Chin Ann. of Math.
8B (4) 1987

INFINITE DIMENSIONAL TRACKING OPTIMAL CONTROL VIA DYNAMIC OUTPUT FEEDBACK*

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Abstract

This paper explores implemantation problems of infinite dimensional linear-quadratic tracking optimal control. Based on the closed-loop result, a new formula of optimal control expressed by past-time state feedback is proved. From this, on the conditions of observability, expressions of optimal control via dynamic output feedback are derived. The main feedback operator functions are given by solution of linear integral equations.

§ 1. Introduction

Let X, U and Y be real Hilbert spaces. We consider an optimal tracking problem for infinite dimensional linear system described by a state equation and an output equation respectively

$$x(t)=e^{At}x_0+\int_0^t e^{A(t-s)}Bu(s)\,ds+\int_0^t e^{A(t-s)}f(s)\,ds, \tag{1.1}$$

$$y(t)=Cx(t), \quad t\geqslant 0, \tag{1.2}$$

with cost function

$$J(u)=\langle M(x(T)-\xi), x(T)-\xi\rangle+\int_0^T [\langle Q(x(t)-\varphi(t)), x(t)-\varphi(t)\rangle + \langle Ru(t), u(t)\rangle]\,dt. \tag{1.3}$$

Assume that $T>0$ is finite and fixed, $f(\cdot)\in L(0, T; X)$, $\varphi(\cdot)\in L^2(0, T; X)$ and $\xi\in X$ are fixed, $x_0\in X$ is an arbitrary initial state, $e^{At}(t\geqslant 0)$ is a C_0-semigroup of boundded linear operators on X, generated by a dense defined and closed operator A, besides, $B\in \mathscr{L}(U; X)$, $C\in\mathscr{L}(X; Y)$. Let $M\in\mathscr{L}(X)$, $Q\in\mathscr{L}(X)$ and $R\in\mathscr{L}(U)$ be self-adjoint such that $M\geqslant 0$, $Q\geqslant 0$ and $R\geqslant \delta_0 I>0$, where δ_0 is a positive constant.

The optimal tracking problem is to find optimal control $u(\cdot)\in L^2(0, T; U)$ which minimizes the cost function $J(u)$ for a given initial state x_0. This problem will be denoted briefly by (TP).

Manuscript received April 9, 1985. Revised September 9, 1985.

* Projects supported by the Science Fund of the Chinese Academy of Sciences.

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In this paper, $e^{A^*t}(t\geqslant 0)$ is the dual C_0-semigroup of e^{At}, all the integrals are in the Bochner sense. Two conventions are made: $e^{At}=e^{A^*t}=0$ for all $t<0$; and in those integrals of operator-valued functions we omit elements being acted.

To this problem, optimal control law expressed by state feedback was discussed by some authors, typically [1, 2]. However in practical systems, due to time lag in signal measurement, transformation and transmission, it is often impossible to provide real-time state feedback. Moreover, it is difficult or sometimes unrealistic to measure all the information of state, especially in the case of infinite dimensional state space. In view of these, there have been several suboptimal approaches like the Luenberger observers, but not applicable or satisfactory to the finite time cases.

As far as we know, optimal control via output feedback is still an attractive open problem, especially in the infinite dimensional case.

In this paper we shall establish optimal control expressed by past-time state feedback and derive from it the dynamic out-put feedback under the assumptions on observability.

§ 2. Closed-Loop Optimal Control

Theorem 1. *For any given $x_0\in X$, there exists a unique optimal control of (TP). $u(\cdot)$ is optimal control if and only if*

$$u(t)=-R^{-1}B^*[e^{A^*(T-t)}M(x(T)-\xi)+\int_t^T e^{A^*(\sigma-t)}Q(x(\sigma)-\varphi(\sigma))d\sigma],\ t\in[0,T], \tag{2.1}$$

where $x(\cdot)$ is the trajectory corresponding to $u(\cdot)$.

The proof of Theorem 1 is similar to [3] Theorem 2, so omitted.

Theorem 2. *For any given $x_0\in X$, a control $u(\cdot)$ is optimal if and only if it is a state feedback given by*

$$u(t)=-R^{-1}B^*[P(t)x(t)+\gamma(t)],\quad t\in[0,T], \tag{2.2}$$

where $x(\cdot)$ is the trajectory corresponding to $u(\cdot)$, and $P(\cdot)$: $[0,T]\to\mathscr{L}(X)$ is a strongly continuous solution of integral Riccati equation

$$P(t)=e^{A^*(T-t)}Me^{A(T-t)}+\int_t^T e^{A^*(\sigma-t)}[Q-P(\sigma)BR^{-1}B^*P(\sigma)]e^{A(\sigma-t)}d\sigma \tag{2.3}$$

such that $P(t)=P^*(t)$, and $\gamma(t)$ is given by the solution of

$$\gamma(t)=-e^{A^*(T-t)}M\xi-\int_t^T e^{A^*(\sigma-t)}[Q\varphi(\sigma)-P(\sigma)f(\sigma)]d\sigma$$
$$-\int_t^T e^{A^*(\sigma-t)}P(\sigma)BR^{-1}B^*\gamma(\sigma)d\sigma,\quad t\in[0,T]. \tag{2.4}$$

The optimal trajectory $x(\cdot)$ satisfies the following relation

$$x(t)=G(t,s)x(s)+g(t,s),\quad 0\leqslant s\leqslant t\leqslant T, \tag{2.5}$$

where $G(t, s)$, $0\leqslant s\leqslant t\leqslant T$, *is the mild evolution operator*[2] *generated by* $A-BR^{-1}B^*P(t)$, *i.e.*,

$$G(t, s)=e^{A(t-s)}-\int_s^t e^{A(t-\eta)}BR^{-1}B^*P(\eta)G(\eta, s)\,d\eta, \quad 0\leqslant s\leqslant t\leqslant T, \tag{2.6}$$

and $g(t, s)$, $0\leqslant s\leqslant t\leqslant T$, *is given by*

$$g(t, s)=\int_s^t G(t, \sigma)\,[f(\sigma)-BR^{-1}B^*\gamma(\sigma)]\,d\sigma, \quad 0\leqslant s\leqslant t\leqslant T. \tag{2.7}$$

Proof The sufficiency part. By an approach similar to [3] Lemma 2 and Theorem 6, we obtain

$$\begin{aligned} J(u)=&\int_0^T \langle R[u(t)+R^{-1}B^*(P(t)x(t)+\gamma(t))],\ u(t)+R^{-1}B^*(P(t)x(t)+\gamma(t))\rangle dt \\ &+\theta(x_0, f(\cdot), \xi, \varphi(\cdot)), \end{aligned} \tag{2.8}$$

where $\theta(x_0, f(\cdot), \xi, \varphi(\cdot))$ is a constant only depending on $\{x_0, f(\cdot), \xi, \varphi(\cdot)\}$:

$$\begin{aligned} \theta(x_0, f(\cdot), \xi, \varphi(\cdot))=&\langle P(0)x_0, x_0\rangle+\langle M\xi, \xi\rangle+\int_0^T\langle Q\varphi(t), \varphi(t)\rangle dt \\ &-\int_0^T\langle R^{-1}B^*\gamma(t), B^*\gamma(t)\rangle dt+2\langle x_0, \gamma(0)\rangle \\ &+2\int_0^T\langle f(t), \gamma(t)\rangle dt. \end{aligned} \tag{2.9}$$

The feedback control (2.2) is admissible because the equation

$$\begin{aligned} x(t)=&e^{At}x_0+\int_0^t e^{A(t-s)}B[-R^{-1}B^*(P(s)x(s)+\gamma(s))]ds \\ &+\int_0^t e^{A(t-s)}f(s)ds, \quad t\in[0,T], \end{aligned} \tag{2.10}$$

admits a unique solution given by

$$x(t)=G(t, 0)x_0-\int_0^t G(t, s)\,[BR^{-1}B^*\gamma(s)-f(s)]ds, \quad t\in[0, T]. \tag{2.11}$$

Thus (2.2) is optimal control and the optimal trajectory is given by (2.11) from which we see that (2.5) is true.

The necessity part. From (2.1) we only need to prove that $\gamma(t)=y(t)-P(t)x(t)$ is a solution of the equation (2.4), where

$$y(t)=e^{A^*(T-t)}M[x(T)-\xi]+\int_t^T e^{A^*(\sigma-t)}Q[x(\sigma)-\varphi(\sigma)]d\sigma. \tag{2.12}$$

This can be done by direct verification and omitted here.

Remark. The strongly continuous solution of Riccati equation (2.3) is unique [3].

§ 3. Past-Time State Feedback Operator

Here we prove a new formula of the optimal control expressed by past-time state feedback where the feedback operator is solution of a linear integral equation.

This is a mediate step leading to dynamic output feedback.

Lemma 1.[3] *Let $P(t)$ and $G(t, s)$ be described as in Theorem 2. Then*

$$P(t)=e^{A^*(T-t)}MG(T, t)+\int_t^T e^{A^*(\sigma-t)}QG(\sigma, t)\,d\sigma. \tag{3.1}$$

Lemma 2. *Let $P(t)$ and $G(t, s)$ be described as in Theorem 2. Then*

$$G(t, s)+\int_s^t e^{A(t-\sigma)}BR^{-1}B^*e^{A^*(T-\sigma)}MG(T, s)\,d\sigma$$

$$+\int_s^t e^{A(t-\sigma)}BR^{-1}B^*\int_\sigma^T e^{A^*(\eta-\sigma)}QG(\eta, s)\,d\eta d\sigma=e^{A(t-s)},\quad 0\leqslant s\leqslant t\leqslant T. \tag{3.2}$$

Proof Substituting (3.1) into (2.6) and using the evolution property of $G(t,s)$, we know that (3.2) is true.

Define operator functions $K(t, s)$ and $W(t, s)$ as follows

$$K(t, s)=e^{A^*(T-t)}MG(T, s)+\int_t^T e^{A^*(\sigma-t)}QG(\sigma, s)\,d\sigma,\quad (t, s)\in[0, T]^2, \tag{3.3}$$

$$W(t, s)=e^{A^*(T-t)}Me^{A(T-s)}+\int_0^T e^{A^*(\sigma-t)}Qe^{A(\sigma-s)}d\sigma,\quad (t, s)\in[0, T]^2. \tag{3.4}$$

Here, in addition to previous convention that $e^{At}=e^{A^*t}=0$ if $t<0$, we make a convention that $G(t, s)=0$ if $t<s$.

Lemma 3. *$K(t, s)$ is a unique strongly continuous solution of the following integral operator equation*

$$K(t, s)+\int_s^T W(t, \sigma)BR^{-1}B^*K(\sigma, s)\,d\sigma=W(t, s),\ (t, s)\in[0, T]^2. \tag{3.5}$$

Proof The strong continuity of $K(t, s)$ is apparent. By means of (3.2) and order exchanging of the integrations, we have

$$\begin{aligned}
&\int_s^T W(t, \sigma)BR^{-1}B^*K(\sigma, s)\,d\sigma\\
&=e^{A^*(T-t)}M\int_s^T e^{A(T-\sigma)}BR^{-1}B^*e^{A^*(T-\sigma)}MG(T,t)\,d\sigma\\
&\quad+e^{A^*(T-t)}M\int_s^T\int_\sigma^T e^{A(T-\sigma)}BR^{-1}B^*e^{A^*(\eta-\sigma)}QG(\eta, s)\,d\eta\, d\sigma\\
&\quad+\int_s^T\left(\int_0^T e^{A^*(\rho-t)}Qe^{A(\rho-\sigma)}\,d\rho\right)BR^{-1}B^*e^{A^*(T-\sigma)}MG(T, s)\,d\sigma\\
&\quad+\int_s^T\left(\int_0^T e^{A^*(\rho-t)}Qe^{A(\rho-\sigma)}\,d\rho\right)BR^{-1}B^*\left(\int_\sigma^T e^{A^*(\eta-\sigma)}QG(\eta, s)\,d\eta\right)d\sigma\\
&=e^{A^*(T-t)}M\{e^{A(T-s)}-G(T, s)\}\\
&\quad+\int_t^T e^{A^*(\rho-t)}Q\left\{\int_s^\rho e^{A(\rho-\sigma)}BR^{-1}B^*e^{A^*(T-\sigma)}MG(T, s)\,d\sigma\right\}d\rho\\
&\quad+\int_t^T e^{A^*(\rho-t)}Q\left\{\int_s^\rho e^{A(\rho-\sigma)}BR^{-1}B^*\int_\sigma^T e^{A^*(\eta-\sigma)}QG(\eta, s)\,d\eta d\sigma\right\}d\rho\\
&=e^{A^*(T-t)}M\,\{e^{A(T-s)}-G(T, s)\}+\int_t^T e^{A^*(\rho-t)}Q\{e^{A(\rho-s)}-G(\rho, s)\}d\rho\\
&=W(t, s)-K(t, s),\quad (t, s)\in[0, T]^2.
\end{aligned} \tag{3.6}$$

Hence $K(t, s)$ is a solution of the equation (3.5).

Now prove the uniqueness. For any fixed $s\in[0, T]$, let $\Delta K(t, s)$ be strongly continuous and such that

$$\Delta K(t, s)+\int_s^T W(t, \sigma)BR^{-1}B^*\Delta K(\sigma, s)d\sigma=0, \quad t\in[0, T]. \tag{3.7}$$

As in [5] (p. 112) we can prove $R^{-1}B^*\Delta K(t, s)=0$ for $t\in[s, T]$. Substitute it into (3.7), then $\Delta K(t, s)\equiv 0$ for $(t, s)\in[0, T]^2$.

Based on the previous results, we establish past-time state feedback formula as follows.

Theorem 3. *For any given $x_0\in X$, the optimal control $u(\cdot)$ of (TP) is given by*

$$u(t)=\begin{cases}-R^{-1}B^*[K(t, t-\delta)x(t-\delta)+\lambda(t, t-\delta)], & t\in[\delta, T],\\ -R^{-1}B^*[K(t, 0)x_0+\lambda(t, 0)], & t\in[0, \delta],\end{cases} \tag{3.9}$$

where $\delta>0$ is any given small constant, $x(\cdot)$ is the corresponding trajectory, $K(t, s)$ is the unique strongly continuous solution of the linear integral equation (3.5), *and*

$$\lambda(t, s)=-e^{A^*(T-t)}M\xi-\int_t^T e^{A^*(\sigma-t)}Q\varphi(\sigma)d\sigma$$
$$+\int_s^T K(t, \sigma)[f(\sigma)-BR^{-1}B^*\gamma(\sigma)]d\sigma. \tag{3.10}$$

Proof From Theorem 1, for any given $x_0\in X$, the optimal control of (TP) uniquely exists and must be given by (2.1). Substituting (2.5) and (2.7) into (2.1), we obtain the following relation satisfied by the optimal control $u(\cdot)$

$$u(t)=-R^{-1}B^*\{(e^{A^*(T-t)}M[G(T, t-\delta)x(t-\delta)+g(T, t-\delta)-\xi]$$
$$+\int_t^T e^{A^*(\eta-t)}Q[G(\eta, t-\delta)x(t-\delta)+g(\eta, t-\delta)-\varphi(\eta)]d\eta\}$$
$$=-R^{-1}B^*\{K(t, t-\delta)x(t-\delta)-e^{A^*(T-t)}M\xi-\int_t^T e^{A^*(\eta-t)}Q\varphi(\eta)d\eta$$
$$+\int_{t-\delta}^T e^{A^*(T-t)}MG(T, \rho)[f(\rho)-BR^{-1}B^*\gamma(\rho)]d\rho$$
$$+\int_t^T e^{A^*(\eta-t)}Q\int_{t-\delta}^{\eta} G(\eta, \rho)[f(\rho)-BR^{-1}B^*\gamma(\rho)]d\rho d\eta\}$$
$$=-R^{-1}B^*\{K(t,t-\delta)x(t-\delta)-e^{A^*(T-t)}M\xi-\int_t^T e^{A^*(\eta-t)}Q\varphi(\eta)d\eta$$
$$+\int_{t-\delta}^T\{e^{A^*(T-t)}MG(T, \rho)$$
$$+\int_t^T e^{A^*(\eta-t)}QG(\eta, \rho)d\eta\}[f(\rho)-BR^{-1}B^*\gamma(\rho)]d\rho\}$$
$$=-R^{-1}B^*\{K(t, t-\delta)x(t-\delta)+\lambda(t, t-\delta)\}, \quad t\in[\delta, T], \tag{3.11}$$

where $K(t, s)$ and $\lambda(t, s)$ are given by (3.3) and (3.10) respectively.

On the interval $[0, \delta]$, we have to make use of the initial state value x_0 in order to get the optimal $u(t)$. Similarly

$$\begin{aligned}u(t) &= -R^{-1}B^*\{K(t,0)x_0 - e^{A^*(T-t)}M\xi - \int_t^T e^{A^*(\sigma-t)}Q\varphi(\sigma)d\sigma \\ &\quad + \int_0^T K(t,\sigma)[f(\sigma) - BR^{-1}B^*\gamma(\sigma)]d\sigma\} \\ &= -R^{-1}B^*\{K(t,0)x_0 + \lambda(t,0)\}, \quad t\in[0,\delta]. \end{aligned} \tag{3.12}$$

By Lemma 3, $K(t, s)$ is characterized by the equation (3.5).

Theorem 4. *If $Q=0$ in (1.3), then for any given $x_0\in X$, the optimal control of (TP) is given by*

$$u(t)=\begin{cases}-R^{-1}B^*\{e^{A^*(T-t)}\sqrt{M}[I+\Lambda(t-\delta)]^{-1}\sqrt{M}e^{A(T-(t-\delta))}x(t-\delta)-l(t,t-\delta)\}, \\ \quad t\in[\delta,T], \\ -R^{-1}B^*\{e^{A^*(T-t)}\sqrt{M}[I+\Lambda(0)]^{-1}\sqrt{M}e^{AT}x_0+l(t,0)\}, \quad t\in[0,\delta],\end{cases} \tag{3.13}$$

where $\delta>0$ is any given small constant, $x(\cdot)$ is the corresponding trajectory,

$$\Lambda(t)=\sqrt{M}\int_t^T e^{A(T-s)}BR^{-1}B^*e^{A^*(T-s)}ds\sqrt{M}, \tag{3.14}$$

and

$$\begin{aligned}l(t,s) &= -e^{A^*(T-t)}M\xi - \int_t^T e^{A^*(\sigma-t)}Q\varphi(\sigma)d\sigma \\ &\quad + \int_s^T e^{A^*(T-t)}\sqrt{M}[I+\Lambda(\sigma)]^{-1}\sqrt{M}e^{A(T-\sigma)}[f(\sigma)-BR^{-1}B^*\gamma(\sigma)]d\sigma.\end{aligned} \tag{3.15}$$

Proof In this case, because of $Q=0$, it is easy to verify that the solution of (3.5) is explicitly given by

$$K(t,s)=e^{A^*(T-t)}\sqrt{M}[I+\Lambda(s)]^{-1}\sqrt{M}e^{A(T-s)}, \quad (t,s)\in[0,T]^2, \tag{3.16}$$

where $\Lambda(\cdot)$ is defined by (3.14). Thus, (3.13) is the consequence of (3.9) with (3.16).

§ 4. Dynamic Output Feedback

Definition 1. *$\{A, C\}$ is continuously observable*[2] *if for any given $\delta>0$, there exists a constant $\alpha>0$ such that*

$$\|Ce^{At}x_0\|_{L^2(0,\delta;Y)}\geqslant\alpha\|x_0\|, \quad \forall x_0\in X. \tag{4.1}$$

Definition 2. *Linear system (1.1)—(1.2) is closed-loop continuously observable with respect to (1.3) if for any t_1, t_2, $0\leqslant t_1<t_2\leqslant T$, there exists a constant $\alpha>0$ depending on t_1, t_2 and such that*

$$\|CG(t,t_1)x_0\|_{L^2(t_1,t_2;Y)}\geqslant\alpha\|x_0\|, \quad \forall x_0\in X, \tag{4.2}$$

where $G(t, s)$ is given by (2.6).

It is obvious that (4.1) and (4.2) are respectively equivalent to the coercive positivity of following operators:

$$F(\delta)=\int_0^\delta e^{A^*t}C^*Ce^{At}\,dt\geqslant\alpha^2I, \tag{4.3}$$

$$H(t_2,\ t_1)=\int_{t_1}^{t_2}G^*(t,\ t_1)C^*CG(t,\ t_1)\,dt\geqslant\alpha^2I. \tag{4.4}$$

Theorem 5. *Assume that linear system* (1.1)—(1.2) *is closed-loop continuously observable with respect to* (1.3). *Then, for any given* $x_0\in X$, *the optimal control of* (*TP*) *is given by the following dynamic output feedback*

$$u(t)=\begin{cases}-R^{-1}B^*\{K(t,\ t-\delta)H^{-1}(t,\ t-\delta)\int_{t-\delta}^{t}G^*(s,\ t-\delta)C^*y(s)\,ds+\psi(t,\ t-\delta)\},\\ t\in[\delta,\ T],\\ -R^{-1}B^*\{K(t,\ 0)x_0+\lambda(t,\ 0)\},\quad t\in[0,\ \delta],\end{cases} \tag{4.5}$$

where $\delta>0$ *is any given small constant,* $y(\cdot)$ *is the output corresponding to the optimal process* $\{u(\cdot),\ x(\cdot)\}$,

$$\psi(t,\ t-\delta)=\lambda(t,\ t-\delta)+K(t,\ t-\delta)H^{-1}(t,\ t-\delta)\int_{t-\delta}^{t}\left\{\int_{\eta}^{t}G^*(s,\ t-\delta)C^*Ce^{A(s-\eta)}ds\right\}$$
$$\cdot\{BR^{-1}B^*[P(\eta)g(\eta,\ t-\delta)+\gamma(\eta)]-f(\eta)\}d\eta,\quad t\in[\delta,\ T], \tag{4.6}$$

and $K(t,\ s)$, $\lambda(t,\ s)$, $H(t,\ s)$, $G(t,\ s)$, $g(t,\ s)$, $P(t)$ *and* $\gamma(t)$ *are determined by* (3.5), (3.10), (4.4), (2.6), (2.7), (2.3) *and* (2.4) *respectively.*

Proof From Theorem 3 and (3.9), we only need to prove (4.5) on $[\delta,\ T]$. Let $\{u(\cdot),\ x(\cdot)\}$ be the unique optimal process for a given x_0 and $y(\cdot)$ be the corresponding output. By (1.1), (1.2), (2.2), (2.5) and (2.6), we have

$$\begin{aligned}y(s)&=Cx(s)\\ &=Ce^{A(s-(t-\delta))}x(t-\delta)-\int_{t-\delta}^{s}Ce^{A(s-\eta)}\{BR^{-1}B^*[P(\eta)x(\eta)+\gamma(\eta)]-f(\eta)\}d\eta\\ &=C\{e^{A(s-(t-\delta))}x(t-\delta)-\int_{t-\delta}^{s}e^{A(s-\eta)}BR^{-1}B^*P(\eta)G(\eta,\ t-\delta)d\eta x(t-\delta)\}\\ &\quad-C\int_{t-\delta}^{s}e^{A(s-\eta)}\{BR^{-1}B^*[P(\eta)g(\eta,\ t-\delta)+\gamma(\eta)]-f(\eta)\}d\eta\\ &=CG(s,\ t-\delta)x(t-\delta)-C\int_{t-\delta}^{s}e^{A(s-\eta)}\{BR^{-1}B^*[P(\eta)g(\eta,t-\delta)+\gamma(\eta)]-f(\eta)\}d\eta,\\ &\quad s\in[t-\delta,\ t].\end{aligned} \tag{4.7}$$

Multiply (4.7) by $G^*(s,\ t-\delta)C^*$ and integrate it for $s\in[t-\delta,\ t]$, then we obtain

$$\begin{aligned}x(t-\delta)&=H^{-1}(t,\ t-\delta)\int_{t-\delta}^{t}G^*(s,\ t-\delta)C^*y(s)\,ds\\ &\quad+H^{-1}(t,\ t-\delta)\int_{t-\delta}^{t}G^*(s,\ t-\delta)C^*C\int_{t-\delta}^{s}e^{A(s-\eta)}\{BR^{-1}B^*[P(\eta)g(\eta,\ t-\delta)\\ &\quad+\gamma(\eta)]-f(\eta)\}d\eta\,ds,\quad t\in[\delta,\ T].\end{aligned} \tag{4.8}$$

Substitute (4.8) into (3.9) and exchange the order of integrations, then we obtain (4.5). On the other hand, all the involved operator functions $P(t)$, $G(t,\ s)$, $g(t,\ s)$, $K(t,\ s)$, $\lambda(t,\ s)$ and $\gamma(t)$ are uniquely determined by our previous results. Therefore, control function (4.5) is uniquely defined and by Theorem 3 is optimal.

An alternative and more useful result of dynamic output feedback is given by the following theorem.

Theorem 6. *If $\{A, C\}$ is continuously observable, then, for any given $x_0 \in X$, the optimal control of (TP) is given by*

$$u(t)=\begin{cases}-R^{-1}B^*\{K(t,t-\delta)F^{-1}(\delta)\int_{t-\delta}^{t}[N(t,s)y(s)+\Pi(t,s)u(s)]ds \\ \quad +\zeta(t,t-\delta)\}, \quad t\in[\delta,T]. \\ -R^{-1}B^*\{K(t,0)x_0+\lambda(t,0)\}, \quad t\in[0,\delta],\end{cases} \tag{4.9}$$

where $\delta>0$ is any given small constant, $y(\cdot)$ is the output corresponding to the optimal process $\{u(\cdot), x(\cdot)\}$,

$$N(t,s)=e^{A^*(s-(t-\delta))}C^*, \tag{4.10}$$

$$\Pi(t,s)=-e^{A^*(s-(t-\delta))}F(t-s)B, \tag{4.11}$$

$$\zeta(t,t-\delta)=\lambda(t,t-\delta)-K(t,t-\delta)F^{-1}(\delta)\int_{t-\delta}^{t}e^{A^*(s-(t-\delta))}F(t-s)f(s)ds, \tag{4.12}$$

and $K(t,s)$, $\lambda(t,s)$, $F(t)$ are determined by (3.5), (3.10), (4.3) respectively.

Proof Let $\delta\in[0,T]$ be sufficiently small. By (1.1) and (1.2), we have

$$y(s)=Cx(s)=C\{e^{A(s-(t-\delta))}x(t-\delta)+\int_{t-\delta}^{s}e^{A(s-\eta)}[Bu(\eta)+f(\eta)]d\eta\}, \quad s\in[t-\delta,t], \tag{4.13}$$

where $\{u(\cdot), x(\cdot)\}$ is the optimal process for a given x_0, and $y(\cdot)$ is the corresponding output. Multiply (4.13) by $e^{A^*(s-(t-\delta))}C^*$ and integrate it for $s\in[t-\delta, t]$, we obtain

$$x(t-\delta)=F^{-1}(\delta)\int_{t-\delta}^{t}e^{A^*(s-(t-\delta))}C^*\{y(s) \\ -C\int_{t-\delta}^{s}e^{A(s-\eta)}[Bu(\eta)+f(\eta)]d\eta\}ds, \quad t\in[\delta,T]. \tag{4.14}$$

Substitute (4.14) into (3.9) and exchange the order of integration. Then, after rearrangement, we obtain (4.9).

On the other hand, we show that the control function $u(\cdot)$ constructed by dynamic output-input feedback (4.9) is unique for the same output $y(\cdot)$. This amounts to that corresponding homogeneous equation

$$\begin{cases}\tilde u(t)=-R^{-1}B^*K(t,t-\delta)F^{-1}(\delta)\int_{t-\delta}^{t}\Pi(t,s)\tilde u(s)ds, \quad t\in[\delta,T], \\ \tilde u(t)=0, \quad t\in[0,\delta],\end{cases} \tag{4.15}$$

admits only zero solution $\tilde u(t)\equiv 0$, $t\in[0,T]$. In fact, since $\tilde u(t)=0$ on $[0,\delta]$, it must be

$$\tilde u(t)=-R^{-1}B^*K(t,t-\delta)F^{-1}(\delta)\int_{\delta}^{t}\Pi(t,s)\tilde u(s)ds, \quad t\in[\delta,2\delta]\cap[0,T], \tag{4.16}$$

where $\|R^{-1}B^*K(t,t-\delta)F^{-1}(\delta)\Pi(t,s)\|_{\mathscr{L}(U)}\leqslant$const., $\delta\leqslant s\leqslant t\leqslant T$, for a given δ.

Hence, $\tilde{u}(t)=0$ on $[\delta, 2\delta]\cap[0, T]$ by the Gronwall inequality. Recursively, it must be $\tilde{u}(t)\equiv 0$ on $[0, T]$.

From Theorem 3 and this uniqueness, we conclude that control given by (4.9) is optimal.

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