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SHAPE-PRESERVING APPROXIMATION BY SPLINE FUNCTIONS

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Abstract

The author establishes the Jackson-type estimates for monotone and convex approximation by spline functions with non-equally spaced knots. These estimaes involve high order modulus of the approximated function and its derivatives. Hence some Bernstein-type theorems can be got conveniently.

§1. Introduction

Let $\Delta_n: =t_0<t_1<\cdots<t_{n+1}=1$ be a partition of $I=[0, 1]$. Given any positive integer m, we call the space

$$S_m(\Delta_n)=\left\{s\in C^{m-1}[0, 1]\ \middle|\ \begin{array}{l}\text{There exist polynomials of degree}\\ m,\ \{s_i\}_{i=0}^n,\ \text{such that}\\ s(x)=s_i(x)\ \text{for}\ x\in[t_i, t_{i+1}].\end{array}\right\}$$

the space of polynomials splines of degree m with knots $t_1, \cdots, t_n$. Given $1\leqslant p\leqslant+\infty$ and any positive integer r, we define

$$L_p^r[0, 1]=\left\{f\ \middle|\ \begin{array}{l}\text{For } 0\leqslant k\leqslant r-1,\ D^kf \text{ is absolutely continuous}\\ \text{on } [0, 1] \text{ and } \|D^rf\|_p<+\infty.\end{array}\right\},$$

where

$$\|g\|_p=\begin{cases}\left[\int_0^1 |g(x)|^p dx\right]^{1/p}, & p<\infty,\\ \max\limits_{0\leqslant x\leqslant 1} |g(x)|, & p=+\infty.\end{cases}$$

In this paper we obtain Jackson type estimates for L_p-approximation by convex (monotone) splines whose knots are not equally spaced. These estimates involve higher order modulus of some derivative of the given function and can not be improved. By using them, we can improve the results obtained in [1, 2, 3, 4].

Given $f\in L_p[0, 1]$, define its r-th L_p-modulus of smoothness by

$$\omega_{r,p}(f, h)=\sup_{0\leqslant t\leqslant h}\|\Delta_t^r f(\cdot)\|_{L_p[0, 1-rt]},$$

where Δ^r is the r-th forward difference, and define

$$E_{m,p}^*(f, \Delta_n)=\inf\left\{\|s-f\|_p\ \middle|\ \begin{array}{l}s \text{ is an arbitrary convex}\\ \text{function in } S_m(\Delta_n).\end{array}\right\}.$$

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We state our main results as follows:

Theorem 1. *Let m be a positive integer and $1\leqslant p\leqslant+\infty$. There exists a positive constant C only depending on m and p such that the following hold.*

(1) *Let $m\geqslant 2$ and $p=+\infty$. For any convex function f in $L^2_\infty[0, 1]$, we have*

$$E^*_{m,\infty}(f, \Delta_n)\leqslant C\bar{\Delta}^2_n\omega_{m-1,\infty}(D^2f, \bar{\Delta}_n), \quad \bar{\Delta}_n=\max_{1\leqslant i\leqslant n}(t_i-t_{i-1}). \tag{1.1}$$

(2) *Let $m\geqslant 3$ and $1\leqslant p<+\infty$. For any convex function f in $L^3_p[0, 1]$,*

$$E^*_{m,p}(f, \Delta_n)\leqslant C\bar{\Delta}^3_n\omega_{m-2,p}(D^3f, \bar{\Delta}_n). \tag{1.2}$$

A more desirable estimate would be

$$E^*_{m,p}(f, \Delta_n)\leqslant C\omega_{m+1,p}(f, \bar{\Delta}_n).$$

As pointed out in [1, 3, 4], this estimate is important in yielding inverse theorem characterizing smoothness properties of f by $E^*_{m,p}(f, \Delta_n)$. Unfortunately, this estimate does not hold. In fact, we have the following theorem.

Theorem 2. *If $1\leqslant p<+\infty$, then*

$$\sup\left\{\frac{E^*_{m,p}(f, \Delta_n)}{\bar{\Delta}^2_n\omega_{2,p}(D^2f, \bar{\Delta}_n)}\ \middle|\ \begin{array}{l}f \text{ is an arbitrary convex function}\\ \text{in } L^2_p[0, 1].\ n \text{ is an arbitrary}\\ \text{positive integer and } \Delta_n \text{ is an}\\ \text{arbitrary partition of } [0, 1].\end{array}\right\}=+\infty. \tag{1.3}$$

If $p=+\infty$, then

$$\sup\left\{\frac{E^*_{m,+\infty}(f, \Delta_n)}{\bar{\Delta}_n\omega_{3,\infty}(Df, \bar{\Delta}_n)}\ \middle|\ \begin{array}{l}f \text{ is an arbitrary convex function}\\ \text{in } L^1_\infty[0, 1].\ n \text{ is an arbitrary}\\ \text{positive integer and } \Delta_n \text{ is an}\\ \text{arbitrary partition of } [0, 1].\end{array}\right\}=+\infty. \tag{1.4}$$

For the monotone approximation we can establish similar results as above. Let

$$E^{**}_{m,p}(f, \Delta_n)=\inf\left\{\|f-s\|_p\ \middle|\ \begin{array}{l}s \text{ is an arbitrary increasing}\\ \text{spline function in } S_m(\Delta_n).\end{array}\right\}.$$

We obtain the following results.

Theorem 3. *Let m be a positive integer and $1\leqslant p\leqslant+\infty$. There exists a positive constant C only depending on m and p such that the following estimates hold.*

(1) *Let $m\geqslant 1$ and $p=+\infty$. For any increasing function f in $L^1_\infty[0, 1]$, then*

$$E^{**}_{m,\infty}(f, \Delta_n)\leqslant C\bar{\Delta}_n\omega_{m,\infty}(Df, \bar{\Delta}_n). \tag{1.5}$$

(2) *Let $m\geqslant 2$ and $1\leqslant p\leqslant+\infty$. For any increasing function f in $L^2_p[0, 1]$, then*

$$E^{**}_{m,p}(f, \Delta_n)\leqslant C\Delta^2_n\omega_{m-1,p}(D^2f, \bar{\Delta}_n). \tag{1.6}$$

Theorem 4. *If $1\leqslant p<+\infty$, then*

$$\sup\left\{\frac{E^{**}_{m,p}(f, \Delta_n)}{\bar{\Delta}_n\omega_{2,p}(Df, \bar{\Delta}_n)}\ \middle|\ \begin{array}{l}f \text{ is an arbitrary increasing function}\\ \text{in } L^1_p[0, 1],\ n \text{ is an arbitrary}\\ \text{positive integer and } \Delta_n \text{ is an}\\ \text{arbitrary partition of } [0, 1].\end{array}\right\}=+\infty. \tag{1.7}$$

If $p=+\infty$, then

$$\sup\left\{\frac{E^{**}_{m,\infty}(f,\Delta_n)}{\omega_{3,\infty}(f,\bar{\Delta}_n)}\;\middle|\;\begin{array}{l}f \text{ is an arbitrary increasing function}\\ \text{in } C[0,1],\ n \text{ is an arbitrary positive}\\ \text{integer and } \Delta_n \text{ is an arbitrary}\\ \text{partition of } [0,1].\end{array}\right\}=+\infty. \tag{1.8}$$

In [1], DeVore conjectured that when the partition Δ_n is a uniform partition,

$$E^{**}_{m,\infty}(f,\Delta_n)\leqslant C\omega_{m+1,\infty}(f,\bar{\Delta}_n),$$

where C depends only on m. Here inequality (1.8) implies a negative answer to DeVore's conjucture.

Because Df is increasing if f is convex, Theorem 1 and Theorem 4 imply Theorem 3 and Theorem 2, respectively. Hence we only prove Theorem 1 and Theorem 4 in this paper. Throughout this paper C_1, C_2, etc. will denote constants independing on f and Δ_n but depending on m and p.

§ 2. Some Lemmas

Let I be an interval and $K_{r,p}(f,t,I,)$ denote the K-functional of Peetre on I, that is

$$K_{r,p}(f,t,I)=\inf_{g\in L_p^r(I)}\{\|f-g\|_{L_p(I)}+t^r\|D^rg\|_{L_p(I)}\}.$$

Lemma 2.1. *There is a positive constant C_1 such that for any function f in $L_p^1[a,b]$*

$$K_{r,\infty}(f,b-a,[a,b])\leqslant C_1[(b-a)^{-1/p}K_{r,p}(f,b-a,[a,b],)$$
$$+(b-a)^{1-1/p}K_{r-1,p}(Df,b-a,[a,b])]. \tag{2.1}$$

Proof It is sufficient to prove that (2.1) holds for the case in which $a=0$ and $b=1$.

It is well known that $K_{r,p}(f,t,I,)$ is equivalent to $\omega_{r,p}(f,t,I,)$. By using [5, Corollary 3.1], we obtain a polynomial q in π_{r-1} such that

$$\|D^i(f-q)\|_{L_p[0,1]}\leqslant C_2K_{r-i,p}(D^if,1,[0,1],\ i=0,1.$$

Hence $K_{r,\infty}(f,1,[0,1],)\leqslant\|f-q\|_{L_\infty[0,1]}\leqslant\|f-q\|_{L_p[0,1]}+\|D^1(f-q)\|_{L_p[0,1]}$

$$\leqslant C_2[K_{r,p}(f,1,[0,1],)+K_{r-1,p}(Df,1,[0,1],)].$$

Let $\Delta_n: 0=t_0<t_1<\cdots<t_n=1$ be a partition of $[0,1]$. The space of piecewise polynomials of degree m with knots $t_0,\cdots,t_n$ is denoted by

$$PS_m(\Delta_n)=\left\{s\;\middle|\;\begin{array}{l}\text{There exist polynomials of degree } m,\ \{s_i\}_{i=0}^{n-1},\\ \text{such that } s(x)=s_i(x) \text{ for } x\in[t_i,t_{i-1}].\end{array}\right\}$$

Lemma 2.2. *Let $0\leqslant r\leqslant m$. There is a positive constant C_3 such that for any $s\in PS_m(\Delta_n)$*

$$\|D^rs\|_{L_p[0,1]}\leqslant C_3\underline{\Delta}_n^{-r}\omega_{r,p}(s,\bar{\Delta}_n),$$

where

$$\underline{\Delta}_n=\min_{1\leqslant i\leqslant n}(t_i-t_{i-1}).$$

This is a corollary to Markoff's inequality and Whitney-type inequality (See [5]). The details of proof is omitted here.

Lemma 2.3. *Let A be a compact set in π_m satisfying that if $f\in A$, then $Df\not\equiv 0$. Let $n\geqslant 2m-1$ and Δ_n: $0=x_0<x_1<\cdots<x_{n+1}=1$ be a partition of* $[0, 1]$. *Then there is a positive number $\delta>0$ depending on n and m such that, for any function $q\in A$, there is an integer i, $1\leqslant i\leqslant n$, such that*

$$|Dq(x)|\geqslant\delta,\ for\ x\in[x_i,\ x_{i+1}].$$

Proof Suppose that δ does not exist. Then there is a sequence of functions, $\{q_i\}$, in A, satisfying that for each i there is a point $x_{j,i}$ in $[x_i,\ x_{i+1}]$ such that

$$|Dq_j(x_{j,i})|<j^{-1}. \tag{2.3.1}$$

A is a compact set. So we can suppose that there is a function q in A such that

$$\|q_j-q\|_\infty\to 0,\ \text{as}\ j\to\infty. \tag{2.3.2}$$

By using Markoff's inequality, we have

$$\|D(q_j-q)\|_\infty\to 0,\ \text{as}\ j\to\infty. \tag{2.3.3}$$

We can also suppose that there is a point $x_{0,1}\in[x_i,\ x_{i+1}]$ such that

$$|x_{0,1}-x_{j,i}|\to 0,\ \text{as}\ j\to\infty. \tag{2.3.4}$$

From (2.3.3) and (2.3.4), it follows that $Dq(x_{0,i})=0$ for $0\leqslant i\leqslant n$. Because $n\geqslant 2m-1$, Dq has at least m zero points. So $Dq\equiv 0$. It implies that $q\notin A$. From (2.3.2), it follows that A is not a closed, contradiction.

For a positive number α and an interval $[-b,\ b]$, we introduce a function set denoted by $B(\alpha,[-b,\ b])$. f is a function in $B(\alpha,\ [-b,\ b])$ if and only if f satisfies the following conditions:

(1) There are two polynomials f_1 and f_2 in π_m such that

$$f(x)=\begin{cases}f_1(x),\ x(-\infty,\ 0],\\ f_2(x),\ x(0,\ +\infty).\end{cases}$$

(2) f_1 and f_2 are convex on $(-b,\ 2/b)$ and $(-2/b,\ b)$, respectively.

(3) Let $\tilde{f}_i(x)=f_i(x)-Df_i(0)x-f_i(0)$, $i=1,\ 2$.

$$e(f)=\max\begin{Bmatrix}\|D\tilde{f}_1\|_{C[-b,0]}, & \|\tilde{f}_1\|_{C[-b,0]},\\ \|D\tilde{f}_2\|_{C[0,b]}, & \|\tilde{f}_2\|_{C[0,b]},\end{Bmatrix}$$

$$\geqslant\alpha\|f_1-f_2\|_{C[-b/2,b/2]}.$$

Lemma 2.4. *Let $m\geqslant 2$ and $b=4m^2$. Let $x_i=i$ for $-b\leqslant i\leqslant b$. Δ_{2b}: $x_{-b}<\cdots<x_b$ is a partition of $[-b,\ b]$. Then there exists a positive real number α such that for any function f in $B(\alpha,\ [-b,\ b])$ there is a function $s\in C^{m-1}(-\infty,\ +\infty)$ satisfying the following properties:*

(1′) *$s=f$ on $(-\infty,\ -b)$ and $(b,\ +\infty)$.*

(2′) *Restricted to $[-b,\ b]$, s is a convex function in $S_m(\Delta_{2b})$.*

(3′) *There is a positive constant C_5 independing on f such that*

$$\|f-s\|_{C(-\infty,+\infty)}\leqslant C_5\|f_1-f_2\|_{C[-b/2,b/2]}.$$

Proof At first we suppose that $e(f)=\|D\tilde{f}_1\|_{C[-b,0]}$.

Let $\Delta'_{2m}: 0=x_0<\cdots<x_{2m}=2m^2$ denote a partition of $[0,2m^2]$. By using [6, Lemma 2.1], we obtain a function $s_1\in C^{m-3}(-\infty,+\infty)$ whose restriction to $[0, 2m^2]$ is a spline function in $S_{m-2}(\Delta'_{2m})$ and for each point $x\in[0, 2m^2]$, $s_1(x)$ is a number between $D^2f_1(x)$ and $D^2f_2(x)$ and whose restriction to $(-\infty, 0)$ and $(2m^2, +\infty)$ are identical with D^2f. Define

$$s_2(x)=f_1(0)+Df_1(0)x+\int_0^x dt\int_0^t S_1(\tau)d\tau.$$

Then s_2 is a convex function on $[-b, b]$ and satisfies that

$$\begin{aligned}\|f-s_2\|_{C[0,b]}&\leqslant 2b^2\|D^2f_2-s_1\|_{C[0,b]}+|f_1(0)-f_2(0)|+|Df_1(0)-Df_2(0)|\\&\leqslant 2b^2\|D^2(f_1-f_2)\|_{C[0,b/2]}+\|f_1-f_2\|_{C[0,b/2]}+\|D(f_1-f_2)\|_{C[0,b/2]}\\&\leqslant C_6\|f_1-f_2\|_{C[0,b/2]}. \qquad (2.4.1)\end{aligned}$$

Let $N_k(\cdot|x_i, \cdots, x_{i+k+1})$ denote the normalized B-splines of order $k+1$ whose support set is $[x_i, x_{i+k+1}]$ and whose knots are $\{x_j\}_{j=i}^{i+k+1}$. Define

$$\begin{aligned}s(x)=s_2(x)+\int_{-\infty}^x dt\int_{-\infty}^t\Big[&y_1N_{m-2}(\tau|x_1, \cdots, x_{l+m-1})\\&+y_2N_{m-2}(\tau|x_{l+m-1}, \cdots, x_{l+2m-2})\Big]d\tau.\end{aligned}$$

If $-b\leqslant l\leqslant -2(m-1)$, then $s=f$ on $(-\infty, -b)$ and $D^2s=D^2f$ on $[b, +\infty)$. Let

$$M(x)=s(x)-s_2(x).$$

It is obvious that we can choose y_1 and y_2 such that $M(b)=f_2(b)-s_2(b)$ and $DM(b)=D(f_2-s_2)(b)$ and

$$\max\{|y_1|, |y_2|\}\leqslant C_7\max\{|(f_2-s_2)(b)|, |D(f_2-s_2)(b)|\}.$$

So $s=f$ on $(b, +\infty)$ and

$$\|D^2M\|_{C(-\infty,+\infty)}\leqslant C_8\|f-s_2\|_{C[0,b]}\leqslant C_9\|f_1-f_2\|_{C[0,b/2]}.$$

Now we choose the integer l such that s is convex on $[-b, b]$. Let

$$A=\{q\in\pi_{m-1}|q(0)=0, \|q\|_{C[-b,0]}=1\}.$$

Then A satisfies the hypothesis of Lemma 2.3. Hence there is a positive constant $\delta>0$ such that, for the polynomial

$$(Df_1(x)-Df_1(0))/\|D\tilde{f}_1\|_{C[-b,0]}$$

in A, there is an integer l with $-b\leqslant l\leqslant -2(m-1)$ such that

$$\min_{x\in[x_l,x_{l+2(m-1)}]}|D^2f_1(x)|\geqslant\delta\|Df_1(0)-Df_1(\cdot)\|_{C[-b,0]}\geqslant\alpha\delta\|f_1-f_2\|_{C[-b/2,b/2]}.$$

Hence

$$\min_{x\in[x_l,x_{l+2(m-1)}]}|D^2s_2(x)|\geqslant\alpha\delta\|f_1-f_2\|_{C[-b/2,b/2]}.$$

By more careful analysis, it can be proved that C_9 depends only on m. Let $\alpha=\delta^{-2}C_9$. The support set of D^2M is contained in $[x_l, x_{l+2(m-1)}]$. Hence $D^2s(x)=D^2(s_2+M)(x)\geqslant 0$, for $x\in[-b, b]$.

The obtained convex spline function s satisfies $(1')$ and $(2')$. We shall show that it also satisfies $(3')$. In fact

$$\|s_2-s\|_{C[-b,b]}=\left|\int_{xl}^{b}dt\int_{xl}^{t}[y_1N_{m-2}(\tau|x_l,\cdots,x_{l+m-1})\right.$$
$$+y_2N_{m-2}(\tau|x_{l+m-1},\cdots,x_{l+2m-2})]d\tau$$
$$\leqslant 8b^2\max\{|y_1|,|y_2|\}$$
$$\leqslant 8b^2C_7\max\{|(f_2-s_2)(b)|,|D(f_2-s_2)(b)|\}$$
$$\leqslant C_7'\|f_1-f_2\|_{C[0,b/2]}$$
$$\|f-s\|_{C(-\infty,+\infty)}=\|f-s\|_{C[-b,b]}\leqslant\|f-s_2\|_{C[0,b]}+\|s_2-s\|_{C[-b,b]}$$
$$\leqslant C_5\|f_1-f_2\|_{C[-b/2,b/2]},$$

where $C_5=C_6+C_7'$.

We have proved the lemma in the case $e(f)=\|D\tilde{f}_1\|_{C[-b,0]}$. The proofs for other cases are almost identical with this case.

Lemma 2.5. *Let $m\geqslant 2$ and $b=4m^2$ and $x_i=\frac{i}{b}$ for $-b\leqslant i\leqslant b$. Let Δ: $-2<-1<0<1<2$ and Δ': $-2\triangleq x_{-b-1}<x_{-b}<\cdots<x_b<x_{b+1}\triangleq 2$ be two partitions of $[-2,2]$. Then there is a positive constant C_{10} such that for any convex function f in $C^2[-2,2]$, at least one of the following propositions is true.*

(1) *There is a convex function s in $PS_m(\Delta)\cap C[-2,2]$, whose restrictions to $[-1,0]$ and $[0,1]$ are linear functions, respectively, satisfying*

$$s(\pm 1)=f(\pm 1) \tag{2.5.1}$$

and

$$Df(-1)\leqslant Ds_+(-1)\leqslant Ds_-(1)\leqslant Df(1), \tag{2.5.2}$$

such that

$$\|s-f\|_{C[-2,2]}\leqslant C_{10}K_{m-1,\infty}(D^2f,1,[-2,2]). \tag{2.5.3}$$

(2) *There is a convex function $s\in S_m(\Delta')$ satisfying*

$$D^is(\pm 1)=D^if(\pm 1),\ i=0,1 \tag{2.5.4}$$

such that

$$\|s-f\|_{C[-2,2]}\leqslant C_{10}K_{m-1,\infty}(D^2f,1,[-2,2]). \tag{2.5.5}$$

Proof For the convenience of statement we suppose that $f(0)=Df(0)=0$. Let $F(x)=f(b^{-1}x)$. By using [5, Theorem 4.2,], we obtain two polynomials q_1 and q_2 in π_{m-2} such that

$$\|D^2F-q_1\|_{C[-2b,b]}\leqslant C_{11}K_{m-1,\infty}(D^2F,1,[-2b,b]); \tag{2.5.6}$$
$$\|D^2E-q_2\|_{C[-b,2b]}\leqslant C_{11}K_{m-1,\infty}(D^2F,1,[-b,2b]). \tag{2.5.7}$$

Define

$$f_1(x)=F(-b)+DF(-b)(x+b)+\int_{-b}^{x}dt\int_{-b}^{x}[q_1(\tau)+C_{11}K_{m-1,\infty}(D^2F,1,[-2b,b])]d\tau$$

and

$$f_2(x)=F(b)+DF(b)(x-b)+\int_{b}^{x}dt\int_{b}^{t}[q_2(\tau)+C_{11}K_{m-1,\infty}(D^2F,1,[-b,2b])]d\tau.$$

Then f_1 and f_2 are convex functions on $[-2b,b]$ and $[-b,2b]$, respectively. They satisfy that

$$D^i f_j((-1)^j b) = D^i F((-1)^j b),\ i=0,\ 1 \text{ and } j=1,\ 2.$$

From (2.5.6) and (2.5.7), it follows that

$$\|f_1 - F\|_{C[-2b,b]} \leqslant C_{12} K_{m-1,\infty}(D^2F,\ 1,\ [-2b,\ b]) \tag{2.5.8}$$

and

$$\|f_2 - F\|_{C[-b,2b]} \leqslant C_{12} K_{m-1,\infty}(D^2F,\ 1,\ [-b,\ 2b]). \tag{2.5.9}$$

Suppose that f_1 and f_2 satisfy the condition (3) in the definition of $B(\alpha,\ [-b,\ b])$, where α is determined in the proof of Lemma 2.4. Let

$$g(x) = \begin{cases} f_1(x), & x \in [-2b,\ 0], \\ f_2(x), & x \in [0,\ 2b]. \end{cases}$$

By Lemma 2.4, we obtain a spline function s_1 which has properties (1′), (2′) and (3′), that is

$$D^i s_1((-1)^j b) = D^i f_j((-1)^j b),\ i=0,\ 1 \text{ and } j=1,\ 2,$$

and

$$\begin{aligned}\|F - s_1\|_{C[-2b,2b]} &\leqslant \|F - g\|_{C[-2b,2b]} + \|g - s_1\|_{C[-2b,2b]} \\ &\leqslant C_{13} K_{m-1,\infty}(D^2F,\ 1,\ [-2b,\ 2b]).\end{aligned}$$

Let $s(x) = s_1(bx)$. Then the proposition (2) holds.

For the case in which f_1 and f_2 do not satisfy the condition (3), that is,

$$\max\begin{Bmatrix} \|Df_1\|_{C[-b,0]}, & \|f_1\|_{C[-b,0]}, \\ \|Df_2\|_{C[0,b]}, & \|f_2\|_{C[0,b]}, \end{Bmatrix} \leqslant \alpha \|f_1 - f_2\|_{C[-b/2,b/2]}, \tag{2.5.10}$$

where α is determined in Lemma 2.4, we shall show that proposition (1) is true.

For $\lambda \in [0,\ 1]$ and $x \in [-2b,\ 2b]$, define

$$U_1(\lambda,\ x) = f_1(-b) + \lambda Df_1(-b)(x+b),$$
$$U_2(\lambda,\ x) = f_2(b) + \lambda Df_2(b)(x-b).$$

At first we suppose that $U_1(1,\ 0) \leqslant U_2(1,\ 0)$. From the fact that $F(0) = DF(0) = 0$, it follows that F is increasing on $[0,\ 2b]$ and decreasing on $[-2b,\ 0]$. Hence we have

$$\begin{aligned}U_2(1,\ 0) &= -bDf_2(b) + f_2(b) = -bDF(b) + F(b) \\ &\leqslant F(0) \leqslant F(-b) = f_1-(b) = U_1(0,\ 0).\end{aligned}$$

It implies that there is a number $\lambda_0 \in [0, 1]$ such that $U_1(\lambda_0, 0) = U_2(1, 0)$. We define

$$s_3(x) = \begin{cases} f_1(x), & x \in [-2b,\ b], \\ U_1(\lambda_0,\ x), & x \in [-b,\ 0], \\ U_2(1,\ x), & x \in [0,\ b], \\ f_2(x), & x \in [b,\ 2b]. \end{cases}$$

It is obvious that s_3 is a convex function and satisfies that

$$Ds_3(b^-) \leqslant DF(b),$$
$$Ds_3((-b^+) \geqslant DF(-b).$$

Now we begin to estimate the error $\|F - s_3\|_{C[-2b,2b]}$.

$$\begin{aligned}\|f_1-U_1(\lambda_0,\ \cdot)\|_{C[-b,0]}&=\max_{x\in[-b,0]}|f_1(x)-f_1(-b)-\lambda_0 Df_1(-b)(x+b)|\\&\leqslant 2b\max_{x\in[-b,0]}|Df_1(x)-\lambda_0 Df_1(-b)|\\&\leqslant 2b(\|Df_1(\cdot)-Df_1(0)\|_{C[-b,0]}\\&\quad+|Df_1(0)-\lambda_0 Df_1(-b)|).\end{aligned}$$

If $Df_1(0)\geqslant 0$ or $0\geqslant Df(0)\geqslant Df_1(-b)$, then

$$|Df_1(0)-\lambda_0 Df_1(-b)|\leqslant\|Df_1(\cdot)-Df_1(0)\|_{C[-b,0]}.$$

By (2.5.8), (2.5.9) and (2.5.10), we have

$$\|f_1-U_1(\lambda_0,\ \cdot)\|_{C[-b,0]}\leqslant C_{14}K_{m-1,\infty}(E^2F,\ 1,\ [-2b,\ 2b]). \tag{2.5.11}$$

If $0\geqslant Df_1(-b)\geqslant Df_1(0)$, then

$$\begin{aligned}|Df_1(0)-\lambda_0 Df_1(-b)|\leqslant|Df_1(0)|&=|Df_1(0)-DF(0)|\\&\leqslant\|D(f_1-F)\|_{C[-2b,b]}.\end{aligned}$$

Hence in this case we can also obtain (2.5.11).

Using the same method, we can obtain

$$\|f_2-U_2(1,\ \cdot)\|_{C[0,b]}\leqslant C_{15}K_{m-1,\infty}(D^2F,\ 1,\ [-2b,\ 2b]).$$

Hence $$\|F-s_3\|_{C[-2b,2b]}\leqslant C_{16}K_{m-1,\infty}(D^2F,\ 1,\ [-2b,\ 2b]).$$

Let $s(x)=s_3(bx)$. Then s is a convex spline function in $PS_m(\Delta)\cap C[-2,2]$. Its restrictions to $[-1,0]$ and $[0,1]$ are linear functions, respectively, and satisfy (2.5.1), (2.5.2) and (2.5.3).

For the case in which $U_1(1,0)>U_2(1,0)$, we can prove this Lemma by the similar method.

§3. Proofs of Theorems

Proof of Theorem 1 At first we suppose that Δ_n is a uniform partition.

Let $a=(2b+1)(4d+1)$, where b is defined in Lemma **2.4** and d is determined in [2, Lemma 4.4]. Let $n\geqslant a-1$. For the convenience of statement we assume that there is a positive integer M such that $n+1=aM$. Let $t_i=ia$, $0\leqslant i\leqslant M-1$, $t_{i,j}=ia+(4d+1)j$, $0\leqslant j\leqslant 2b$, $t_{i,j,k}=ia+(4d+1)j+k$, $0\leqslant k\leqslant 4d$, $I_i=[t_i,\ t_{i+1}]$, $I_{i,j}=[t_{i,j},\ t_{i,j+1}]$, $I_{i,j,k}=[t_{i,j,k},\ t_{i,j,k+1}]$.

Let $F(x)=f(n^{-1}x)$. By applying Lemma 2.5 to F on each interval I_i, we obtain a convex spline function s_i on I_i, which at least has one of the following properties.

Property 1. Restrictions of s_i to $[t_i,\ t_{i,b}]$ and $[t_{i,b},\ t_{i+1}]$ are linear functions, respectively, and satisfy that

$$s_i(t_{i+k})=F(t_{i+k}),\ k=0,\ 1, \tag{3.1}$$

$$DF(t_i)\leqslant Ds_i(t_i)\leqslant Ds_i(t_{i+1})\leqslant DF(t_{i+1}), \tag{3.2}$$

$$\|s_i-F\|_{C(I_i)}\leqslant D_{17}K_{m-1,\infty}(D^2F,\ 1,\ [t_{i-1,b},\ t_{i+1,b}]). \tag{3.3}$$

There exist two polynomials $q_{i,1}$ and $q_{i,2}$ in π_m satisfying

$$\|q_{i,1}-F\|_{C[t_{i-1,b},t_i]}\leqslant C_{18}K_{m-1,\infty}(D^2F,\ 1,\ [t_{i-1,b},\ t_{i+1,b}), \tag{3.4}$$

$$\|q_{i,2}-F\|_{C[_{i-1},t_{i+1,b}]}\leqslant C_{18}K_{m-1,\infty}(D^2F,\ 1,\ [t_{i-1,b},\ t_{i+1,b}]), \tag{3.5}$$

$$D^jq_{i,1}(t_i)=D^jF(t_i),\ j=0,\ 1, \tag{3.6}$$

$$D^jq_{i,2}(t_{+1})=D^jF(t_{i+1}),\ j=0,\ 1. \tag{3.7}$$

Property 2. s_i is a function in $C^{m-1}(I_i)$ and its restrictions to $I_{i,j}(j=0,\ \cdots,\ 2b)$ are polynomials of degree m. s_i satisfies

$$D^js_i(t_{i+k})=D^jF(t_{i+k}),\ j,\ k=0,\ 1, \tag{3.8}$$

$$\|s_i-F\|_{C(I_i)}\leqslant C_{19}K_{m-1,\infty}(D^2F,\ 1,\ [t_{i-1,b},\ t_{i+1,b}]). \tag{3.9}$$

There exist two polynomials $p_{i,1}$ and $p_{i,2}$ in π_m satisfying

$$D^jp_{i,1}(t_i)=D^js_i(t_i),\ j=0,\ \cdots,\ m-1;$$

$$D^jp_{i,2}(t_{i+1})=D^js_i(t_{i+1}),\ j=0,\ \cdots,\ m-1;$$

$$\|p_{i,1}-F\|_{C[t_{i-1,b},\ t_i]}\leqslant C_{20}K_{m-1,\infty}(D^2F,\ 1,\ [t_{i-1,b},\ t_{i+1,b}]);$$

$$\|p_{i,2}-F\|_{C[t_{i+1},\ t_{i+1,b}]}\leqslant C_{20}K_{m-1,\infty}(D^2F,\ 1,\ [t_{i-1,b},\ t_{i+1,b}]).$$

Define $\tilde{s}(x)=s_i(x)$ for $x\in I_i$, $i=0,\ \cdots,\ M-1$. Then (3.1), (3.2) and (3.8) imply the fact that $\tilde{s}$ is convex on $[0,\ n]$. $\tilde{s}$ is a spline function of degree m and $D^{m-1}\tilde{s}$ is contiuous except finite knots. Now we use [2, Lemma 4.1 and Lemma 4.3] repeately to smooth $\tilde{s}$ to C^{m-1}.

The points on which $D^{m-1}\tilde{s}$ is discontinuous can be divided into four parts:

$A_1=\{t_i|$Both s_i and s_{i-1} have property 2$\}$,

$A_2=\{t_i|$Both s_i and s_{i-1} have property 2$\}$,

$A_3=\{t_i|$One of functions s_i and s_{i-1} has property 1 and the other has property 2$\}$,

$A_4=\{t_{i,b}|s_i$ has property 1.$\}$.

Suppose that $t_i\in A_1$. By [2, Lemma 4.1], we obtain a convex spline function g_k satisfying that

$$g_i(x)=s(x),\ x\in[0,\ n]\setminus[t_{i-1,2b,2d},\ \ t_{i,0,2d+1}]$$

and restricted to $[t_{i-1,2b},\ t_{i,1}]$, g_i is a function in $C^1[t_{i-1,2b},\ t_{i,1}]$.

$$\begin{aligned}\|s-g_i\|_{C[t_{i-1,2b},\ t_{i,1}]}&\leqslant|Ds(t_i^+)-Ds(t_i^-)|\\&\leqslant|Ds(t_i^+)-DF(t_i))|+|(DF(t_i)-Ds(t_i^-))|.\end{aligned}$$

By using (3.6), (3.3) and (3.4) and Markoff's inequality, we have

$$\begin{aligned}|Ds(t_i^-)-DF(t_i)|&=|Ds_{i-1}(t_i)-DF(t_i)|=|Ds_{i-1}(t_i)-Dq_{i,1}(t_i)|\\&\leqslant C_{21}\|s_{i-1}-q_{i,1}\|_{C(I_{i-1,2b,2d})}\\&\leqslant C_{22}K_{m-1,\infty}(D^2F,\ 1,\ [t_{i-2,b},t_{i,b}]).\end{aligned}$$

We also have

$$|DF(t_i)-Ds(t_i^+)|\leqslant C_{23}K_{m-1,\infty}(D^2F,\ 1,\ [t_{i-2,b},t_{i,b}]).$$

Hence

$$\|s-g_i\|_{C[t_{i-1,2b},t_{i,1}]}\leqslant C_{24}K_{m-1,\infty}(D^2F,\ 1,\ [t_{i-2,b},\ t_{i,b}]). \tag{3.10}$$

s_i and s_{i-1} are linear functions on I_i and I_{i-1}, respectively. In this case we improve the proof of [2, Lemma 4.1] and see the fact that restricted to $[t_{i-1,2b,2d}, t_{i,0,2d+1}]$, g_i is a polynomial of degree 2. So $D^{m-1}g_i$ is continuous on every point in $[t_{i-1,2b}, t_{i,1}]\setminus\{t_{i-1,2b,2d}, t_{i,0,2d+1}\}$.

Using [2, Lemma 4.3], we obtain a convex spline function Q_i satisfying that

$$Q_i(x)=g_i(x),\ x\in[0, n]\setminus[t_{i,0,d+1}, t_{i,0,3d+1}]$$

and restricted to each interval $I_{i,0,k}$, $0\leqslant k\leqslant 4d$, Q_i is a polynomial of degree m and restricted to $[t_i, t_{i,1}]$, Q_i is a function in $C^{m-1}[t_i, t_{i,1}]$.

$$\begin{aligned}\|Q_i-g_i\|_{C[t_i,t_{i,1}]}&\leqslant C_{25}\sum_{j=2}^{m-1}|D^jg_i(t^+_{i,0,2d+1})-D^jg_i(t^-_{i,0,2d+1})|\\&\leqslant C_{25}\sum_{j=2}^{m-1}|D^js_i(t_{i,0,2d+1})-D^jg_i(t_{i,0,2d+1})|\\&\leqslant C_{26}\|s-g_i\|_{C[t_{i-1,2b},t_{i,1}]}\\&\leqslant C_{27}K_{m-1,\infty}(D^2F, 1, [t_{i-2,b}, t_{i,b}]).\end{aligned}$$

Now Q_i is continuous on every point in $[t_{i-1,2b}, t_{i,1}]\setminus\{t_{i-1,2b,2d}\}$. By using the same method we can smooth Q_i to $C^{m-1}[t_{i-1,2b}, t_{i,1}]$. Hence we can obtain a convex spline function Q_i satisfying that

$$Q_i(x)=s(x),\ x\in[0, n]\setminus[t_{i-1,2b}, t_{i,2b}]$$

and
$$\|Q_i-s\|_{C[t_{i-1,2b},t_{i,2b}]}\leqslant C_{28}K_{m-1,\infty}(D^2F, 1, [t_{i-2,b}, t_{i+2,b}]).$$

Restricted to each interval of $I_{i-1,j,k}$, $(0\leqslant j\leqslant 2b,\ 0\leqslant k\leqslant 4d+1)$ and $I_{i,0,k}(0\leqslant k\leqslant 4d+1)$, Q_i is a polynomial of degree m.

We have proved that if $t_i\in A_1$, then restriction of s to $[t_{i-1,2b}, t_{i,2b}]$ can be smoothed to $C^{m-1}[t_{i-1,2b}, t_{i,2b}]$. In other cases we can use the similar smoothing method to smooth s to C^{m-1} locally. In such a way we obtain a convex spline function G which has a good local estimate

$$\|F-G\|_{C(I_{i,j,k})}\leqslant C_{29}K_{m-1,\infty}(D^2F, 1, [t_{i-2,b}, t_{i+2,b}]).$$

Applying Lemma 2.1, we have

$$\|F-G\|_{L_p(I_{i,j,k})}\leqslant C_{30}K_{m-2,p}(D^3F, 1, [t_{i-2,b}, t_{i+2,b}]).$$

Let $s(x)=G(nx)$ for $x\in[0, 1]$. Then s is a convex function in $S_m(\Delta_n)$ such that if f is a convex function in $C^2[0, 1]$, then

$$\|f-s\|_{C[0,1]}\leqslant C_{31}n^{-2}K_{m-1,\infty}(D^2f, n^{-1}, [0, 1]),$$

if f is a convex function in $L^3_p[0, 1]$, then

$$\|f-s\|_{L_p[0,1]}\leqslant C_{32}n^{-3}K_{m-2,p}(D^3f, n^{-1}, [0, 1]).$$

The K-functional is equivalent to higher order modulus of smoothness. In other words we have proved Theorem 1 if Δ_n is a uniform partition.

If there is a positive constant $\sigma>0$ such that

$$\frac{\overline{\Delta}_n}{\underline{\Delta}_n}\leqslant\sigma \text{ for all } n,$$

then we say that Δ_n is a quasi-uniform partition. It is a obvious fact that any partition can be thinned out to get a quasi-uniform partition Δ_n^*, that is

$$\Delta_n^* \subseteq \Delta_n \quad \text{and} \quad \sigma_1 \bar{\Delta}_n \leqslant \underline{\Delta}_n^* \leqslant \bar{\Delta}_n^* \leqslant \sigma_2 \bar{\Delta}_n,$$

where σ_1 and σ_2 do not depend on Δ_n. So we can assume that Δ_n is a quasi-uniform partition. Let $\tilde{\Delta}_n$ denote the uniform partition of $[0, 1]$. Then there is a convex function $\tilde{s}$ in $S_{m+1}(\tilde{\Delta}_n)$ such that

$$\|f-\tilde{s}\|_{L_p[0,1]} \leqslant C_{33} n^{-3} \omega_{m-2,p}(D^3 f, n^{-1}) \tag{3.11}$$

and

$$\|f-\tilde{s}\|_{C[0,1]} \leqslant C_{34} n^{-2} \omega_{m-1,\infty}(D^2 f, n^{-1}). \tag{3.12}$$

By virtue of [6, Theorem 2.1], we obtain a function s_1 in $S_{m-2}(\Delta_n)$ such that

$$0 \leqslant s_1(x) \leqslant D^2\tilde{s}(x), \quad x \in [0, 1]$$

and

$$\|L^2\tilde{s} - s_1\|_{L_p[0,1]} \leqslant C_{35} \bar{\Delta}_n^{m-1} \|D^{m+1}\tilde{s}\|_{L_p[0,1]}.$$

Define $s_2(x) = \tilde{s}(0) + D\tilde{s}(0)x + \int_0^x dt \int_0^t s_1(\tau)d\tau$. Then $\tilde{s} - s_2$ and s_2 are convex functions and s_2 is in $S_m(\Delta_n)$. Define

$$s(x) = s_2(x) + S_{\Delta_n, m}(\tilde{s} - s_2),$$

where $S_{\Delta_n, m}$ is Schonberg's variation diminishing spline operator. Then s is a convex spline function in $S_m(\Delta_n)$.

$$\|\tilde{s} - s\|_{L_p[0,1]} \leqslant C_{36} \bar{\Delta}_n^2 \|D^2(\tilde{s} - s_2)\|_{L_p[0,1]} \leqslant C_{37} \bar{\Delta}_n^{m+1} \|D^{m+1}\tilde{s}\|_{L_p[0,1]}.$$

By using Lemma 2.2, we have

$$\|\tilde{s} - s\|_{L_1[0,1]} \leqslant C_{38} \omega_{m+1,p}(\tilde{s}, \bar{\Delta}_n) \leqslant C_{39}(\|\tilde{s} - f\|_{L_p[0,1]} + \omega_{m+1,p}(f, \bar{\Delta}_n)).$$

Because Δ_n is a quasi-uniform partition, we have $n^{-1} \sim \bar{\Delta}_n$. From (3.11) and (3.12), it follows that

$$\|f - s\|_{L_p[0,1]} \leqslant C_{40} \bar{\Delta}_n^3 \omega_{m-2,p}(D^3 f, \bar{\Delta}_n),$$
$$\|f - s\|_{C[0,1]} \leqslant C_{41} \bar{\Delta}_n^2 \omega_{m-1,\infty}(D^2 f, \bar{\Delta}_n).$$

The proof is completed.

Proof of Theorem 4 For the convenience of statement we assume that Δ_n is a uniform partition.

Let a be a positive number determined later. Let

$$P_n(x) = (x-a)x.$$

$DP_n(x) = 2x - a$. So P_n is incresing on $[a, 1]$. Define

$$f_n(x) = \begin{cases} 0, & x \in [0, a], \\ P_n(x), & x \in [a, 1]. \end{cases}$$

Then f_n is an increasing function in $L_p^1[0, 1]$. Let s be an arbitrary increasing function in $S_m(\Delta_n)$. Then $Ds(0) \geqslant 0$ and

$$\begin{aligned} a = |DP_n(0)| &\leqslant |D(P_n - s)(0)| \leqslant \|D(P_n - s)\|_{C[0,n^{-1}]} \\ &\leqslant C_{42} n^{1/p} \|D(P_n - s)\|_{L_p[0,n^{-1}]} \leqslant C_{43} n^{1+(1/p)} \|P_n - s\|_{L_p[0,n^{-1}]} \\ &\leqslant C_{43} n^{1+(1/p)} (\|P_n - f_n\|_{L_p[0,1]} + \|f_n - s\|_{L_p[0,1]}). \end{aligned}$$

where the third inequality follows from the uniqueness of norm in finite dimensional vector space and the fourth inequality follows from Markoff's inequality because restricted to $[0, n^{-1}]$, P_n and s are two polynomials of degree of m. Let $a=[C_{42}(n+1)\cdot n^{1+(1/p)}]^{-p/(p+1)}$. Then we have

$$\|f_n-P_n\|_{L^q[0,1]}=\|P_n\|_{L_p[1,a]}\leqslant a^{2+(1/p)}$$
$$\leqslant(n+1)^{-1}(\|P_n-f_n\|_{L_p[0,1]}+\|f_n-s\|_{L_p[0,1]}).$$

Hence
$$n\|P_n-f_n\|_{L_p[0,1]}\leqslant\|f_n-s\|_{L_p[0,1]}.$$

It follows that

$$E^{**}_{m,p}(f_n, \Delta_n)\geqslant n\|P_n-f_n\|_{L_p[0,1]}. \tag{3.13}$$

On the other hand, we have

$$\begin{aligned} n^{-1}\omega_{2,p}(Df_n, n^{-1}) &\leqslant 4n^{-1}\|D(f_n-P_n)\|_{L_p[0,1]} \\ &\leqslant 4n^{-1}\|DP_n\|_{L_p[0,a]} \\ &\leqslant C_{43}a^{-1}n^{-1}\|P_n\|_{L_p[0,a]}. \\ &\leqslant C_{43}a^{-1}n^{-1}\|P_n-f_n\|_{L_p[0,1]}. \end{aligned}$$

Hence
$$\frac{E^{**}_{m,p}(f_n, \Delta_n)}{n^{-1}\omega_{2,p}(Df_n, n^{-1})}\geqslant C_{43}^{-1}\, an^{-2}\sim n^{1/(p+1)},\ \text{as } n\to+\infty.$$

If $1\leqslant p<\infty$, it implies (1.7). If $p=+\infty$, we have

$$\omega_{3,\infty}(f_n, n^{-1})\leqslant 16\|f_n-P_n\|_{C[0,1]}.$$

(1.8) follows from (3.13) and this inequality.

Remark. Using the same method, we can prove the following theorem.

Theorem 5. *Let $m\geqslant 2$ and $1\leqslant p\leqslant+\infty$. Then there is a positive constant C such that for any convex function f on $[0, 1]$,*

$$E^{*}_{m,p}(f, \Delta_n)\leqslant C\bar{\Delta}_n\omega_{2,p}(Df, \bar{\Delta}_n).$$

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References

[1] DeVore, R. A., Monotone approximation by splines, *SIAM J. Math. Anal.*, **8** (1977), 891—905.

[2] Beatson, R. K., Convex approximation by splines, *SIAM J. Math. Anal.*, **12** (1981), 547—559.

[3] Leviatan, D. and Mhasker, H. N., The rate of monotone spline approximation in L_p-norm, *SIAM J. Math. Anal.*, **13** (1982), 866—872.

[4] Chui, C. K., Smith, P. W. and Ward, J. D., Degree of L_p-approximation by monotone splines, *SIAM J. Math, Anal.*, **11** (1980), 436—447.

[5] DeVore, R. A., Degree of Approximation, Approximation Theory II, G. G. Lorentz, C. K. Chui and L. L. Schumaker, eds, Academic Press, New York, (1976), 117—162.

[6] Beatson, R. K., Restricted range approximation by splines and variational inequalities, *SIAM J. Numer. Anal.*, **19** (1982), 372—380.