1$ such that

$$f(x_{i,j})-p_i(x_{i,j})=\varepsilon(-1)^j\|f-p_i\|\cdot\Delta(x_{i,0}, \cdots, x_{i,j-1}, x_{i,j+1}, \cdots, x_{i,n}),$$
$$0\leqslant j\leqslant n,\ 1\leqslant i\leqslant 2. \tag{7}$$

As every nonzero g in G has at most n zeros, for any $y_0, \cdots, y_r$, $r\leqslant n$, we have

$$\dim G|_{(y_i:0\leqslant i\leqslant r)}\geqslant r. \tag{8}$$

Thus there exist $y_{i,0}, \cdots, y_{i,s_i}$ in $\{x_{i,j}: 0\leqslant j\leqslant n\}$ such that

$$\dim G|_{(y_{i,j}:0\leqslant j\leqslant s_i, j\neq k)}=\dim G|_{(y_{i,j}:0\leqslant j\leqslant s_i)}=s_i,\quad 0\leqslant k\leqslant s_i,\ 1\leqslant i\leqslant 2. \tag{9}$$

From (7) and (9), we obtain

$\{y_{i,j}: 0\leqslant j\leqslant s\}\subset Z(p-p_i)$, for every p in $P_G(f)$, $i=1, 2$. So

$$\{y_{i,j}: 0\leqslant j\leqslant s_i,\ 1\leqslant i\leqslant 2\}\subset Z(p_1-p_2). \tag{10}$$

Because $p_1\neq p_2$, $Z(p_1-p_2)$ contains at most n zeros. From (8), (9) and (10), we get

$$\{y_{1,j}\colon 0\leqslant j\leqslant s_1\}=\{y_{2,j}\colon 0\leqslant j\leqslant s_2\}.$$

Now we may assume that $x_{1,0}=y_{1,0}=y_{2,0}=x_{2,0}$. From (8) and (9), we have

$$\dim G|_{\{x_{i,j}:1\leqslant j\leqslant n\}}=n,\ 1\leqslant i\leqslant 2. \tag{11}$$

Let $\qquad A_{i,j}=\{x\colon \dim G|_{\{x,x_{i,1},\cdots,x_{i,n}\}}=n-1,\ 1\leqslant j\leqslant n,\ 1\leqslant i\leqslant 2.$

As every nonzero g in G has at most n zeros, $A_{i,j}$ contains at most n points. Let

$$A=\bigcup_{i=1}^{2}\bigcup_{j=1}^{n}A_{i,j}.$$

Then A is a finite subset of X. There exists an open neighborhood V of $x_{1,0}=x_{2,0}$ such that

$$A\cap V=\{x_{1,0}\}=\{x_{2,0}\}. \tag{12}$$

Select z_k in $V\setminus A$, such that

$$\lim_{k\to\infty}z_k=x_{1,0}=x_{2,0},$$

$$p_1(z_k)-p_2(z_k)\neq 0.$$

By the semi-definiteness of G, we obtain for k large enough

$$\begin{aligned}&\Delta(z_k, x_{i,1}, \cdots, x_{i,j-1}, x_{i,j+1}, \cdots, x_{i,n})\\ &=\Delta(x_{i,0}, \cdots, x_{i,j-1}, x_{i,j+1}, \cdots, x_{i,n}),\quad 1\leqslant j\leqslant n,\ 1\leqslant i\leqslant 2.\end{aligned} \tag{13}$$

It follows from (7) that

$$(p_1(x_{1,j})-p_2(x_{1,j}))(f(x_{1,j})-p_1(x_{1,j}))\leqslant 0,\quad 1\leqslant j\leqslant n, \tag{14}$$

$$(p_2(x_{2,j})-p_1(x_{2,j}))(f(x_{2,j})-p_2(x_{2,j}))\leqslant 0,\quad 1\leqslant j\leqslant n. \tag{15}$$

Now, if $(p_1(z_k)-p_2(z_k))\cdot\varepsilon_1\cdot\Delta(x_{1,0}, \cdots, x_{1,n})<0$, then from (13), (14) and (7) we get

$$\begin{aligned}&(p_1(x_{1,j})-p_2(x_{1,j}))\cdot\varepsilon_1(-1)^j\cdot\Delta(z_k, x_{1,1}, \cdots, x_{1,j-1}, x_{1,j+1}, \cdots, x_{1,n})\\ &\leqslant 0,\quad 1\leqslant j\leqslant n.\end{aligned}$$

Thus

$$\begin{aligned}0&>\varepsilon_1(p_1(z_k)-p_2(z_k))D(x_{1,1}, \cdots, x_{1,n})\\ &\quad+\varepsilon_1\sum_{j=1}^{n}(p_1(x_{1,j})-p_2(x_{1,j}))\cdot(-1)^j\cdot D(z_k, x_{1,1}, \cdots, x_{1,j-1}, x_{1,j+1}, \cdots, x_{1,n})\\ &=\varepsilon_1\cdot\det\begin{pmatrix}p_1(z_k)-p_2(z_k) & p_1(x_{1,1})-p_2(x_{1,1}) & \cdots & p_1(x_{1,n})-p_2(x_{1,n})\\ g_1(z_k) & g_1(x_{1,1}) & \cdots & g_1(x_{1,n})\\ & \cdots\cdots & & \\ g_n(z_k) & g_n(x_{1,1}) & \cdots & g_n(x_{1,n})\end{pmatrix}\\ &=0.\end{aligned}$$

This is impossible.

If $(p_1(z_k)-p_2(z_k))\cdot\varepsilon_1\cdot\Delta(x_{1,1}, \cdots, x_{1,n})>0$, then from (7) and (10), we obtain

$$(p_2(z_k)-p_1(z_k))\cdot\varepsilon_2\cdot\Delta(x_{2,1}, \cdots, x_{2,n})<0.$$

Similarly, this can also lead to a contradiction.

From Lemma 8, Lemma 9 and Theorem 10, we come to the following conclusion:

Corollary 11. *Suppose that G is an n-dimensional weak Chebyshev Z-subspace and B is a nowhere dense subset of X. If G satisfies Haar condition on $X\setminus B$ and every* *nonzero g in G has at most n zeros, then P_G has a unique pointwise-Lipschitz-continuous semi-additive selection.*

Now we have our characterization theorem:

Theorem 12. *If X is a locally connected and compact Hausdorff space and G is an n-dimensional Z-subspace of $C(X)$, then the following statements are mutually equivalent:*

(1) *P_G has a unique pointwise-Lipschitz-continuous semi-additive selection,*

(2) *P_G has a continuous selection,*

(3) *every nonzero g in G has at most n zeros and has at most $n-1$ zeros with sign changes,*

(4) *G is a weak Chebyshev subspace and every nonzero g in G has at most n zeros,*

(5) *G is semi-definite and every f in $C(X)$ has a unique alternation element,*

(6) *G is semi-definite and every nonzero g in G has at most n zeros.*

Proof It is trivial that (1) implies (2). And Theorem B in section 1 states that (2) implies (3). By the theorem of Brown in [4], we know that the locally connectedness of X and (3) imply that X is homeomorphic to a union of several closed intervals. Thus, from the Lemma 2.2 in [3], we know that all conditions in Corollary 11 is satisfied by G so, (1), (4), (6) hold. And by Lemma 9, Theorem 10, we can also see that (3) implies (5). Using the same approach in [7], we can show that (5) implies (3). And it is trivial that (4) or (6) implies (3).

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