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ON THE UPPER BOUND OF THE NUMBER OF PRINES IN ARITHMETIC PROGRESSION

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Abstract

Let a and q be relatively prime positive integers and $\pi(x; q, a)$ stand for the number of primes $p\leqslant x$ congruent to a and q. H. Iwanice proved that

$$\pi(x;\ q,\ a)<\frac{(2+\varepsilon)x}{\varphi(q)\log D} \tag{1}$$

for any $\varepsilon>0$, $x>x_0(\varepsilon)$ and $q\leqslant x^{\frac{9}{20}-\varepsilon}$, where $D=x\,q^{-\frac{3}{8}}$.

The author applies an improved estimation of the error term in the linear sieve, proves that for any $\varepsilon>0$, $x>x_0(\varepsilon)$ and $q\leqslant x^{\frac{5}{11}-\varepsilon}$, (1) is true.

§1. Introduction

Let a and q be relatively prime positive integers and $\pi(x;\ q,\ a)$ stand for the number of primes $p\leqslant x$ congruent to a mod q.

In 1930 E. C. Titchmarsh[1] used Brun's sieve to prove that if $q<x^{1-\varepsilon}$ then

$$\pi(x;\ q,\ a)\ll\frac{x}{\varphi(q)\log x}.$$

Recently, H. Iwaniec[2] proved that

$$\pi(x;\ q,\ a)<\frac{(2+\varepsilon)x}{\varphi(q)\log D} \tag{1}$$

for any $\varepsilon>0$, $x>x_0(\varepsilon)$ and $q\leqslant x^{9/20-\varepsilon}$, where $D=xq^{-3/8}$.

In this paper we have the following theorem:

Theorem. *For any* $\varepsilon>0$, $x>x_0(\varepsilon)$ and $q\leqslant x^{5/11-\varepsilon}$, (1) is true.

§2. A Character Soms Approach

Lemma 2.1 (Burguss). *For any $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that*

$$\sum_{l\leqslant L}\chi(l)\ll Lq^{-\delta}$$

for all non-principal charactors $\chi(\bmod q)$ and all $L\geqslant q^{3/8+\varepsilon}$, where q is any positive integer.

Given $q<x$ we consider the sequence

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$$\mathscr{A}^q=\{l\leqslant x;\ l\equiv a(\bmod q)\}$$

and for $(d, q)=1$ we denote

$$r(\mathscr{A}^q, d)=|\{l\in\mathscr{A}^q;\ l\equiv 0(\bmod q)\}|-x/qd.$$

An application of sieve method leads to (see [3, 4]) the following Lemma.

Lemma 2.2 *For any positive number s, $x>x_0(s)$ we have*

$$\pi(x; q, a)<\frac{(2+Cs)x}{\varphi(q)\log D}+R(\mathscr{A}^q, D),$$

where c is an absolute constant and

$$R(\mathscr{A}^q, D)=\sum_{(D)}\sum_{\nu<D^s}{}' C_{(D)}(\nu, s)\sum_{\substack{D_i\leqslant p_i<D_i^{1+s}\\ i\leqslant l}} r(\mathscr{A};\ \nu p_1\cdots p_l),$$

where (D) denotes a set of all subsequences of $\{D^{s^2(1+s^7)^n}, n\geqslant 0\}$ including the empty subsequence, for which $D_1\geqslant D_2\geqslant\cdots\geqslant D_l$ and

$$D_1D_2\cdots D_{2r}D_{2r+1}^3\leqslant D^{1-s_1}\quad\left(0\leqslant r\leqslant\frac{1}{2}(l-1)\right),$$

s_1 is a suitable constant. Moreover, Σ' indicates that ν and p_i $(1\leqslant i\leqslant l)$ are respected by the conditions

$$\nu\,|\,P(D^{s^2}),\qquad p_i\,|\,P(z).$$

Finally the coefficients $C_{(D)}(\nu, s)$ depend at most on (D), ν, s and satisfy

$$|C_{(D)}(\nu, s)|\leqslant 1.$$

By Lemma 2.2 the proof of Theorem reduces to showing that

$$R(\mathscr{A}^q;\ D)\ll\frac{x^{1-\delta}}{\varphi(q)}. \tag{2.1}$$

Let

$$R_k(x; D)=\sum_{(D)}\sum_{\nu=D^s} C_{(D)}(\nu, s)\sum_{\substack{D_i\leqslant p_i<D^{1+s}\\ 1\leqslant i\leqslant l}} r_k(x;\ \nu p_1\cdots p_e),$$

where

$$r_k(x; d)=\mathscr{A}_k(x; d)-x/qd,$$

$$\mathscr{A}_k(x; d)=\frac{1}{k!}\sum_{\substack{l\leqslant x\\ l\equiv a(\bmod q),\, l\equiv o(\bmod l)}}\left(\log\frac{x}{l}\right)^k.$$

We deduce from (2.5) in [2] the follwing implication:

if $R_k(x; D)\ll x^{1-\delta}/\varphi(q)$, then $R_{k-1}(x; D)\ll x^{1-\frac{\delta}{2}}/\varphi(q)$.

Therefore the proof of (2.1) reduces to showing that

$$R_4(x; D)\ll x^{1-\delta}/\varphi(q), \tag{2.2}$$

subject to $\{D_i\}\in(D)$ with any $s>0$ and some $\delta=\delta(s)>0$. By the orthogonality of characters we have for $(d, q)=1$

$$r_4(x; d)=\frac{1}{24\varphi(q)}\sum_{\substack{\chi(\bmod q)\\ \chi\neq\chi_0}}\bar{\chi}(a)\chi(d)\sum_{b\leqslant x/d}\chi(b)\left(\log\frac{x}{bd}\right)^4+O\left(\frac{x^{\varepsilon}}{q}\right).$$

The series $\{D_1, \cdots, D_l\}$ can divide into j parts, their multiples are $M_1, \cdots, M_j$ respectively. Hence letting $L=x/M_1, \cdots, M_j$

$$B(s, \chi)=\sum_{l\leqslant L}\chi(l)l^{-s},\quad M_i(s, \chi)=\sum_{M_i<m\leqslant 2M_i}a_m^{(i)}\chi(m)m^{-s},\ i\leqslant j,$$

if $M_i = D_i, \cdots, D_{ik}, 1 \leqslant i, \cdots, i_k \leqslant l$,

$$a_m^{(i)} = \sum_{p_{i1} \cdots p_{ik} = m} a_{p_{i1}} a_{p_{i2}} \cdots a_{p_{ik}}, \tag{2.3}$$

Now the proof of (2.2) reduces to estimating that

$$\frac{1}{2\pi i} \int_{(\frac{1}{2})} \frac{x^s}{s^5 \varphi(q)} \sum_{\chi \neq \chi_0} \bar{\chi}(a) B(s, \chi) M_1(s, \chi) \cdots M_j(s, \chi) ds.$$

It is sufficient to show that

$$\sum_{\chi \neq \chi_0} |B(s, \chi) M_1(s, \chi) \cdots M_j(s, \chi)| \ll |s|^3 x^{1/2-\delta}. \tag{2.4}$$

We have trivial estimate

$$|B(s, \chi)| \leqslant 2L^{1/2}; \ |M_i(s, \chi)| \leqslant M_i^{1/2} \quad (i \leqslant j),$$

thus the characters $\chi \neq \chi_0$ for which one of the above bounds is less than $(\varphi(q)x^\delta)^{-1}$ can be neglected. The set of remaining character $\chi \neq \chi_0$ can be classified into $\ll (\log x)^{j+1}$ subsets $S(U_1, \cdots, U_j, W)$ of characters satisfying simultaneous conditions

$$W < |B(s, \chi)| \leqslant 2W; \ U_i < |M_i(s, \chi)| \leqslant 2U_i, \quad i \leqslant j.$$

where $W = 2^{1-u} L^{1/2}, U_i = 2^{-v_i} M_i^{1/2}, u, v_i = 1, 2, \cdots, [2\log x]$.

It is, therefore, sufficient to show that for every U_i, W in question

$$W U_1 \cdots U_j |S(U_1, \cdots, U_j, W)| \ll |s|^3 x^{1/2-\delta}. \tag{2.5}$$

Here $|s|$ stands for the cardinality of S. By the mean-square theorem we deduce that (see [2])

$$|S(U_1, \cdots, U_j, W)| \ll M_i U_i^{-2} + q U_i^{-2}, \tag{2.6}$$

$$|S(U_1, \cdots, U_j, W)| \ll q W^{-4} |s| (\log qL|s|)^6, \tag{2.7}$$

$$|S(U_1, \cdots, U_j, W)| \ll M_i U_i^{-2} + q^{1+\varepsilon} M_i U_i^{-6} (1 \leqslant i \leqslant j), \tag{2.8}$$

$$|S(U_1, \cdots, U_j, W)| \ll (L^2 W^{-4} + q^{1+\varepsilon} L^2 W^{-12}) (\log L)^6. \tag{2.9}$$

By partial summation we deduce that, unless $S(U_1, \cdots, U_j, W)$ is empty,

$$W \ll |s| L^{1/2} x^{-3\delta}, \ \delta = \delta(\varepsilon) > 0, \tag{2.10}$$

subject to $L \geqslant q^{3/8+\varepsilon}$.

§ 3. Proof of Theorem

Let $t_0 = \log q / \log x + \varepsilon'$, $9/20 - \varepsilon \leqslant t_0 < 5/11$. ε, ε' are the suitable positive constants.

Lemma 3.1 *If $j=2$, let $L > q^{1/2+\varepsilon}$, $M_1 \geqslant q$, $M_2 \geqslant q^{1/2}$. Then* (2.5) *is true.*

The series $\{D_1, \cdots, D_l, L\}$ divides into $j+1$ parts, their multiples are $M_1, \cdots, M_j$, L_0, respectively. If $L_0 \leqslant q^{1/2}$, by (2.6) we deduce that

$$|S(U_1, \cdots, U_j, W)| \ll W^{-4}(L_0^2 + q) q^\varepsilon \ll W^{-4} q^{1+\varepsilon}.$$

Therefore we agree $L_0 \leqslant q^{1/2}$.

Lemma 3.2. *If $j=2$, let $M_1 \geqslant q$, $M_2 \geqslant q$, $L_0 \geqslant q^\varepsilon$. Then* (2.5) *is true.*

Lemma 3.3. *If $j=2$, we have*

$$R(\mathscr{A}^q, D) \ll \frac{x^{\frac{1}{2}+h(a,b)}}{\varphi(q)} + \frac{x^{1-\delta}}{\varphi(q)}, \tag{3.1}$$

where $a=\log M_1/\log x$, $b=\log M_2/\log x$, $\sigma=1-a-b$, $\sigma_0=\log L_0/\log x$,

$$h(a, b) = \begin{cases} \frac{1}{2}-\eta, & \text{if } a\geqslant b>t_1, & (3.2) \\ A, & \text{if } a\geqslant t_0>b, & (3.3) \\ B, & \text{if } t_0\geqslant a>b, & (3.4) \end{cases}$$

η is a suitable positive constant,

$$A=\begin{cases} t_0/2+a/2+\min\{b/4k, (\sigma/6+b/12k)\}, & \text{if } \sigma<t_0/k, \quad k\geqslant 2, & (3.5) \\ t_0/2+a/2+\min\{b/8, (\sigma/6+b/24)\}, & \text{if } \quad k<2. & (3.6) \end{cases}$$

$$B=t_0+\min\{b/8, \sigma/6+b/24\}.$$

Let $\theta_i=\log D_i/\log x$, $d=\log D/\log x$. Let M_{σ_0}, m_{σ_0} be maximun and minimun of a if $j=2$, $\sigma=\sigma_0$ and (2.5) is true. Let $\{D_i\}\in(D)$ be q-admissible if there exists a combination of $\{D_i\}$ that satisfies (2.5).

Lemma 3.4. *If there exist i_0 and k $(1\leqslant i_0\leqslant l,\ 1\leqslant k\leqslant l\leqslant l-i_0)$ such that $\theta_{i_0}<t_0/2$, $\theta_{i_0}+k<M_{\theta_{i_0}}-m_{\theta_{i_0}}$ and let g be the sum of some numbers in $\{1-\sum_{j=1}^{l}\theta_j,\ \theta_1, \cdots, \theta_{i_0-1}\}$ such that $g<M_{\theta_{i_0}}$ and $g+\sum_{j=i_0+k}^{l}\theta_j>m_{\theta_0}$, we have $\{D_i\}$ is q-admissible.*

Now we are ready to prove Theorem. It is sufficient to show that all of $\{D_i\}\in(D)$ are q-admissible.

It follows that $\{D_i\}$ is q-admissible if $\sum_{i=1}^{l}\theta_i<1-t_0/2$ by Lemmas 3.1 and 3.3. Therefore we suppose $\sum_{i=1}^{l}\theta_i\geqslant 1-t_0/2$ and consider four cases.

Case 1. $\sum_{j=1}^{5}\theta_i<t_0$. In this case $\theta_5<M_{\theta_5}-m_{\theta_5}$, we get $\{D_i\}$ is q-admissible.

Case 2. $t_0\leqslant\sum_{i=1}^{5}\theta_i<\frac{115}{122}-\frac{60}{61}t_0$. Let $a=\sum_{i=1}^{5}\theta_i$, $\sigma=\theta_6$. we have $\{D_i\}$ is q-admissible by Lemma 3.3.

Case 3. $\frac{115}{122}-\frac{60}{61}t_0\leqslant\sum_{i=1}^{5}\theta_i<1/2$. Let $a=\sum_{i=1}^{5}\theta_i$, $\sigma=\theta_7$ we get $\theta_7<t_0/5$ and then $\{D_i\}$ is q-admissible.

Case 4. $\sum_{i=1}^{5}\theta_i\geqslant 1/2$. We only discuss the case of $\sum_{i=1}^{4}\theta_i\geqslant 1/2$ and $\sum_{i=1}^{3}\theta_i\geqslant\frac{9}{4}-\frac{9}{8}t_0-\frac{3}{2}d$. The other cases follow from Lemmas 3.3 and 3.4. We consider four cases.

Case 4.1 $\theta_1+\theta_2<\frac{104}{115}t_0$. If $\theta_5<M_{\theta_3}-m_{\theta_3}$, we take $i_0=3$ and $g=\theta_1+\theta_2$. By Lemma 3.4 we obtain $\{D_i\}$ is q-admissible. If $\theta_5\geqslant M_{\theta_3}-m_{\theta_3}$, we have $\theta_1+\theta_2\geqslant\frac{2}{3}\sum_{i=1}^{3}\theta_i$ and then $\theta_1+\theta_2+\theta_5>m_{\theta_3}$. It is enough to consider that $\theta_1+\theta_2+\theta_5\geqslant M_{\theta_3}$. Let $a=\theta_1+\theta_2+\theta_5$ and $\sigma=\theta_6$. Therefore we obtain $\{D_i\}$ is q-admissible by Lemma 3.3.

Case 4.2. $\frac{104}{115}t_0 \leqslant \theta_1+\theta_2 < \frac{11}{10}-\frac{6}{5}t_0-\frac{1}{10}d$. If $\theta_3 < t_0/5$, let $g=\theta_1+\theta_2$. By Lemma 3.4 we have $\{D_i\}$ is g-admissible. If $\theta_3 \geqslant t_0/5$, the result follows from Lemma 3.3.

Case 4.3. $\frac{1}{2} > \theta_1+\theta_2 \geqslant \frac{11}{10}-\frac{6}{5}t_0-\frac{1}{10}d$ $(>t_0)$. We have $\{D_i\}$ is g-admissible by Lemma 3.3.

Case 4.4. $\theta_1+\theta_2 \geqslant 1/2$. Let $t'=\frac{29}{9}-\frac{20}{27}d-\frac{89}{18}t_0$. If $\theta_3<t'$, let $g=\theta_2+(1-\sum_{i=1}^{l}\theta_i)$. We have $\{D_i\}$ is g-admissible by Lemma 3.4. If $\theta_3 \geqslant t'$ and $\theta_4 < t'$, we have $\{D_i\}$ is g-admissible in the same way. If $\theta_3 \geqslant t'$ and $\theta_4 \geqslant t'$, we have $\theta_5 < t'$ and then $\theta_1+\theta_3+\theta_4 < M_{\theta_5}$. Therefore we obtain $\{D_i\}$ is g-admissible by Lemma 3.4.

The Theorem follows.

References

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