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SADDLE VALUES AND INTEGRABILITY CONDITIONS OF QUADRATIC DIFFERENTIAL SYSTEMS

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Abstract

In the first section of the paper, the first three saddle values R_1, R_2, R_3 of the real quadratic differential system (QDE) are computed by use of the method with which Poincaré researchs on Hopf bifurcation. In the second section, by applying the method and results of Dulac seeking integrability conditions of QDE it is proved that the system is integrable if and only if $R_1=R_2=R_3=0$, and it is also true when the system is complex. The integrability conditions given here can be applied much more easily then Dulac's. In the last part of the paper, it is pointed out that iR_1, iR_2, iR_3 are the first three focal values of the weak focus of the complex system. By the formulae of R_1, R_2, R_3 and the result in section 2, one can easily give the formulae of the focal values for the real QDE with Bautin form and give a new proof of Bautin's famous result.

In this paper, first we compute the first three saddle values R_1, R_2, R_3 of weak saddle of QDE with Poincaré's method, then, using the method and results of Dulac, prove that $R_1=R_2=R_3=0$ is the necessary and sufficient condition of integrability and it is also true for the complex system. At last, we point out that the formulae of the saddle values of real system are different from the formulae of the focal values of complex system with the same form only by a pure imaginary factor i. Thus we can give a new proof of Bautin's famous result.

Remark In other paper, we have proved that the real QDE with weak saddle has no closed and singular closed orbits which consist of only separatrices and saddles when $R_1=0$, $R_2^2+R_3^2\neq 0$.

§ 1. Computation of the Saddle Values

In this section, we consider the real system with weak saddle

$$\begin{aligned}\dot{x}&=x+a_1x^2+a_2xy+a_3y^2,\\ \dot{y}&=-y+b_1y^2+b_2xy+b_3x^2.\end{aligned}\tag{1.1}$$

We will seek the polynomial transformation as follows:

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$$
\begin{aligned}
x&=u+a_{20}u^2+a_{11}uv+a_{02}v^2+\cdots=u+\sum_{2\leqslant i+j\leqslant r} a_{ij}u^iv^j,\\
y&=v+b_{20}v^2+b_{11}vu+b_{02}u^2+\cdots=v+\sum_{2\leqslant i+j\leqslant r} b_{ij}v^iu^j,
\end{aligned}
\tag{1.2}
$$

where a_{ij}, b_{ij} are coefficients to be determinated such that (1.1) is transformed into

$$
\begin{aligned}
\dot{u}&=u+c_1u^2v+c_2u^3v^2+\cdots+c_ku^{k+1}v^k+O(|u|^{k+2}\cdot|v|^{k+1}),\\
\dot{v}&=-v+d_1v^2u+d_2v^3u^2+\cdots+d_kv^{k+1}u^k+O(|v|^{k+2}\cdot|u|^{k+1}).
\end{aligned}
\tag{1.3}
$$

Definition 1 *Let R_0 be the divergence at $O(0,0)$, $R_i=k_i(c_i+d_i)$, where $i>0$, k_i's are proper positive constants. Then O is called a weak saddle of order j and R_j is the j-th saddle value of saddle O if $R_i=0$ for any $0\leqslant i<j$ and $R_j\neq 0$,*

Definition 2 *Let $W^s(W^u)$ be a stable (unstable) manifold of the saddle O. If $W^s=W^u$, then the loop $\{O\}\cup W^s$ is called the homoclinic loop bifurcation (simply by HLB). Moreover, if $\bar{R}_i=0$ for any $0\leqslant i<j$, $\bar{R}_j\neq 0$, then this HLB is of order j,* where $\bar{R}_{2k+1}=R_k$ and $\bar{R}_{2k}$ (see [2]). We *also say the HLB is degenerate when $j>0$.*

Theorem 1[2,3]. *If system (1.1) has a degenerate HLB of order k, then any perturbation of (1.1) has at most k limit cycles and, for any $i\leqslant k$, there exists a perturbation with exactly i limit cycles.*

For the proof see [2].

Now we compute the saddle values of weak saddle O of system (1.1).

Differentiate (1.2)

$$
\begin{aligned}
\dot{x}&=\dot{u}+(\sum ia_{ij}u^{i-1}v^j)\dot{u}+(\sum ja_{ij}u^iv^{j-1})\dot{v},\\
\dot{y}&=\dot{v}+(\sum jb_{ij}v^iu^{j-1})\dot{u}+(\sum ib_{ij}v^{i-1}u^j)\dot{v}.
\end{aligned}
\tag{1.4}
$$

First substitute (1.2) into the right hand of (1.1), then substitute (1.1) and (1.3) into (1.4), we have

$$
\begin{aligned}
&(u+\sum a_{ij}u^iv^j)+a_1(u+\sum a_{ij}u^iv^j)^2+a_2(u+\sum a_{ij}u^iv^j)\\
&\quad\cdot(v+\sum b_{ij}v^iu^j)+a_3(v+\sum b_{ij}v^iu^j)^2\\
&=(1+\sum ia_{ij}u^{i-1}v^j)(u+c_1u^2v+c_2u^3v^2+\cdots+c_ku^{k+1}v^k)\\
&\quad+\sum ja_{ij}u^iv^{j-1}(-v+d_1v^2u+d_2v^3u^2+\cdots+d_kv^{k+1}u^k)+\text{h.o.t.}
\end{aligned}
\tag{1.5$_1$}
$$

$$
\begin{aligned}
&-(v+\sum b_{ij}v^iu^j)+b_1(v+\sum b_{ij}v^iu^j)^2\\
&\quad+b_2(u+\sum a_{ij}u^iv^j)(v+\sum b_{ij}v^iu^j)+b_3(u+\sum a_{ij}u^iv^j)^2\\
&=\sum jb_{ij}v^iu^{j-1}(u+c_1u^2v+c_2u^3v^2+\cdots+c_ku^{k+1}v^k)\\
&\quad+(1+\sum ib_{ij}v^{i-1}u^j)(-v+d_1v^2u+d_2v^3u^2+\cdots+d_kv^{k+1}u^k)+\text{h.o.t.}
\end{aligned}
\tag{1.5$_2$}
$$

Comparing the coefficients of two sides, we have

$$
\begin{aligned}
&a_{20}=a_1,\ a_{11}=-a_2,\ a_{02}=-\frac{1}{3}a_3,\\
&b_{20}=-b_1,\ b_{11}=b_2,\ b_{02}=\frac{1}{3}b_3;
\end{aligned}
\tag{1.6}
$$

$$
c_1=2a_1a_{11}+a_2(b_{11}+a_{20})+2a_3b_{02},
$$

$$a_{30}=\frac{1}{2}(2a_1a_{20}+a_2b_{02}),$$

$$a_{12}=-\frac{1}{2}[2a_1a_{02}+a_2(a_{11}+b_{20})+2a_3b_{11}],$$

$$a_{03}=-\frac{1}{4}(a_2a_{02}+2a_3b_{20});$$

$$a_{40}=\frac{1}{3}[a_1(2a_{30}+a_{20}^2)+a_2(b_{03}+a_{20}b_{02})+a_3b_{02}^2],$$

$$a_{31}=2a_1(a_{21}+a_{20}a_{11})+a_2(b_{12}+a_{30}+a_{20}b_{11}+a_{11}b_{02})$$
$$+2a_3(b_{03}+b_{02}b_{11})-2a_{20}c_1,$$

$$a_{22}=-a_1(2a_{12}+a_{11}^2+2a_{20}a_{02})-a_2(b_{21}+a_{21}+a_{11}b_{11}+a_{20}b_{20}+a_{02}b_{02})$$
$$-a_3(2b_{12}+b_{11}^2+2b_{20}b_{02})+a_{11}(c_1+d_1),$$

$$a_{13}=-\frac{1}{3}[a_1(2a_{03}+2a_{11}a_{02})+a_2(b_{30}+a_{12}+a_{11}b_{20}+a_{02}b_{11})$$
$$+a_3(2b_{21}+2b_{11}b_{20})-2a_{02}d_1],$$

$$a_{04}=-\frac{1}{5}[a_1a_{02}^2+a_2(a_{03}+a_{02}b_{20})+a_3(b_{20}^2+2b_{30})];$$

$$c_2=2a_1C_{21}+a_2C_{22}+2a_3C_{23}-a_{21}(2c_1+d_1),$$

$$a_{41}=\frac{1}{2}[2a_1A_4^1+a_2B_4^1+2a_3C_4^1-3a_{30}c_1],$$

$$a_{23}=-\frac{1}{2}[2a_1A_2^3+a_2B_2^3+2a_3C_2^3-a_{12}(c_1+2d_1)],$$

$$a_{14}=-\frac{1}{4}[2a_1A_1^4+a_2B_1^4+2a_3C_1^4-3a_{03}d_1],$$

where

$$C_{21}=a_{22}+a_{11}a_{21}+a_{02}a_{30}+a_{20}a_{12},$$

$$C_{22}=b_{22}+a_{31}+a_{20}b_{21}+a_{11}b_{12}+a_{02}b_{03}+a_{30}b_{20}+a_{21}b_{11}+a_{12}b_{02},$$

$$C_{23}=b_{13}+b_{20}b_{03}+b_{11}b_{12}+b_{02}b_{21},$$

$$A_4^1=a_{31}+a_{20}a_{21}+a_{30}a_{11},$$

$$B_4^1=b_{13}+a_{40}+a_{20}b_{12}+a_{30}b_{11}+a_{11}b_{03}+a_{21}b_{02},$$

$$C_4^1=b_{04}+b_{02}b_{12}+b_{11}b_{03},$$

$$A_2^3=a_{13}+a_{20}a_{03}+a_{11}a_{12}+a_{02}a_{21},$$

$$B_2^3=b_{31}+a_{22}+a_{20}b_{30}+a_{11}b_{21}+a_{02}b_{12}+a_{03}b_{02}+a_{12}b_{11}+a_{21}b_{20},$$

$$C_2^3=b_{22}+b_{20}b_{12}+b_{11}b_{21}+b_{02}b_{30},$$

$$A_1^4=a_{04}+a_{02}a_{12}+a_{11}a_{03},$$

$$B_1^4=b_{40}+a_{13}+a_{02}b_{21}+a_{11}b_{30}+a_{12}b_{20}+a_{03}b_{11},$$

$$C_1^4=b_{31}+b_{11}b_{30}+b_{20}b_{21};$$

$$a_{42}=2a_1A_4^2+a_2B_4^2+a_3C_4^2-3a_{31}c_1-a_{31}d_1-2a_{20}c_2,$$

$$a_{33}=-2a_1A_3^3-a_2B_3^3-2a_3C_3^3+2a_{22}(c_1+d_1)+a_{11}(c_2+d_2),$$

$$b_{24}=\frac{1}{3}[b_1C_4^2+b_2B_4^2+2b_3A_4^2-b_{13}(3c_1+d_1)-2b_{02}c_2],$$

$$c_3=2a_1C_{31}+a_2C_{32}+2a_3C_{33}-a_{32}(3c_1+2d_1)-a_{21}(2c_2+d_2),$$

where
$$A_4^2=a_{32}+a_{20}a_{22}+a_{11}a_{31}+a_{02}a_{40}+a_{30}a_{12}+a_{21}a_{21}/2,$$
$$B_4^2=b_{23}+a_{41}+a_{20}b_{22}+a_{11}b_{13}+a_{02}b_{04}+a_{30}b_{21}+a_{21}b_{12}$$
$$+a_{12}b_{03}+a_{40}b_{20}+a_{31}b_{11}+a_{22}b_{02},$$
$$C_4^2=2b_{14}+2b_{20}b_{04}+2b_{11}b_{13}+2b_{02}b_{22}+2b_{03}b_{21}+b_{12}^2,$$
$$A_3^3=a_{23}+a_{20}a_{13}+a_{11}a_{22}+a_{02}a_{31}+a_{30}a_{03}+a_{21}a_{12},$$
$$B_3^3=b_{32}+a_{32}+a_{20}b_{31}+a_{11}b_{22}+a_{02}b_{13}+a_{30}b_{30}+a_{21}b_{21}$$
$$+a_{12}b_{12}+a_{03}b_{03}+a_{31}b_{20}+a_{22}b_{11}+a_{13}b_{02},$$
$$C_3^3=b_{23}+b_{20}b_{13}+b_{11}b_{22}+b_{02}b_{31}+b_{30}b_{03}+b_{21}b_{12},$$
$$C_{31}=a_{33}+a_{20}a_{23}+a_{11}a_{32}+a_{02}a_{41}+a_{30}a_{13}+a_{21}a_{22}+a_{12}a_{31}+a_{03}a_{40},$$
$$C_{32}=b_{33}+a_{42}+a_{20}b_{32}+a_{11}b_{23}+a_{02}b_{14}+a_{30}b_{31}+a_{21}b_{22}+a_{12}b_{13}+a_{03}b_{04}$$
$$+a_{40}b_{30}+a_{31}b_{21}+a_{22}b_{12}+a_{13}b_{03}+a_{41}b_{20}+a_{32}b_{11}+a_{23}b_{02},$$
$$C_{33}=b_{24}+b_{20}b_{14}+b_{11}b_{23}+b_{02}b_{32}+b_{30}b_{04}+b_{21}b_{13}+b_{12}b_{22}+b_{03}b_{31}.$$

If we denote

$$A=(a_1,\ a_2,\ a_3),\ B=(b_1,\ b_2,\ b_3),$$
$$A_k=(a_{20},\ a_{11},\ a_{02},\ a_{30},\ \cdots,\ a_{k0},\ \cdots,\ a_{0k}),$$
$$B_k=(b_{20},\ b_{11},\ b_{02},\ \cdots,\ b_{k0},\ \cdots,\ b_{0k}),$$
$$C=(c_1,\ c_2,\ c_3),\ D=(d_1,\ d_2,\ d_3),$$
$$a_{ij}=a_{ij}(A,\ A_k,\ B_k,\ C,\ D),\ c_i=c_i(A,\ A_k,\ B_k,\ C,\ D),$$
$$b_{ij}=b_{ij}(B,\ A_k,\ B_k,\ C,\ D),$$

then from (1.5), it follows that

$$b_{ij}=-a_{ij}(B,\ B_k,\ A_k,\ D,\ C),\ d_i=c_i(B,\ B_k,\ A_k,\ D,\ C),$$
$$a_{ij}=-b_{ij}(A,\ B_k,\ A_k,\ D,\ C). \tag{1.7}$$

Proposition 1. *In the transformation* (1.2), *the coefficients* $a_{21}, b_{21}, a_{32}, b_{32}$ *can be chosen arbitrarily.*

Proof Compare the coefficients of the same power in two sides of (1.5), the proposition follows immediately.

Proposition 2 R_1, R_2 *and* R_3 *are independent of* a_{21}, b_{21}, a_{32} *and* b_{32}.

Proof By (1.7) and the expressions c_1 and R_1, it is easy to see that R_1 is independent of these four numbers.

Again by (1.7) and by the expressions c_2 and R_2, evidently, R_2 does not depend on a_{32} and b_{32}. Now we show R_2 has no relation to a_{21} (we can prove that R_2 has no relation to b_{21} similarly).

In the expression of c_2, the coefficient of a_{21} is

$$F_1=2a_1(-a_2+a_{11})+a_2(b_2+2a_1+b_{11})+2a_3\cdot\frac{1}{3}\cdot 2b_3-(2c_1+d_1)$$
$$=-2a_1a_2+2a_2b_2+\frac{4}{3}a_3b_3-2c_1-d_1=-d_1,$$

and in the expression of d_2, the coefficient of a_{21} is

$$F_2=2b_1b_2+b_2(-a_2+b_{20})+2b_3a_{02}$$
$$=b_1b_2-a_2b_2-\frac{2}{3}a_3b_3=d_1.$$

So, in the expression of R_2, the coefficient of a_{21} is zero.

Similarly, we can show R_3 is not relevant to the choice of a_{21}, b_{21}, a_{32} and b_{32}.

By Propositions 1 and 2, we take

$$a_{21}=b_{21}=a_{32}=b_{32}=0. \tag{1.8}$$

If consider a_{ij}, c_i as the functions of $A=(a_1, a_2, a_3)$ and $B=(b_1, b_2, b_3)$: $a_{ij}=a_{ij}(A,B)$, $c_i=c_i(A, B)$, then by (1.6), (1.7) and the expressions of a_{ij} and c_i we obtain

$$b_{ij}=(-1)^{i+j+1}a_{ij}(B, A),\ d_i=-c_i(B, A). \tag{1.9}$$

Using (1.6)—(1.9), we can get $a_{ij}(A,B)$, $c_i(A,B)$, $d_i(A,B)$ one by one as follows.

$$\begin{aligned}
&a_{30}=a_1^2+\frac{1}{6}a_2b_3,\\
&a_{21}=0,\\
&a_{12}=\frac{1}{3}a_1a_3+\frac{1}{2}a_2^2+\frac{1}{2}a_2b_1-a_3b_2,\\
&a_{03}=\frac{1}{12}a_2a_3+\frac{1}{2}a_3b_1.
\end{aligned} \tag{1.10}$$

$$\begin{aligned}
&c_1=-a_1a_2+a_2b_2+\frac{2}{3}a_3b_3,\\
&d_1=b_1b_2-a_2b_2-\frac{2}{3}a_3b_3.
\end{aligned} \tag{1.11}$$

The following expressions are obtained by using the relations $R_1=k_1(c_1+d_1)=0$ and $R_2=k_2(c_2+d_2)=0$.

$$\begin{aligned}
&a_{40}=a_1^3+\frac{7}{18}a_1a_2b_3+\frac{1}{36}a_2b_2b_3+\frac{1}{27}a_3b_3^2,\\
&a_{31}=a_1^2a_2-\frac{1}{2}a_1a_2b_2+\frac{1}{2}a_2b_2^2+\frac{1}{3}a_2b_1b_3-\frac{7}{6}a_2^2b_3+\frac{5}{6}a_3b_2b_3-\frac{1}{3}a_1a_3a_3,\\
&a_{22}=-2a_1a_2^2+a_1a_3b_2+a_2^2b_2+\frac{19}{9}a_2a_3b_3-2a_3b_2^2,\\
&a_{13}=\frac{1}{18}a_1a_2a_3-\frac{1}{3}a_1a_3b_1-\frac{1}{3}a_2b_1^2+\frac{11}{18}a_2a_3b_2-\frac{1}{6}a_2^3-\frac{1}{2}a_2^2b_1+\frac{4}{27}a_3^2b_3,\\
&a_{04}=-\frac{1}{45}a_1a_3^2-\frac{1}{60}a_2^2a_3-\frac{1}{6}a_2a_3b_1-\frac{3}{5}a_3b_1^2-\frac{1}{15}a_3^2b_2.
\end{aligned} \tag{1.12}$$

$$\begin{aligned}
c_2=&-2a_1^2a_2^2+\frac{2}{3}a_1^2a_3b_2+3a_1a_2^2b_2+\frac{37}{9}a_1a_2a_3b_3-\frac{1}{3}a_1a_3b_1b_3-2a_1a_3b_2^2\\
&+2a_2^3b_3-a_2^2b_1b_3-a_2^2b_2^2-\frac{175}{36}a_2a_3b_2b_3-\frac{8}{27}a_3^2b_3^2+\frac{4}{3}a_3b_2^3,\\
d_2=&\,2a_1^2a_2^2-\frac{2}{3}a_2b_1^2b_3-3a_1a_2^2b_2-\frac{37}{9}a_1a_2a_3b_3+\frac{1}{3}a_1a_3b_1b_3+2a_2^2b_1b_3\\
&-2a_3b_2^3+a_1a_3b_2^2+a_2^2b_2^2+\frac{175}{36}a_2a_3b_2b_3+\frac{8}{27}a_3^2b_3^2-\frac{4}{3}a_2^3b_3.
\end{aligned} \tag{1.13}$$

$$a_{41}=2a_1^3a_2-\frac{13}{12}a_1^2a_2b_2-\frac{11}{15}a_1^2a_3b_3-\frac{5}{3}a_1a_2^2b_3+a_1a_2b_2^2+\frac{2}{3}a_1a_2b_1b_3$$
$$+\frac{5}{3}a_1a_3b_2b_3-\frac{1}{2}a_2^2b_2b_3-\frac{22}{45}a_2a_3b_3^2+\frac{1}{12}a_2b_2^3+\frac{2}{15}a_3b_1b_3^2+\frac{4}{15}a_3b_2^2b_3,$$
$$a_{32}=0,$$
$$a_{23}=\frac{31}{36}a_1^2a_2a_3-\frac{1}{6}a_1^2a_3b_1+\frac{7}{6}a_1a_2^3+\frac{7}{4}a_1a_2^2b_1-\frac{77}{18}a_1a_2a_3b_2-\frac{1}{6}a_1a_2b_1^2 \quad (1.14)$$
$$-\frac{7}{27}a_1a_3^2b_3-\frac{3}{4}a_2^3b_2-\frac{215}{72}a_2^2a_3b_3-\frac{13}{36}a_2a_3b_1b_3+\frac{5}{2}a_2a_3b_2^2+\frac{43}{18}a_3^2b_2b_3,$$
$$a_{14}=\frac{1}{15}a_1^2a_3^2-\frac{173}{240}a_1a_2^2a_3+\frac{23}{24}a_1a_2a_3b_1+\frac{1}{30}a_1a_3^2b_2$$
$$+\frac{3}{10}a_1a_3b_1^2+\frac{1}{24}a_2^4+\frac{1}{4}a_2^3b_1+\frac{1}{16}a_2^2a_3b_2+\frac{11}{24}a_2^2b_1^2$$
$$+\frac{25}{72}a_2a_3^2b_3+\frac{1}{4}a_2b_1^3-\frac{5}{12}a_3^2b_1b_3-\frac{2}{3}a_3^2b_2^2.$$

$$A_4^2=-2a_1a_3b_2^2+\frac{5}{4}a_2^3b_3-\frac{1}{4}a_2^2b_1b_3+A^*,$$
$$A^*=-\frac{5}{2}a_1^2a_2^2+\frac{1}{2}a_1^2a_2b_1+\frac{3}{2}a_1a_2^2b_2+\frac{64}{27}a_1a_2a_3b_3$$
$$-\frac{1}{2}a_2^2b_2^2-\frac{109}{108}a_2a_3b_2b_3-\frac{1}{81}a_3^2b_3^2,$$
$$B_4^2=\frac{11}{6}a_1^3a_2-a_1^3b_1+\frac{10}{3}a_1^2a_2b_2-\frac{23}{30}a_1^2a_3b_3-\frac{95}{24}a_1a_2^2b_3$$
$$+\frac{1}{18}a_1a_2b_1b_3+\frac{1}{6}a_1a_2b_2^2-\frac{4}{5}a_1a_3b_2b_3-\frac{1}{6}a_1b_1^2b_3+\frac{131}{72}a_2^2b_2b_3$$
$$+\frac{737}{270}a_2a_3b_3^2-\frac{1}{5}a_2b_2^3-\frac{23}{135}a_3b_1b_3^2-\frac{951}{360}a_3b_2^2b_3,$$
$$C_4^2=\frac{1}{2}a_1^3b_2+\frac{31}{12}a_1^2a_2b_3-\frac{3}{5}a_1^2b_1b_3+\frac{11}{6}a_1^2b_2^2-\frac{331}{360}a_1a_2b_2b_3$$
$$-\frac{5}{6}a_1a_3b_3^2+2a_1b_2^3+a_2^2b_3^2-\frac{7}{5}a_2b_1b_3^2-\frac{199}{72}a_2b_2^2b_3$$
$$-\frac{109}{108}a_3b_2b_3^2+\frac{1}{5}b_1^2b_3^2+\frac{2}{3}b_2^4,$$
$$A_3^3=\frac{2}{3}a_1^2a_2a_3+3a_1a_2^3+\frac{5}{4}a_1a_2^2b_1-\frac{9}{2}a_1a_2a_3b_2-\frac{1}{2}a_1a_2b_1^2-\frac{7}{4}a_2^3b_2$$
$$-\frac{169}{36}a_2^2a_3b_3-\frac{7}{18}a_2a_3b_1b_3+\frac{13}{3}a_2a_3b_2^2+\frac{19}{9}a_3^2b_2b_3$$
$$=A_3^3(A,\ B),$$
$$B_3^3=B_3^3(A,\ B)=B_3^3(B,\ A),$$
$$C_3^3=A_3^3(B,\ A)=-\frac{1}{2}a_1^3a_2+\frac{5}{4}a_1^2a_2b_2-\frac{9}{2}a_1a_2^2b_3$$
$$+\frac{2}{3}a_1a_2b_1b_3+3a_1a_2b_2^2-\frac{7}{18}b_1a_3b_2b_3+\frac{13}{3}a_2^2b_2b_3$$
$$+\frac{19}{9}a_2a_3b_3^2-\frac{7}{4}a_2b_2^3-\frac{169}{36}a_3b_2^2b_3.$$

$$
\begin{aligned}
a_{42}= & -\frac{5}{6} a_1^3 a_3 b_2+\frac{11}{6} a_1^2 a_3 b_2^2-\frac{203}{24} a_1 a_2^3 b_3+\frac{20}{9} a_1 a_2^2 b_1 b_3 \\
& -\frac{1}{6} a_1 a_2 b_1^2 b_3-\frac{2}{3} a_1 a_3 b_2^3+\frac{299}{72} a_2^3 b_2 b_3+\frac{1427}{270} a_2^2 a_3 b_3^2 \\
& -\frac{272}{135} a_2 a_3 b_1 b_3^2+\frac{1}{5} a_3 b_1^2 b_3^2+\frac{2}{3} a_3 b_2^4+a^*, \\
b_{24}= & -\frac{1}{6} a_1^4 a_2+\frac{13}{9} a_1^3 a_2 b_2+\frac{17}{9} a_1^2 a_2 b_2^2-\frac{149}{270} a_1^2 a_3 b_2 b_3 \\
& +\frac{1}{18} a_1 a_2 b_2^3-\frac{14}{9} a_1 a_3 b_2^2 b_3+\frac{7}{18} a_2^3 b_3^2+\frac{7}{18} a_2^2 b_1 b_3^2 \\
& -\frac{7}{15} a_2 b_1^2 b_3^2-\frac{2}{9} a_2 b_2^4-\frac{1351}{1080} a_3 b_2^3 b_3+\frac{1}{15} b_1^3 b_3^2+b^*, \qquad (1.15) \\
a_{33}= & -\frac{1}{3} a_1^3 a_2 a_3+\frac{13}{2} a_1^2 a_2 a_3 b_2-\frac{44}{3} a_1 a_2 a_3 b_2^2-\frac{31}{9} a_1 a_3^2 b_2 b_3 \\
& +\frac{7}{2} a_2 a_3 b_2^3+\frac{169}{18} a_3^2 b_2^2 b_3+e^*, \\
b_{33}= & -a_{33}(B, A)=\frac{44}{3} a_1 a_2^3 b_3-\frac{13}{2} a_1 a_2^2 b_1 b_3+\frac{1}{3} a_1 a_2 b_1^2 b_3 \\
& -\frac{7}{2} a_2^3 b_2 b_3-\frac{169}{18} a_2^2 a_3 b_3^2+\frac{31}{9} a_2 a_3 b_1 b_3^2+f^* .
\end{aligned}
$$

Here $a_2 a^*$, $a_3 b^*$, $a_1 e^*$, $a_2 f^*$ are symmetric with respect to A and B, so by (1.9) and the expressions of c_3, d_3 and R_3, R_3 is independent of a^*, b^*, e^* and f^*.

$$
\begin{aligned}
C_{31}= & \frac{1}{3} a_1^3 a_2 a_3+\frac{73}{36} a_1^2 a_2 a_3 b_2+\frac{1}{45} a_1^2 a_3^2 b_3-\frac{71}{6} a_1 a_2 a_3 b_2^2 \\
& -a_1 a_3^2 b_2 b_3-\frac{11}{18} a_2^4 b_3-\frac{1}{2} a_2^3 b_1 b_3+\frac{1}{9} a_2^2 b_1^2 b_3 \\
& +\frac{107}{36} a_2 a_3 b_2^3+\frac{127}{15} a_3^2 b_2^2 b_3+C_{31}^*, \\
C_{32}= & -\frac{35}{36} a_1^3 a_3 b_2+\frac{229}{72} a_1^2 a_3 b_2^2+\frac{40}{3} a_1 a_2^3 b_3-\frac{23}{6} a_1 a_2^2 b_1 b_3 \\
& -\frac{61}{36} a_1 a_3 b_2^3-\frac{7}{2} a_2^3 b_2 b_3-\frac{10351}{1080} a_2^2 a_3 b_3^2+\frac{747}{270} a_2 a_3 b_1 b_3^2 \\
& +\frac{5}{54} a_3 b_1^2 b_3^2-\frac{37}{72} a_3 b_2^4+C_{32}^*, \\
C_{33}= & -\frac{5}{12} a_1^4 a_2+\frac{59}{72} a_1^3 a_2 b_2+\frac{79}{18} a_1^2 a_2 b_2^2-\frac{167}{270} a_1^2 a_3 b_2 b_3 \\
& +\frac{121}{72} a_1 a_2 b_2^3-\frac{43}{18} a_1 a_3 b_2^2 b_3-\frac{29}{18} a_2^3 b_3^2+\frac{49}{18} a_2^2 b_1 b_3^2 \\
& -\frac{23}{30} a_2 b_1^2 b_3^2-\frac{53}{36} a_2 b_2^4-\frac{701}{135} a_3 b_2^3 b_3+\frac{1}{45} b_1^3 b_3^2+C_{33}^*,
\end{aligned}
$$

where $a_1 C_{31}^*$, $a_2 C_{32}^*$, $a_3 C_{33}^*$ are symmetric with respect to A and B.

$$
\begin{aligned}
c_3= & -\frac{1}{6}a_1^4a_2a_3+\frac{85}{18}a_1^3a_2a_3b_2+\frac{2}{45}a_1^3a_3^2b_3-\frac{281}{24}a_1^2a_2a_3b_2^2 \\
& -\frac{437}{135}a_1^2a_3^2b_2b_3+\frac{109}{9}a_1a_2^4b_3-\frac{29}{6}a_1a_2^3b_1b_3+\frac{2}{9}a_1a_2^2b_1^2b_3 \\
& +\frac{137}{18}a_1a_2a_3b_2^3+\frac{547}{45}a_1a_3^2b_2^2b_3-\frac{7}{2}a_2^4b_2b_3-\frac{13831}{1080}a_2^3a_3b_3^2+\frac{739}{90}a_2^2a_3b_1b_3^2 \\
& -\frac{389}{270}a_2a_3b_1^2b_3^2-\frac{83}{24}a_2a_3b_2^4-\frac{1402}{135}a_3^2b_2^3b_3+\frac{2}{45}a_3b_1^3b_3^2+c^*, \\
d_3= & \frac{1}{6}a_1a_2b_1^3b_3-\frac{85}{18}a_1a_2^2b_1^2b_3-\frac{2}{45}a_3b_1^3b_3^2+\frac{281}{24}a_1a_2^3b_1b_3+\frac{437}{135}a_2a_3b_1^2b_3^2 \\
& -\frac{109}{9}a_1a_2a_3b_2^3+\frac{29}{6}a_1^2a_2a_3b_2^2-\frac{2}{9}a_1^3a_2a_3b_2-\frac{137}{18}a_1a_2^4b_3-\frac{547}{45}a_2^2a_3b_1b_3^2 \\
& +\frac{7}{2}a_2a_3b_2^4+\frac{13831}{1080}a_3^2b_2^3b_3-\frac{739}{90}a_1a_3^2b_2^2b_3+\frac{389}{270}a_1^2a_3^2b_2b_3 \\
& +\frac{83}{24}a_2^4b_2b_3+\frac{1402}{135}a_2^3a_3b_3^2-\frac{2}{45}a_1^3a_3^2b_3-c^*,
\end{aligned}
\tag{1.16}
$$

where $c^*=2a_1C_{31}^*+a_2C_{32}^*+2a_3C_{33}^*$.

By (1.11), (1.13) and (1.16), the formulae R_1, R_2, R_3 are obtained as follows.

Theorem 2. $R_1=b_1b_2-a_1a_2$,

$$R_2=a_2b_3(2a_2-b_1)(a_2+2b_1)-a_3b_2(2b_2-a_1)(b_2+2a_1),$$

$$R_3=(a_1b_1+a_2b_2-5a_3b_3)[a_2b_3(4b_1^2-a_2^2)-a_3b_2(4a_1^2-b_2^2)].$$

Proof By Definition 1 and (1.11), and taking $k_1=1$, we get $R_1=b_1b_2-a_1a_2$ immediately.

By (1.13) and taking $k_2=3$,

$$
\begin{aligned}
R_2=3(c_2+d_2) &= -6R_1\left(a_2b_2+\frac{14}{9}a_3b_3\right)+a_2b_3(2a_2^2+3a_2b_1-2b_1^2) \\
&\quad +b_2a_3(-2b_2^2-3b_2a_1+2a_1^2) \\
&=a_2b_3(2a_2-b_1)(a_2+2b_1)-a_3b_2(2b_2-a_1)(b_2+2a_1).
\end{aligned}
$$

Take $k_3=24$, then $R_3=24(c_3+d_3)$. Denote

$$
\begin{aligned}
r=r(A,B)=24a_3\Big(&-\frac{1}{6}a_1^4a_2+\frac{9}{2}a_1^3a_2b_2-\frac{55}{8}a_1^2a_2b_2^2-\frac{97}{54}a_1^2a_3b_2b_3 \\
&-\frac{9}{2}a_1a_2b_2^3+\frac{71}{18}a_1a_3b_2^2b_3+\frac{1}{24}a_2b_2^4+\frac{523}{216}a_3b_2^3b_3\Big),
\end{aligned}
$$

$$r_1=a_3b_2(2a_1^2-3a_1b_2-2b_2^2),\ r_2=r_1(B,A)=a_2b_3(2b_1^2-3a_2b_1-2a_2^2).$$

Then $R_2=0$ means $r_1=r_2$. By $a_1a_2=b_1b_2$ and the expression of $r(A,B)$, we have

$$
\begin{aligned}
r(A,B)&=24a_3\left[-\frac{1}{6}a_1^4a_2-\frac{1}{8}a_1^2a_2b_2^2+\frac{1}{24}a_2b_2^4+\frac{5}{6}a_1^2a_3b_2b_3-\frac{5}{24}a_3b_2^3b_3\right] \\
&\quad +24\left(\frac{9}{4}a_1a_2-\frac{71}{54}a_3b_3\right)r_1 \\
&=-a_3b_2(4a_1^2-b_2^2)(a_1b_1+a_2b_2-5a_3b_3)+\left(54a_1a_2-\frac{284}{9}a_3b_3\right)r_1.
\end{aligned}
$$

From (1.16), it follows that

$$R_3=24(c_3+d_3)=r(A, B)-r(B, A)$$
$$=(a_1b_1+a_2b_2-5a_3b_3)[a_2b_3(4b_1^2-a_2^2)-a_3b_2(4a_1^2-b_2^2)].$$

Thus we have finished the proof of Theorem 2.

§ 2. Saddle Values and Integrability Conditions

In this section, we apply the method and results of Dulac[4] to prove that $R_1=R_2=R_3=0$ is the necessary and sufficient condition of integrability of system (1.1), where the variables and coefficients may be real or complex. Because R_1, R_2, R_3 are formulated with the coefficients of the same equation, the integrability conditions we give here can be applied more conveniently.

Theorem *The system* (1.1) (*real or complex*) *has an analytic integral if and only if* $R_1=R_2=R_3=0$.

Proof First we prove the sufficiency.

We look for the integral of Dulac form of system (1.1)

$$\phi=xy+\phi_3+\phi_4+\cdots \tag{2.1}$$

where ϕ_j' s are homogeneous polynomials of degree j of x, y.

Let $P_2=a_1x^2+a_2xy+a_3y^2$, $Q_2=b_1y^2+b_2xy+b_3x^2$. Then

$$\frac{d\phi}{dt}=\frac{\partial\phi}{\partial x}\dot{x}+\frac{\partial\phi}{\partial y}\dot{y}$$
$$=\left(y+\frac{\partial\phi_3}{\partial x}+\frac{\partial\phi_4}{\partial x}+\cdots\right)(x+P_2)+\left(x+\frac{\partial\phi_3}{\partial y}+\frac{\partial\phi_4}{\partial y}+\cdots\right)(-y+Q_2).$$

By

$$yP_2+xQ_2+\frac{\partial\phi_3}{\partial x}x-\frac{\partial\phi_3}{\partial y}y=0, \tag{2.2}$$

we have

$$\phi_3=-\frac{1}{3}b_3x^3-(a_1+b_2)x^2y+(a_2+b_1)xy^2+\frac{1}{3}a_3y^3. \tag{2.3}$$

We select ϕ_4 such that

$$\frac{\partial\phi_3}{\partial x}P_2+\frac{\partial\phi_3}{\partial y}Q_2+x\frac{\partial\phi_4}{\partial x}-y\frac{\partial\phi_4}{\partial y}=0,$$

then
$$\frac{d\phi}{dt}=-(b_1b_2-a_1a_2)x^2y^2+\text{h.o.t.}$$

Thus we have got the first integrability condition of Dulac:

$$R_1=b_1b_2-a_1a_2=0.$$

According to [4], the coefficients a_1 and b_1 are divided into following three different cases.

1. $a_1b_1\neq 0$.

Let us introduce the transformation

$$u=hx,\ v=ky. \tag{2.4}$$

Then
$$\dot{u}=u+a_1hu^2+a_2kuv+a_3h^{-1}k^2v^2,$$
$$\dot{v}=-v+b_1kv^2+b_2huv+b_3h^2k^{-1}u^2. \quad (2.5)$$

It is easy to see that if we take $h=a_1^{-1}$, $k=b_1^{-1}$, then $a_1h=b_1k=1$, $a_2k=b_2h$. So, we may as well assume that $a_1=b_1=1$, $a_2=b_2$. Now $R_2=a_2(a_2+2)(2a_2-1)(b_3-a_3)$. Thus, $R_2=0$ exactly is the second condition of integrability (simply denoted by CI) of Dulac under the first case (i.e., the first CI of eq. (A) of [4]).

Moreover, if $a_3=b_3$, or $a_2=0$, or $a_2=-2$, then the system (1.1) is integrable (cf. [4] (A_1)—(A_3)). It is easy to see that we always have $R_3=0$ in these cases. If $a_2=1/2$, then $R_3=\frac{75}{32}(b_3-a_3)(1-4a_3b_3)$. When $a_3\neq b_3$, $R_3=0$ is equivalent to $a_3b_3=1/4$. By [4] (cf. the eq. (A_4)), this is the last condition of integrability under the first case.

2. $b_1=0$, $a_1\neq 0$.

Now $R_1=0$ means $a_2=0$.

i) If $a_3=0$, then system (1.1) can be rewritten into linear equation of y. Evidently, we have $R_2=R_3=0$.

ii) If $a_3\neq 0$, then by (2.4), (2.5), we may assume $a_1=a_3=1$.

Because $R_2=-b_2(2b_2-1)(b_2+2)$, we see that $R_2=0$ exactly is the second *CI* under the second case (cf. [4], eq. (B)).

When $b_2=-2$ or $b_2=0$, the system (1.1) is integrable (cf. [4] (B_2), (B_3)) and $R_3=0$. When $b_2=1/2$, $R_3=75\,b_3/8$. By [4] (γ), $R_3=0$ is the last *CI* under the second case.

The case $b_1\neq 0$, $a_1=b_2=0$ may be discussed similarly.

3. $a_1=b_1=0$.

Now $R_2=2(a_2^3b_3-a_3b_2^3)$ and $R_3=(5a_3b_3-a_2b_2)R_2/2$. By the *CI* of [4] equation (c), $R_2=0$ is the second and last *CI* under the third case.

Thus, the system (1.1) is integrable when $R_1=R_2=R_3=0$.

Now we prove the necessity.

Let $\phi(x, y)=c$ be the general integral of (1.1). By the analyticity of ϕ, $\phi(x, y)$ may be written as
$$\phi=ax+by+dx^2+exy+fy^2+\phi_3+\phi_4+\cdots.$$
From
$$\frac{d\phi}{dt}=\frac{\partial\phi}{\partial x}(x+P_2)+\frac{\partial\phi}{\partial y}(-y+Q_2)=0,$$
we get $a=b=0$, $d=f=0$. Then by (2.2), we have $e\neq 0$, and may assume $e=1$. At last, by the proof of sufficiency, we obtain $R_1=R_2=R_3=0$.

The proof of the theorem is competed.

§ 3. The Formulae of Real Saddle Values and the Formulae of Cemplex Focal Values

In the last section, we point out that the formulae iR_1, iR_2, iR_3, exactly are the formulae of focal values of complex system

$$\begin{aligned}\dot{x}&=ix+a_1x^2+a_2xy+a_3y^2,\\ \dot{y}&=-iy+b_1y^2+b_2xy+b_3x^2,\end{aligned}\tag{3.1}$$

where $i^2=-1$.

First we give a definition of complex focal values.

Under the complex coordinate, (3.1) may be written as $\dot{z}=iz+F(z,\bar{z})$. Then by the polynomial transformation

$$z=u+\sum_{2\le i+j\le k}a_{ij}u^iv^j,\ v=\bar{u},$$

it becomes

$$\dot{u}=iu+c_1u^2v+c_2u^3v^2+\cdots+c_ku^{k+1}v^k+\text{h. o. t.}\tag{3.2}$$

Definition. *The singular point O is a weak focus of order k and* Re (c_k) *is the k-th focal values, if* $\mathrm{Re}(c_1)=\cdots=\mathrm{Re}(c_{k-1})=0$, $\mathrm{Re}(c_k)\neq 0$.

Then, by the similarity of the real system (1.1), (1.3) to the complex system (3.1) and it's Poincaré normal form (3.2) respectively, and by the Poincaré method we applied in the first section to compute R_1, R_2, R_3, we conclude that the formulae R_1, R_2, R_3 of weak saddle O of the real system (1.1) and the formulae $\mathrm{Re}(c_1)$, $\mathrm{Re}(c_2)$, $\mathrm{Re}(c_3)$ of weak focus O of the complex system (3.1) have the same form. Moreover, we can conclude that $\mathrm{Re}(c_j)=ik_jR_j$, where k_j's are proper positive constants, because the linear part of (3.1) ((3.2)) is different from that of (1.1) ((1.3)) only by a pure imaginary factor i.

Now we give a very simple proof of the corner stone of Bautin's famous result, as a check of the preceding conclusion and an application of the former two sections.

Write the real *QDE* with weak focus into Bautin's form:

$$\begin{aligned}\dot{x}&=-y-\lambda_3x^2+(2\lambda_2+\lambda_5)xy+\lambda_6y^2,\\ \dot{y}&=x+\lambda_2x^2+(2\lambda_3+\lambda_4)xy-\lambda_2y^2.\end{aligned}\tag{3.3}$$

Then by transformation

$$u=x+iy,\ v=x-iy,\ \tau=it,\tag{3.4}$$

such that (3.3) becomes complex system with the same form of (1.1):

$$\begin{aligned}\frac{du}{d\tau}&=u+a_1u^2+a_2uv+a_3v^2,\\ \frac{dv}{d\tau}&=-v+b_1v^2+b_2uv+b_3u^2,\end{aligned}\tag{3.5}$$

where $a_1=-(\lambda_5+Mi)/4$, $a_2=(\lambda_3-\lambda_6)i/2$, $a_3=(4\lambda_2+\lambda_5+Ni)/4$,

$b_1=(\lambda_5-Mi)/4$, $b_2=a_2$, $b_3=(-4\lambda_2-\lambda_5+Ni)/4$,

$M=\lambda_3+\lambda_4-\lambda_6$, $N=3\lambda_3+\lambda_4+\lambda_6$.

Now, by the formulae of R_1, R_2, R_3, we can give the formulae of focal values of (3.3) very conveniently.

Let $\bar{\nu}_3=i\pi R_1$, $\bar{\nu}_5=\dfrac{1}{3}\pi iR_2$, $\bar{\nu}_7=\dfrac{5}{24}\pi iR_3$. Then

$$\bar{\nu}_3 = i\pi(b_1b_2 - a_1a_2) = i\pi a_2(b_1 - a_1) = -\pi\lambda_5(\lambda_3 - \lambda_6)/4. \tag{3.6}$$

Evidently, $\bar{\nu}_3$ is the first focal value of the weak focus O of (3.3), and when $R_1 = 0$, $R_3R_2 \neq 0$, we have $i(\lambda_3 - \lambda_6)/2 = a_2 = b_2 \neq 0$, $\lambda_5 = 0$, hence, $a_1 = b_1 = -Mi/4$.

$$\begin{aligned}
\bar{\nu}_5 &= \frac{1}{3}\pi i[a_2b_3(2a_2 - b_1)(a_2 + 2b_1) - a_3b_2(2b_2 - a_1)(b_2 + 2a_1)] \\
&= \frac{1}{3}\pi i a_2(2a_2 - a_1)(a_2 + 2a_1)(b_3 - a_3) \\
&= -\frac{\pi}{6}(\lambda_3 - \lambda_6)\left[(\lambda_3 - \lambda_6)i + \frac{1}{4}Mi\right]\cdot\left[\frac{1}{2}(\lambda_3 - \lambda_6)i - \frac{1}{2}Mi\right]\cdot(-2\lambda_2) \\
&= \frac{1}{24}\pi\lambda_2\lambda_4(\lambda_3 - \lambda_6)(\lambda_4 + 5\lambda_3 - 5\lambda_6)
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
\bar{\nu}_7 &= \frac{5}{24}\pi i(a_1b_1 + a_2b_2 - 5a_3b_3)[a_2b_3(4b_1^2 - a_2^2) - a_3b_2(4a_1^2 - b_2^2)] \\
&= \frac{5}{24}\pi i a_2(a_1^2 + a_2^2 - 5a_3b_3)(4a_1^2 - a_2^2)(b_3 - a_3) \\
&= -\frac{5}{48}\pi(\lambda_3 - \lambda_6)\cdot\frac{1}{16}[-M^2 - 4(\lambda_3 - \lambda_6)^2 - 5(4\lambda_2 + Ni)(-4\lambda_2 + Ni)] \\
&\quad \cdot\frac{1}{4}[-M^2 + (\lambda_3 - \lambda_6)^2]\cdot(-2\lambda_2) \\
&= -\frac{5\pi}{32\cdot 48}\lambda_2(\lambda_3 - \lambda_6)[(\lambda_3^2 + \lambda_4^2 + \lambda_6^2 + 2\lambda_3\lambda_4 - 2\lambda_3\lambda_6 - 2\lambda_4\lambda_6) \\
&\quad + (4\lambda_3^2 + 4\lambda_6^2 - 8\lambda_3\lambda_6) - 80\lambda_2^2 - 5(9\lambda_3^2 + \lambda_4^2 + \lambda_6^2 + 6\lambda_3\lambda_4 + 6\lambda_3\lambda_6 + 2\lambda_4\lambda_6)] \\
&\quad \cdot(-\lambda_4^2 - 2\lambda_3\lambda_4 + 2\lambda_4\lambda_6) \\
&= -\frac{5\pi}{12\times 32}\lambda_2\lambda_4(\lambda_3 - \lambda_6)[-10\lambda_3^2 - \lambda_4^2 - 7\lambda_3\lambda_4 - 10\lambda_3\lambda_6 - 3\lambda_4\lambda_6 - 20\lambda_2^2] \\
&\quad \cdot(-\lambda_4 - 2\lambda_3 + 2\lambda_6).
\end{aligned}$$

When $R_1 = R_2 = 0$, $R_3 \neq 0$, by (3.6), (3.7), we have $\lambda_4 + 5\lambda_3 - 5\lambda_6 = 0$. It follows that

$$\bar{\nu}_7 = -\frac{25}{32}\pi\lambda_2\lambda_4(\lambda_3 - \lambda_6)^2(\lambda_3\lambda_6 - 2\lambda_6^2 - \lambda_2^2). \tag{3.8}$$

Evidently, (3.6) — (3.8) exactly are the formulae of focal values (9.40) formulated in [1].

By section 2, when $\bar{\nu}_3 = \bar{\nu}_5 = \bar{\nu}_7 = 0$, system (3.3) is integrable. Then by the Taylor expansion of the displacement function, we get the lemma 9.2 of [1]. From this, we can obtain Bautin's famous result ([1] Th 9.3) with no much difficulty.

Remark. At last, we point out that the formulae of focal values W_1, W_2, W_3 of the real system

$$\begin{aligned}
\dot{x} &= -y + a_1x^2 + a_2xy + a_3y^2, \\
\dot{y} &= x + b_1y^2 + b_2xy + b_3x^2
\end{aligned} \tag{3.9}$$

are different from that of saddle values R_1', R_2', R_3' of the real system

$$\begin{aligned}
\dot{x} &= y + a_1x^2 + a_2xy + a_3y^2, \\
\dot{y} &= x + b_1y^2 + b_2xy + b_3x^2
\end{aligned} \tag{3.10}$$

although the form of (3.9) is similar to that of (3.10), and (3.9) can be transformed into (3.1) through complex transformation (3.4), (3.10) can be transformed into (1.1) through Joyal transformation

$$u=x+jy,\ v=x-jy,\ j^2=1 \tag{3.11}$$

which have a form similar to (3.4). This can be shown by changing R_1 into R_1' through (3.11). Also, we cannot transform R_i to W_i or reverse through real transformation (3.11). This paper confirm that we can deduce R_i and W_i from one to one only through complex transformation (3.4) or it's reverse. But it also needs a great deal of computation no less than that we do in section 1 to deduce R_i from W_i. This is because: (i) through (3.4), the coefficients a_i, b_i in (3.9) will be the sum of six or four terms; (ii) the complexity of the formulae of W_i, particularly, W_2 (W_3) is of degree 6 (degree 8); (iii) it is very difficult to cancel a positive definite quadric form factor such that R_2 (R_3) is of degree 4 (degree 6) and have a simple form.

The relationship between W_i and R_i shows a deep duality between focus and saddle.

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