Chin. Ann. of Math.
8B (4) 1987

GL_2 OVER FULL RINGS

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Abstract

In this paper the authors generalize the result of B. R. McDonald. The following result is obtained.

Let R be a (1, 6)-full ring, or (3, 3)-full ring, or (2, 4)-full ring, and R has the property T. G is an $SL_2(R)$-normal subgroup of $GL_2(R)$. Then there is an ideal A in R such that

$$SC(R, A)\subseteq G\subseteq GC(R, A).$$

The problem of GL_2 over full rings was discussed in [1]. In this paper we do not demand that 2 be a unit and we discuss normal subgroups of GL_2 over full rings. We generalize the results of [1, 2, 3].

Suppose R is a commutative ring with 1. $U(R)$ denotes the multiplicative group of units of R. Suppose m and n are integers and $m>1$, $n\geqslant 2$. For any $m\times n$ matrix $A=(a_{ij})$ whose element a_{ij} satisfies the equations

$$\sum_{j=1}^{n} a_{ij}R=R,\ i=1,\ \cdots,\ m,$$

if there exist $b\in R$, $u_i\in U(R)$, $i=1, \cdots, m$, such that

$$A(1,\ b,\ \cdots,\ b^{n-1})^T=(u,\ \cdots,\ u_m)^T,$$

we call R an (m, n)-full ring.

We know ϕ-surjective rings and Von Neumann regular rings are (m, n)-full rings for some positive integers m and n. Particularly, primitive rings are (m, n)-full rings, for any integers $m\geqslant 1$, $n\geqslant 2$. If R is an (m, n)-full ring, then it must be an (s, t)-full ring where $1\leqslant s\leqslant m$, $2\leqslant t\leqslant n$. We know that a (1, 2)-full ring is a ring of stable range 1[1].

$GL_2(R)$ denotes multiplicative group of all 2×2 inverse matrices over R. $SL_2(R)=\{\sigma\in GL(R)\mid \det\sigma=1\}$.

Suppose A is an ideal of R, π_A denotes a natural ring homomorphism from R into R/A. Then the following map

$$\lambda_A:\ \sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\longrightarrow \lambda_A\sigma=\begin{pmatrix} \pi_A a & \pi_A b \\ \pi_A c & \pi_A d \end{pmatrix}$$

Manuscript received March 29, 1985. Revised August 6, 1985.

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is a group homomorphism from $GL_2(R)$ into $GL_2(R/A)$.

When A is a proper ideal of R, denote

$$GC_2(R, A)=\{\sigma\in GL_2(R)\mid \lambda_A\sigma\in \text{ center of } GL_2(R/A)\},$$
$$SC_2(R, A)=\{\sigma\in SL_2(R)\mid \lambda_A\sigma=1\}.$$

Particalarly, $GC_2(R, R)=GL_2(R)$, $SC_2(R, R)=SL_2(R)$ and it is clear that $GC_2(R, A)\lhd GL(R)$, $SC_2(R, A)\lhd GL(R)$.

If $\sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in GL_2(R)$, we call ideal $O(\sigma)=(a-d)R+bR+cR$ the order of σ. We have $O(p^{-1}\sigma p)=O(\sigma)$, $\forall p\in GL_2(R)$. If G is a subgroup of $GL_2(R)$, we call $O(G)=\sum\limits_{\sigma\in G} O(\sigma)$ the order of G. The e_{ij} denotes the 2×2 matrix over R where position (i, j) is 1, but other positions are all zero, $1\leqslant i, j\leqslant 2$. Let $\tau_{ij}(a)=I+ae_{ij}$, $i\neq j$, $1\leqslant i, j\leqslant 2$. We have $\tau_{ij}(a)\in SL_2(R)$, $\forall a\in R$, and call $\tau_{ij}(a)$ the transvection. If $a\in U(R)$, Δ_a denotes the matrix $\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}$. Obviously, $\Delta_a\in SL_2(R)$.

Lemma 1. *Suppose R is a (1, 2)-full ring, A is ideal of R. G_A denotes $SL_2(R)$-normal subgroup generated by all $\tau_{12}(a)$, $\forall a\in A$. Then $G_A=SL_2(R, A)$.*

Proof If $\tau_{12}(a)\in G_A$, then $\tau_{12}(a)=\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}^{-1}\tau_{12}^{-1}(a)\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}^{-1}\in G_A$. Let $\sigma\in SC_2(R,A)$. Then there exists $a_{ij}\in A$ such that $\sigma=\begin{pmatrix} 1+a_{11} & a_{12} \\ a_{21} & 1+a_{22}\end{pmatrix}$ and $\det\sigma=1$.

Since $(1+a_{11})R+a_{21}R=R$, there exists $x\in R$ such that $(1+a_{11})+xa_{21}\in U(R)$. Let $\sigma_1=\tau_{12}(x)\sigma\tau_{12}^{-1}(x)=\begin{pmatrix} 1+b_{11} & b_{12} \\ b_{21} & 1+b_{22}\end{pmatrix}$. Then $\sigma_1\in SC_2(R,A)$, $1+b_{11}\in U(R)$. We have

$$\sigma_1=\tau_{21}((1+b_{11})^{-1}b_{21})\,[\tau_{21}(-1)\tau_{21}((1+b_{11})^{-1}b_{11})\tau_{12}(b_{11})\tau_{21}^{-1}(-1)]$$
$$\times\tau_{12}((1+b_{11})^{-1}(b_{12}-b_{11})).$$

So $\sigma_1\in G_A$ and $SC_2(R, A)\subseteq G$. Reverse inclusion is obvious.

Definition. *Suppose R is a commutative ring with 1 and w is any element of R. If the $SL_2(R)$-normal subgroup generated by $\tau_{12}(w)$ includes all the $\tau_{12}(wr)$, $\forall r\in R$, we say that R has property T.*

In [3] we know the full ring which has not property T is existent.

Lemma 2. (1) *Suppose R is a commutative ring with 1. If each element of R can be denoted by algebrac sum of squares of some units of R, then R has property T.*

(2) *Suppose $\{F_t\mid t\in\Lambda\}$ is a variety consisted of fields F_t. If $F_t\neq F_2$, F_3, $\forall t\in\Lambda$, then driect product $R=\prod\limits_{t\in\Lambda} F_t$ has property T.*

Proof Let G_w denote $SL_2(R)$-normal subgroup in $GL_2(R)$ generated by $\tau_{12}(\omega)$.

(1) It is clear that every element r of R has the form $\pm x_1^2\pm\cdots\pm x_n^2, x_i\in U(R)$, $i=1, \cdots, n$. Thus

$$\tau_{12}(wr) = (\Delta_{x_1}\tau_{12}^{\pm1}(\omega)\Delta_{x_1}^{-1})\cdots(\Delta_{x_n}\tau_{12}^{\pm1}(\omega)\Delta_{x_n}^{-1}) \in G_\omega.$$

(2) We take $\omega = (\cdots, \omega_t, \cdots) \in \prod_{t\in\Lambda} F_t$ where $\omega_t \in F_t$, $t \in \Lambda$. Let G_{ω_t} be the $SL_2(F_t)$-normal subgroup in $GL_2(F_t)$ generated by $\tau_{12}(\omega_t)$. Then $G_\omega = \prod_{t\in\Lambda} G_{\omega_t}$. From simple property of $PSL_2(F_t)$ we can see $\tau_{12}(\omega r) \in G_\omega$.

Corollary. *If R is a (1, 4)-full ring and $2 \in U(R)$, then R has property T.*

Proof According to the property of (1, 4)-full rings we can obtain $a_r \in R$ such that $ra_r - a_r^2 \in U(R)$, $\forall r \in R$. Thus a_r, $r - a_r \in U(R)$ and $r = a_r + (r - a_r)$. Similarly, there exists $b \in R$ such that $a_r b + b^3 \in U(R)$, then b, $a_r + b^2 \in U(R)$. So $2^{-1}b$, $2^{-1}a_r b^{-1}$, $2^{-1}b^{-1}(a_r + b^2) \in U(R)$ and

$$a_r = 2[(2^{-1}b + 2^{-1}a_r b^{-1})^2 - (2^{-1}b) - (2^{-1}a_r b^{-1})^2].$$

Similarly, $r - a_r$ can be denoted by algebrac sum of squares of some units of R and so can r.

Theorem 1. *Suppose R is a (1, 6)-full ring, or (3, 3)-full ring, or (2, 4)-full ring, $\sigma \in GL_2(R)$ and $O(\sigma) = R$. Then $SL_2(R) \subseteq G_\sigma$ where G_σ is the $SL_2(R)$-normal subgroup in $GL_2(R)$ geneyated by σ.*

Proof Let $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have $(a-d)R + bR + cR = R$. Thus there exists $x \in R$ such that

$$c_1 = c + (a-d)x - bx^2 \in U(R).$$

Let $\sigma_1 = \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$. Then $\sigma_1 \in G_\sigma$.

Since $\tau = \begin{pmatrix} c_1 & -a_1 \\ & 1 \end{pmatrix} \in GL_2(R)$, we have

$$\sigma_2 = \tau\sigma\tau^{-1} = \begin{pmatrix} 0 & -\det\sigma \\ 1 & a+d \end{pmatrix} \in \tau G_\sigma \tau^{-1}.$$

We can take $y \in R$ such that $y(y^4 - 1) \in U(R)$. According to the property of full rings, then

$$\sigma_3 = \Delta_y^{-1}\sigma_2\Delta_y\sigma_2^{-1} = \begin{pmatrix} y^{-2} & 0 \\ (a+d)(y^2-1)\det\sigma^{-1} & y^2 \end{pmatrix} \in \tau G_\sigma \tau^{-1},$$

$$\tau_{21}(r) = \tau_{21}(r(1-y^4)^{-1})\sigma_3\tau_{21}^{-1}(r(1-y^4)^{-1})\sigma_3^{-1} \in \tau G_\sigma \tau^{-1}, \quad \forall r \in R.$$

Hence $SL_2(R) \subseteq \tau G_\sigma \tau^{-1}$ and $SL_2(R) \subseteq G_\sigma$.

From now on we always assume that R has the property T.

Lemma 3. *Let R be a (1, 2)-full ring, G an $SL_2(R)$-normal subgroup of $GL_2(R)$.*

(1) *If $\tau_{12}(a) \in G$, then $SO_2(R, aR) \subseteq G$.*

(2) *Let $\{A_i \mid i \in \Lambda\}$ be a veriety of ideals of R, $A = \sum_{i\in\Lambda} A_i$. If $SO_2(R, A_i) \subseteq G$, $\forall i \in \Lambda$, then*

$$SO_2(R, A) \subseteq G.$$

Proof (1) It can be proved by property T and Lemma 1.

(2) It is obvious by (1) of Lemma 3.

Lemma 4. *Let R be a (1, 4)-full ring, or (2, 3)-one, or (3, 2)-one. If $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in G$, where G is an $SL_2(R)$-normal subgroup of $GL_2(R)$, then $SC_2(R, O(\sigma))\subseteq G$.*

Proof We can take $x\in R$ such that $x(x^2-1)\in U(R)$. Since $\tau_{12}(d^{-1}(x^2-1)b)=\Delta_x\sigma\Delta_x^{-1}\sigma^{-1}\in G$ and R has property T, we can see $\tau_{12}(b)\in G$.

Obviously, $\tau_{12}(d-a)=\sigma\tau_{12}^{-1}(d)\sigma^{-1}\tau_{12}(d)\in G$. The result follows from Lemma 3.

Lemma 5. *R and G are both the same as in Lemma 4, $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in G$, and there exists $y\in R$ such that $1-cy\in U(R)$. Then $\tau_{12}(y_i)\in G$, $i=1, 2$, where $y_1=y(cy-2)[d(1-cy)^{-2}-a]$, $y_2=c^4y^4-4c^3y^3+6c^2y^2-4cy$.*

Proof We take

$$\tau=\begin{pmatrix} 1-cy & dy(cy-2)(1-cy)^{-1} \\ 0 & (1-cy)^{-1} \end{pmatrix}.$$

It is clear that

$$\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix}=\tau\sigma\tau^{-1}\sigma^{-1}\in G,$$

where $a_1=(1-cy)^2$, $b_1=y(cy-2)[d(1-cy)^{-2}-a]$, $d_1=(1-cy)^{-2}$. Thus $d_1-a_1=(1-cy)^{-2}(c^4y^4-4c^3y^3+6c^2y^2-4cy)$. The lemma follows from Lemma 4.

lemma 6. *Let R and G be as in Lemma 4, $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in G$ and $\tau_{12}(c)\in G$. Then $SC_2(R, O(\sigma))\subseteq G$.*

Proof According to the definition of R, there exists a $z\in R$ such that $d_1=d+cz\in U(R)$. We have

$$\sigma_1=\tau_{12}^{-1}(z)\sigma\tau_{12}(z)=\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\in G,$$

where $c_1=c$.

It is clear by the hypothesis and the property T that $\tau_{12}(c_1d_1^{-1})\in G$. Thus

$$\sigma_2=\begin{pmatrix} a_1-b_1c_1d_1^{-1} & b_1 \\ 0 & d_1 \end{pmatrix}=\sigma_1\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\tau_{12}(c_1d_1^{-1})\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1}\in G.$$

So $SC_2(R, O(\sigma_2))\subseteq G$ by Lemma 4. But $O(\sigma)=O(\sigma_1)=O(\sigma_2)=cR$. Hence $SC_2(R, O(\sigma))\subseteq G$.

Lemma 7. *Let R and G be as in Lemma 4. $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in G$ and there exists a positive integer n such that $\tau_{12}(c^n)\in G$. Then*

$$SC_2(R, O(\sigma))\subseteq G.$$

Proof The result is obviously true when $n=1$ by Lemma 6. Proceed by induction on n. Suppose Lemma is true when $n=k(k\geqslant 1)$. We observe the σ. Obviously we can suppose $d\in U(R)$ under the meaning of σ_1 in Lemma 6. According to the property of full rings there exists $x\in R$ such that $x(x^2-1)\in U(R)$. Thus we have

$$\sigma_1=\Delta_x\sigma\Delta_x^{-1}\sigma^{-1}=\begin{pmatrix}1+bcux^2 & -abux^2\\ cdu & 1-bcu\end{pmatrix}\in G,$$

where
$$u=(x^{-2}-1)\det\sigma^{-1}\in U(R).$$

Also
$$\sigma_2=\tau_{12}(1)\sigma_1\tau_{12}^{-1}(1)\sigma_1^{-1}=\begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix}\in G,$$

where
$$a_2=1+cdu+c^2d^2u^2+bc^2dx^2u^2,$$
$$b_2=-cdu-2bcx^2u-bc^2dx^2u^2-b^2c^2x^4u^2,$$
$$c_2=c^2d^2u^2,$$
$$d_2=1-cdu-bc^2du^2x^2.$$

If $\tau_{12}(c^{k+1})\in G$, then $\tau_{12}(c_2^k)=\tau_{12}((c^{k-1}d^{2k}u^{2k})c^{k+1})\in G$ by the property T. By the supposition of induction we have $SC_2(R, O(\sigma_2))\in G$. Since $O(\sigma_2)=cR$, we have also $\tau_{12}(c)\in G$. Then $SC_2(R, O(\sigma))\in G$ by Lemma 6.

Theorem 2. *Let R be a (1, 6)-full ring, or (3, 3)-full ring, or (2, 4)-full ring and have the property T. Let G be an $SL_2(R)$-normal subgroup of $GL_2(R)$. Then there exists an ideal A of R such that*

$$SC_2(R, A)\subseteq G\subseteq GC_2(R, A).$$

Proof Without loss of generality, we can assume $\sigma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in G$ with $d\in U(R)$. It is clear from the definition of the full ring that we can take $x\in R$ with $x(x^2-1)\in U(R)$. Thus

$$\sigma_1=\Delta_x\sigma\Delta_x^{-1}\sigma^{-1}=\begin{pmatrix}1+c_1ux^2 & b_1\\ c_1 & 1-c_1u\end{pmatrix}\in G,$$

where $c_1=cd(x^{-2}-1)\det\sigma^{-1}$, $u=bd^{-1}$.

Also
$$\sigma_2=\tau_{12}(2)\sigma_1\tau_{12}^{-1}(2)\sigma_1^{-1}=\begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix}\in G,$$

where $a_2=1+4c_1^2+2c_1(1+c_1ux^2)$, $c_2=2c_1^2$, $d_2=1-2c_1(1+c_1ux^2)$.

Obviously we can take $y\in R$ such that
$$y(1-c_1^2y)(1-2c_1^2y)(1-2c_1^2y+2c_1^4y^2)\in U(R).$$
Since $1-c_2y=1-c_1^2y\in U(R)$ for σ_2 and
$$8c_1^2y(c_1^2y-1)(2c_1^4y^2-2c_1^2y+1)=c_2^4y^4-4c_2^3y^3+6c_2^2y^2-4c_2y,$$
we have $\tau_{12}(8c_1^2)\in G$ by Lemma 5. But $c_2^3=(2c_1^2)^3=8c_1^6$, then $\tau_{12}(c_2^3)\in G$ by the property T. According to Lemma 7 $\tau_{12}(c_2)\in G$. Since $c_2=2c_1^2=2c^2d^2(x^{-2}-1)^2\det\sigma^{-2}$, we have $\tau_{12}(2c^2)\in G$.

Take $z\in R$ such that $z(1-cz)\in U(R)$. By Lemma 5 $\tau_{12}(c^4z^4-4c^3z^3+6c^2z^2-4cy)\in G$.

Thus $$\tau_{12}(c^5z^4)\in G.$$

Then $$\tau_{12}(c^5)\in G.$$

By Lemma 7, we have $SO_2(R, O(\sigma))\subseteq G$.

Take $A=O(G)$. According to Lemma 3, $SO_2(R, A)\subseteq G$. Obviously, $G\subseteq GO_2(R, A)$, the result follows.

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