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ON HERMITIAN OPERATOR IN TENSOR PRODUCT SPACE

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Abstract

V is an n-dim unitary space. $\otimes^k V$ is the k-th tensor product space with the customary induced inner product. $\forall \mathscr{L}\in L(\otimes^k V)$,

$$W^{\perp}(\mathscr{L})=\{(\mathscr{L}x^{\otimes}, x^{\otimes})\,|\,x^{\otimes}=x_1\otimes\cdots\otimes x_k,\ x_1, \cdots, x_k\ \ o.n\}$$

is called the numerical range of $\mathscr{L}$. Wang Boying proved in [11] that if $\mathscr{L}=A_1\otimes\cdots\otimes A_k$, $A_i\in L(V)$, $i=1, \cdots, k$, $k<n$, then $\mathscr{L}$ is a hermitian operator if and only if $W^{\perp}(\mathscr{L})\subseteq\mathbb{R}$. He conjectured that if $k=n\geqslant 3$, then the similar proposition would hold, too.

In this paper the following results are obtained.

1) If $k=n\geqslant 3$, then $\mathscr{L}$ is a hermitian operator if and only if $W^{\perp}(\mathscr{L})\subseteq\mathbb{R}$.

2) The only exception is $k=n=2$ and

$$A_1\cong a\begin{pmatrix}1 & a_{12}\\ -\bar{b}_{12} & a_{22}\end{pmatrix},\quad A_2\cong b\begin{pmatrix}\bar{a}_{22} & b_{12}\\ -\bar{a}_{12} & 1\end{pmatrix}$$

where a, b are real.

§ 1. Introduction

Let V be an n-dimensional unitary space in which the inner product of column vectors x, y is defined as $(x, y)=y^*x$. $A_i\in M_n(\mathbb{C})$ is a linear operator on V. $\otimes^k V$ denotes the k-th tensor product space with the usually inner product. $x^{\otimes}=x_1\otimes\cdots\otimes x_k$, $y^{\otimes}=y_1\otimes\cdots\otimes y_k$ are decomposable tensors in $\otimes^k V$, where $x_i, y_i\in V$, $i=1, \cdots, k$. $\mathscr{L}=A_1\otimes\cdots\otimes A_k$ is a linear operator on $\otimes^k V$.

According to the definition of the induced inner product ([1, p. 47])

$$(\mathscr{L}x^{\otimes}, y^{\otimes})=\prod_{i=1}^{k}(A_i x_i, y_i).$$

In the case $k\leqslant n$, Fan[2,3,4], Goldberg[5,6], and Marcus[7,8] defined the numerical range of $\mathscr{L}$ as

$$W^{\perp}(\mathscr{L})=\{(\mathscr{L}x^{\otimes}, x^{\otimes}):\ x_1, \cdots, x_k\ \ \text{o. n.}\},$$

where "o. n." means "orthonormal", and studied its properties.

If $k=1$, as Donoghue pointed out[9], $\mathscr{L}$ is hermitian if and only if $W^{\perp}(\mathscr{L})\subseteq\mathbb{R}$.

Recently Bo-Ying Wang obtained the result that if $k<n$, then $\mathscr{L}$ is hermitian

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if and only if $W^{\perp}(\mathscr{L})\subseteq\mathbb{R}$[10]. At the same time he cited an instance to illustrate that the statement is not true in the case $k=n=2$. But he conjectured that the same statement might be true in the case $k=n\geqslant 3$.

In this paper we shall prove that his conjecture is right.

§2. Preliminaries

Lemma 1. *Let* $A=\begin{pmatrix}1 & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$, $B=\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & 1\end{pmatrix}\in M_2(\mathbb{C})$, $\mathscr{L}=A\otimes B$. *Then* $W^{\perp}(\mathscr{L}-\mathscr{L}^*)=0$ *if and only if either* i) *or* ii) *holds:*

i) *Both A and B are hermitian;*

ii) $a_{21}=-\bar{b}_{12}$, $a_{12}=-\bar{b}_{21}$, $a_{22}=\bar{b}_{11}$.

Proof Every 2×2-unitary matrix may be written as

$$\begin{pmatrix}\cos\theta u_1u_3 & -\sin\theta u_1u_4\\ \sin\theta u_2u_3 & \cos\theta u_2u_4\end{pmatrix},$$

where $\theta\in\mathbb{R}$, $u_i\in\mathbb{C}$, $|u_i|=1$, $i=1, 2, 3, 4$. Therefore each pair of o. n. vectors in 2-dimensional unitary space may be written as

$$x=(\cos\theta u_1u_3, \sin\theta u_2u_3)^T,\ y=(-\sin\theta u_1u_4, \cos\theta u_2u_4)^T.$$

Thus $W(\mathscr{L}-\mathscr{L}^*)=0$ if and only if $x^*Axy^*By=x^*A^*xy^*B^*y$, i.e.,

$$\begin{aligned}&[(\cos^2\theta+a_{22}\sin^2\theta)+a_{21}\sin\theta\cos\theta u_1u_2^{-1}+x_{12}\sin\theta\cos\theta u_1^{-1}u_2]\\ &\quad\cdot[(b_{11}\sin^2\theta+\cos^2\theta)-b_{21}\sin\theta\cos\theta u_1u_2^{-1}-b_{12}\sin\theta\cos\theta u_1^{-1}u_2]\\ &=[(\cos^2\theta+\bar{a}_{22}\sin^2\theta)+\bar{a}_{12}\sin\theta\cos\theta u_1u_2^{-1}+\bar{a}_{21}\sin\theta\cos\theta u_1^{-1}u_2]\\ &\quad\cdot[(\bar{b}_{11}\sin^2\theta+\cos^2\theta)-\bar{b}_{12}\sin\theta\cos\theta u_1u_2^{-1}-\bar{b}_{21}\sin\theta\cos\theta u_1^{-1}u_2].\end{aligned}$$

Because u_1, u_2 are independent, we obtain

$$(a_{21}b_{21}-\bar{a}_{12}\bar{b}_{12})\sin^2\theta\cos^2\theta=0,$$

$$\begin{aligned}&[a_{21}(b_{11}\sin^2\theta+\cos^2\theta)-b_{21}(\cos^2\theta+a_{22}\sin^2\theta)-\bar{a}_{12}(\bar{b}_{11}\sin^2\theta+\cos^2\theta)\\ &\quad+\bar{b}_{12}(\cos^2\theta+\bar{a}_{22}\sin^2\theta)]\sin\theta\cos\theta=0,\end{aligned}$$

$$\begin{aligned}&(a_{21}b_{12}+a_{12}b_{21}-\bar{a}_{21}\bar{b}_{12}-\bar{a}_{12}\bar{b}_{21})\sin^2\theta\cos^2\theta-(\cos^2\theta+a_{22}\sin^2\theta)(b_{11}\sin^2\theta+\cos^2\theta)\\ &\quad+(\cos^2\theta+\bar{a}_{22}\sin^2\theta)(\bar{b}_{11}\sin^2\theta+\cos^2\theta)=0.\end{aligned}$$

Since θ is arbitrary, we get

$$a_{21}b_{21}=\bar{a}_{12}\bar{b}_{12}, \tag{1}$$

$$a_{21}-b_{21}=\bar{a}_{12}-\bar{b}_{12}, \tag{2}$$

$$a_{21}b_{11}-b_{21}a_{22}=\bar{a}_{12}\bar{b}_{11}-\bar{b}_{12}\bar{a}_{22}, \tag{3}$$

$$a_{22}+b_{11}-a_{21}b_{12}-a_{12}b_{21}=\bar{a}_{22}+\bar{b}_{11}-\bar{a}_{12}\bar{b}_{21}-\bar{a}_{21}\bar{b}_{12}, \tag{4}$$

$$a_{22}b_{11}=\bar{a}_{22}\bar{b}_{11}. \tag{5}$$

From (1) and (2), either $a_{21}=\bar{a}_{12}$, $b_{21}=\bar{b}_{12}$ or $a_{21}=-\bar{b}_{12}$, $b_{21}=-\bar{a}_{12}$.

If $a_{21}=\bar{a}_{12}$, $b_{21}=\bar{b}_{12}$, we get $a_{21}b_{12}+a_{12}b_{21}\in\mathbb{R}$. Thus, we get $a_{22}+b_{11}\in\mathbb{R}$, $a_{22}b_{11}\in\mathbb{R}$ from (4) and (5). Hence, either $a_{22}, b_{11}\in\mathbb{R}$, or $\bar{a}_{22}=b_{11}\in\mathbb{C}\backslash\mathbb{R}$. If a_{22}, $b_{11}\in\mathbb{R}$, we get i). If $\bar{a}_{22}=b_{11}\in\mathbb{C}\backslash\mathbb{R}$, (3) implies $(a_{21}+b_{21})(\bar{a}_{22}-a_{22})=0$. Hence $a_{21}+b_{21}=0$. We get ii).

If $a_{21}=-\bar{b}_{12}$, $b_{21}=-\bar{a}_{12}$, similarly, we get either a_{22}, $b_{11}\in\mathbb{R}$, or $\bar{a}_{22}=b_{11}\in\mathbb{C}\backslash\mathbb{R}$. If a_{22}, $b_{11}\in\mathbb{R}$, (3) implies $(\bar{a}_{12}+\bar{b}_{12})(b_{11}-a_{22})=0$. Then either from $b_{11}=a_{22}$ we get ii), or from $\bar{a}_{12}=-\bar{b}_{12}=a_{21}$ we get i). If $\bar{a}_{22}=b_{11}\in\mathbb{C}\backslash\mathbb{R}$, we get ii).

On the other hand, if i) or ii) holds, it is easy to show that $W^{\perp}(\mathscr{L}-\mathscr{L}^*)=0$.

Lemma 2. (Marcus-Wang[11]). *Let $A_i\in L(V)=M_n(\mathbb{C})$, $i=1, \cdots, k$, $1\leqslant k\leqslant n$, $\mathscr{L}=A_1\otimes\cdots\otimes A_k$. Then $W^{\perp}(\mathscr{L})=0$ if and only if there exists an i such that $A_i=0$.*

Lemma 3. *Let A, B, C be 3×3 non-scalar matrices. Then there exists a 3×3 unitary matrix U such that a'_{ij}, b'_{ij}, $c'_{ij}\neq0$, $i, j=1, 2, 3$, where $U^*AU=(a'_{ij})$, $U^*BU=(b'_{ij})$, $U^*CU=(c'_{ij})$.*

Proof Denote

$$U=\begin{pmatrix} x_1u_1u_4 & y_1u_1u_5 & z_1u_1u_6 \\ x_2u_2u_4 & y_2u_2u_5 & z_2u_2u_6 \\ x_3u_3u_4 & y_3u_3u_5 & z_3u_3u_6 \end{pmatrix}, \tag{6}$$

where $x_1=\cos\theta\cos\alpha$, $x_2=\cos\theta\sin\alpha$, $x_3=\sin\theta$,

$$y_1=\cos\varphi\sin\alpha+\sin\varphi\sin\theta\cos\alpha,\ y_2=-\cos\varphi\cos\alpha+\sin\varphi\sin\theta\sin\alpha,\ y_3=-\sin\varphi\cos\theta,$$
$$z_1=\sin\varphi\sin\alpha-\cos\varphi\sin\theta\cos\alpha,\ z_2=-\sin\varphi\cos\alpha-\cos\varphi\sin\theta\sin\alpha,\ z_3=\cos\varphi\cos\theta, \tag{7}$$

$\varphi, \theta, \alpha\in\mathbb{R}$, $u_l\in\mathbb{C}$, $|u_l|=1$, $l=1, \cdots, 6$. It is easy to verify that U is a unitary matrix for arbitrary φ, θ, α, u_l.

If the statement does not hold, as a polynomial in u_l,

$$\prod_{i,j=1}^{3}\Big(a'_{ij}\prod_{l=1}^{6}u_l\Big)\Big(b'_{ij}\prod_{l=1}^{6}u_l\Big)\Big(c'_{ij}\prod_{l=1}^{6}u_l\Big)=0.$$

Therefore there exists at least a factor being zero as a polynomial in u_l. Without loss of generality we only consider the case $a'_{11}\prod_{l=1}^{6}u_l=0$ and the case $b'_{12}\prod_{l=1}^{6}u_l=0$.

If
$$\begin{aligned}0&=a'_{11}\prod_{l=1}^{6}u_l\\ &=[(a_{11}\cos^2\theta\cos^2\alpha+a_{22}\sin^2\theta\sin^2\alpha+a_{33}\sin^2\theta)u_1u_2u_3\\ &\quad+a_{21}\cos^2\theta\cos\alpha\sin\alpha u_1^2u_3+a_{31}\cos\theta\sin\theta\cos\alpha u_1^2u_2\\ &\quad+a_{12}\cos^2\theta\cos\alpha\sin\alpha u_2^2u_3+a_{32}\cos\theta\sin\theta\sin\alpha u_1u_2^2\\ &\quad+a_{13}\cos\theta\sin\theta\cos\alpha u_2u_3^2+a_{23}\cos\theta\sin\theta\sin\alpha u_1u_3^2]u_4u_5u_6,\end{aligned}$$

we get $a_{ij}=0$. This is impossible.

Smilarly, if

$$\begin{aligned}0&=a'_{12}\prod_{l=1}^{6}u_l\\ &=[a_{11}\cos\theta\cos\alpha(\cos\varphi\sin\alpha+\sin\varphi\sin\theta\cos\alpha)-a_{33}\sin\varphi\sin\theta\cos\theta\\ &\quad+a_{22}\cos\theta\sin\alpha(-\cos\varphi\cos\alpha+\sin\varphi\sin\theta\sin\alpha)]u_1u_2u_3u_5^2u_6\\ &\quad+a_{12}\cos\theta\cos\alpha(-\cos\varphi\cos\alpha+\sin\varphi\sin\theta\sin\alpha)u_2^2u_3u_5^2u_6\end{aligned}$$

$$-a_{13}\sin\varphi\cos^2\theta\cos\alpha u_2u_3^2u_5^2u_6$$
$$+a_{21}\cos\theta\sin\alpha(\cos\varphi\sin\alpha-\sin\varphi\sin\theta\cos\alpha)u_1^2u_3u_5^2u_6$$
$$-a_{23}\sin\varphi\cos^2\theta\sin\alpha u_1u_3^2u_5^2u_6$$
$$+a_{31}\sin\theta(\cos\varphi\sin\alpha+\sin\varphi\sin\theta\cos\alpha)u_1^2u_2u_5^2u_6$$
$$+a_{32}\sin\theta(-\cos\varphi\cos\alpha+\sin\varphi\sin\theta\sin\alpha)u_1u_2^2u_5^2u_6,$$

we get $a_{ij}=0$ $i\neq j$, and

$$(a_{11}-a_{22})\cos\varphi\cos\theta\sin\alpha\cos\alpha+(a_{11}\cos^2\alpha+a_{22}\sin^2\alpha-a_{33})\sin\varphi\sin\theta\cos\theta=0,$$

which implies that $a_{11}=a_{22}=a_{33}$, i. e., $A=a_{11}I$. This is impossible.

Lemma 4. *Let $A, B, C\in M_3(\mathbb{C})$ be non-scalar. Then*

$$W^\perp(A\otimes B\otimes C-A^*\otimes B^*\otimes C^*)=0$$

if and only if $A=aA_1$, $B=bB_1$, $C=cC_1$, where A_1, B_1, C_1 are hermitian and $0\neq abc\in\mathbb{R}$.

Proof The "if" part is clear.

The "only if" part: We may assume that $a_{ij}, b_{ij}, c_{ij}\neq 0$. Take $e_1=(1, 0, 0)^T$, $e_2=(0, 1, 0)^T$, $e_3=(0, 0, 1)^T$. Since

$$(Ae_1, e_1)(Be_2, e_2)(Ce_3, e_3)=(A^*e_1, e_1)(B^*e_2, e_2)(C^*e_3, e_3),$$

we get $a_{11}b_{22}c_{32}=\bar a_{11}\bar b_{22}\bar c_{33}\neq 0$. Hence

$$W^\perp((a_{11}^{-1}A)\otimes(b_{22}^{-1}B)\otimes(c_{33}^{-1}C)-(a_{11}^{-1}A)^*\otimes(b_{22}^{-1}B)^*\otimes(c_{33}^{-1}C)^*)=0.$$

It is enough to show that $a_{11}^{-1}A$, $b_{22}^{-1}B$, $c_{33}^{-1}C$ are hermitian. Thus we may assume that $a_{11}=b_{22}=c_{33}=1$.

First, let x, y, z be the column vectors of the matrix (6), which are o. n.. Thus

$$(Ax, x)=(a_{11}x_1^2+a_{22}x_2^2+a_{33}x_3^2)+a_{21}x_1x_2u_1u_2^{-1}+a_{31}x_1x_3u_1u_3^{-1}+a_{12}x_1x_2u_1^{-1}u_2$$
$$+a_{32}x_2x_3u_2u_3^{-1}+a_{13}x_1x_3u_1^{-1}u_3+a_{23}x_2x_3u_2^{-1}u_3.$$

Similarly we may get the expressions of (By, y), (Cz, z), (A^*x, x), (B^*y, y), (C^*z, z). The following polynomials are equal:

$$u_1^3u_2^3u_3^3(Ax, x)(By, y)(Cz, z)=u_1^3u_2^3u_3^3(A^*x, x)(B^*y, y)(C^*z, z).$$

Comparing their coefficients of $u_1^6u_3^3$, $u_1^6u_2u_3^2$ and $u_1^5u_3^4$, we get

$$\begin{aligned}&x_1x_2y_1y_2z_1z_2(a_{21}b_{21}c_{21}-\bar a_{12}\bar b_{12}\bar c_{12})=0,\\&x_1x_2y_1y_2z_1z_3(a_{21}b_{21}c_{31}-\bar a_{12}\bar b_{12}\bar c_{13})+x_1x_2y_1y_3z_1z_2(a_{21}b_{31}c_{21}-\bar a_{12}\bar b_{13}\bar c_{12})\\&\quad+x_1x_3y_1y_2z_1z_2(a_{31}b_{21}c_{21}-\bar a_{13}\bar b_{12}\bar c_{12})=0,\\&x_1x_2y_1y_2z_2z_3(a_{21}b_{21}c_{23}-\bar a_{12}\bar b_{12}\bar c_{32})+x_1x_2y_2y_3z_1z_2(a_{21}b_{23}c_{21}-\bar a_{12}\bar b_{32}\bar c_{12})\\&\quad+x_2x_3y_1y_2z_1z_2(a_{23}b_{21}c_{21}-\bar a_{32}\bar b_{12}\bar c_{12})=0.\end{aligned}\tag{8}$$

Substituting in (8) by (7) and noticing that φ, θ, α are arbitrary, we obtain

$$a_{21}b_{21}c_{21}=\bar a_{12}\bar b_{12}\bar c_{12},\ a_{21}b_{21}c_{31}=\bar a_{12}\bar b_{12}\bar c_{13},\ a_{21}b_{21}c_{23}=\bar a_{12}\bar b_{12}\bar c_{32}.$$

Secondly it is easy to show that

$$W^\perp\left[\begin{pmatrix}1 & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\otimes\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & 1\end{pmatrix}-\begin{pmatrix}1 & a_{12}\\ x_{21} & a_{22}\end{pmatrix}^*\otimes\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & 1\end{pmatrix}^*\right]=0,$$

which implies that $a_{21}b_{21}=\bar a_{12}\bar b_{12}$ according to (1). But $a_{21}b_{21}\neq 0$. Hence $c_{ij}=\bar c_{ji}$, $i\neq j$,

$i, j=1, 2, 3$. Similarly we can obtain $a_{ij}=\bar{a}_{ji}$, $b_{ij}=\bar{b}_{ji}$, $i\neq j$, $i, j=1, 2, 3$.

Finally it is enough to show that a_{ii}, b_{ii}, c_{ii}, $i=1, 2, 3$, are real. Otherwise, if $a_{22}\bar{\in}\mathbb{R}$, we have $b_{11}=\bar{a}_{22}$, according to the proof of Lemma 1. Denote $A_2=a_{22}^{-1}A$, $B_2=b_{11}^{-1}B$, $C_2=C$. Then it is clear that $W^{\perp}(B_2\otimes A_2\otimes C_2-B_2^*\otimes A_2^*\otimes C_2^*)=5$ and $b'_{11}=a'_{22}=c'_{33}=1$. From the foregoing proof, we get $b'_{21}=b_{11}^{-1}b_{21}=\bar{b}'_{12}=\bar{b}_{11}^{-1}\bar{b}_{12}$. Since $b_{21}=\bar{b}_{12}\neq 0$, it implies that $b_{11}=\bar{b}_{11}$. This is absurd and shows that b_{11} is real. Similarly we can prove that other diagonal elements of A, B, C are real, too. Therefore A, B, C are hermitian.

Obviously the restrictive condition that A, B, C are not scalar is not necessary since we have the following more general result.

Lemma 5. *Let $A, B, C\in M_3(\mathbb{C})$. Then a necessary and sufficient condition of $W^{\perp}(A\otimes B\otimes C-A^*\otimes B^*\otimes C^*)=0$ is $A\otimes B\otimes C=A^*\otimes B^*\otimes C^*$.*

Proof It is enough to show that the necessarity holds if $C=cI$. Let $A_1=cA$, $B_1=B$, we get $W^{\perp}(A_1\otimes B_1\otimes I-A_1^*\otimes B_1^*\otimes I^*)=0$, which implies that $W^{\perp}(A_1\otimes B_1-A_1^*\otimes B_1^*)=0$. Hence $A_1\otimes B_1=A_1^*\otimes B_1^*$, according to the result proved by B. Y. Wang[10, Th1]. It is easy to show that $A\otimes B\otimes C=A^*\otimes B^*\otimes C^*$.

§ 3. Main Results

Theorem 1. *Let $0\neq A_i\in M_n(\mathbb{C})$, $i=1, \cdots, n$, $n\geqslant 3$. Then*

$$W^{\perp}(A_1\otimes\cdots\otimes A_n-A_1^*\otimes\cdots\otimes A_n^*)=0$$

if and only if $A_i=a_iA'_i$, where each A'_i is hermitian and $\prod_{i=1}^{n}a_i\in\mathbb{R}$.

Proof The "if" part is clear.

The "only if" part: According to Marcus-Wang Theorem (foregoing Lemma 2), we may assume that ${}_ka_{kk}=1$ and prove that each $A_k=({}_ka_{ij})$ is hermitian.

In the case $n=3$, the required result may be obtained out of Lemma 5. If the statement is true in the case $n(\geqslant 3)$, we consider the case $n+1$.

Let $x_{n+1}=e_{n+1}=(0, \cdots, 0, 1)^T$ and $x_1, \cdots, x_n, x_{n+1}$ o. n.. Then we get

$$\prod_{i=1}^{n+1}(A_ix_i, x_i)=\prod_{i=1}^{n+1}(A_i^*x_i, x_i).$$

Since ${}_{n+1}a_{n+1,n+1}=1$, it implies that

$W^{\perp}(A_1(n+1|n+1)\otimes\cdots\otimes A_n(n+1|n+1)-A_1^*(n+1|n+1)\otimes\cdots\otimes A_n^*(n+1|n+1))=0$. According to the inductive assumption, $A_1(n+1|n+1)=A_1^*(n+1|n+1)$. Similarly we may get $A_1(n|n)=A_1^*(n|n)$, $A_1(n-1|n-1)=A_1^*(n-1|n-1)$. Therefore A_1 is hermitian. Using the same method we may prove that each A_k is hermitian, too.

Theorem 2. *Let $A_i\in M_n(\mathbb{C})$, $i=1, \cdots, n$, $n\geqslant 3$. Then $W^{\perp}(A_1\otimes\cdots\otimes A_n-A_1^*\otimes\cdots\otimes A_n^*)=0$ if and only if $A_1\otimes\cdots\otimes A_n=A_1^*\otimes\cdots\otimes A_n^*$.*

It suffices to point out that if some $A_i=0$, the statement is true obviously.

Thus we may prove the following result which affirmatively solves the open question cited by B. Y. Wang.

Theorem 3. *Let* $A_i \in M_n(\mathbb{C}) = L(V)$, $i=1, \cdots, n$, $n = \dim V \geq 3$. $\mathscr{L} = A_1 \otimes \cdots \otimes A_n$. *Then* $\mathscr{L}$ *is a hermitian operator on* $\otimes^n V$ *if and only if* $W^\perp(\mathscr{L}) \subseteq \mathbb{R}$.

Proof Since $2\mathscr{L} = (\mathscr{L}+\mathscr{L}^*) - i[i(\mathscr{L}-\mathscr{L}^*)]$ and both $(\mathscr{L}+\mathscr{L}^*)$ and $i(\mathscr{L}-\mathscr{L}^*)$ are hermitian, $W^\perp(\mathscr{L}) \subseteq \mathbb{R}$ if and only if $W^\perp(\mathscr{L}-\mathscr{L}^*) = 0$. But $W^\perp(\mathscr{L}-\mathscr{L}^*) = 0$ if and only if $\mathscr{L}$ is hermitian according to Theorem 2. The theorem is proved.

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