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AN UPPER BOUND ON AVERAGE TOUCHING NUMBER OF A VORONOI PARTITION

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Abstract

A Voronoi partition is decided bythe configurations of N centerepoints in n dimensiona Euclidean space. The total number of nearest neighbor points for a given centerpoint in th partition is called its touching number. It is shown that the average touching number for al points in a Voronoi partition is not greater than the n dimensional kissing number, that is, th maximum uumber of unit spheres that can touch a given unit sphere without overlapping.

§1. Introduction

The Voronoi partition[1] in packing and covering theory is widely applied ir fields of computer science, physics, chemistry and so on[2,3,4].

There is a set of finite or enumerable points $\{y_1, y_2, \cdots, y_N, \cdots\}$. A Vor partition $V(y_1, y_2, \cdots, y_N, \cdots)$ is a sequence $\{S_1, S_2, \cdots, S_N, \cdots\}$ of convex polyt covering R^n (n-dimensional Euclidean space). The partition satisfies the follov conditions:

$$\begin{cases} \bigcup_i S_i = R^n, \\ M_n(S_i \cap S_j) = 0 \quad \text{for all } i \neq j, \\ S_i = \{x \mid \|x - y_i\| \leqslant \|x - y_j\|, \quad \text{for all } j \neq i\}, \end{cases}$$

where M_n is the n-dimensional Euclidean measurement, $\|\cdot\|$ is the Euclidean r of a vector. S_i is called a Voronoi region or a Voronoi polytope, y_i is called centerpoint of S_i. Clearly, a Voronoi partition of N centerpoints $V(y_1, y_2, \cdots, y$ decided by the places of the N points.

We define the halfspace H_{ij} by

$$H_{ij} = \{x \mid \|x - y_i\| \leqslant \|x - y_j\|, \ i, \ j = 1, 2, \cdots, N, \ i \neq j\},$$

then

$$S_i = \bigcap_{j \neq i} H_{ij}.$$

Let L_{ij} denote the boundary of H_{ij}, then

$$L_{ij} = \{x \mid \|x - y_i\| = \|x - y_j\|, \ i \neq j\}.$$

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$$M_{n-1}(L_{ij}\cap S_i)>0,$$
hen L_{ij} is called an effective boundary and y_j is called a neighbor point of y_i (or s_i). o, the number of neighbor points of a Voronoi Polytope is also the number of its oundary faces. Klee[5] gave the tightest possible upper bound $K(n,N$ (of the number f the boundary faces for any Voronoi partition with N centerpoints.

$$\begin{cases}K(2,N)=2(3N-6), & N>2,\\ K(n,N)=2\binom{N}{2}, & n\geqslant 4.\end{cases}$$

Vhen $n\geqslant 4$, he proved that there exists a Voronoi partition such that each L_{ij} is an fective one.

For each Voronoi region S_i, the nearest neighbor points of y_i is called touching oints of y_i, whose numder is denoted by m_i, the distance of its touching points from is denoted by d_i. Studying the Voronoi partition for a lattice structure[7] is uivalent to considering the packing with congruent spheres of radius $d_i/2$ about l centerpoints y_i. $m=m_i$ is defined as the touching number (or kissing number) of congruent spheres packing. To seek the upper bound of the number is a significant oblem in packing and covering theory[1]. [6] studied the upper bound τ_n to the aximum number of spheres with nonoverlapping interiors that touch a given here in R^n. They give the known values on known bounds on τ_n, the n-dimensional ssing number for several values of n. Obviously, $m\leqslant\tau_n$. From Table 1 in [6] it easy to know that there are some lattices whose kissing numbers arrive at τ_n when $=1, 2, 3, 8, 24$.

§ 2. Statement of Main Result

This paper concerns an arbitrary Voronoi partition. In general, m_i and d_i may ffer from one centerpoint to another. It is simple to construct a partition in which ch point has only one touching point. For example, in R^1 let

$$y_1=0 \text{ and } y_{k+1}=y_k+1/k \text{ for } k=2,\ 3,\ \cdots,\ N-1.$$

en y_{k+1} is the unique touching point of y_k for $k=1$ 2, $\cdots$, $N-1$ and y_{N-1} is the ique touching point of y_N. It is also possible for a centerpoint to have $N-1$ iching points. (Let Y_0 be the origion and let all remaining $N-1$ points lie on the face of a sphere in R^n). We denote the average touching number of all centerpoints a Voronoi partition by $\overline{m}$ and the tightest lower bound of d_i by d_0. For N enough ge and for a fixed dimension n, it is not possible for all centerpoints to simulta- ously have $N-1$ touching points. We shall see that there is, in fact, an upper bound on the average touching number of a Voronoi partition that is independent of the size N of the partition.

A touching point y_i of a centerpoint y has separation c_i if c_i is the minimum distance between y_i and any other touching point y_j of y. Thus,

$$C_i=\min_{j\neq i}\|y_i-y_j\|.$$

Evidently, $c_i \geqslant d_i$. The following theorem will be proved.

Theorem. *The average touching number of any Voronoi partition in R^n is less than or equal to τ_n.*

This bound appears to be the tightest possible upper bound, for example, wł $n=1$, 2, 3, 8, 24, there exist optimal congruent sphere packings whose touchi numbers are equal to τ_n.

§ 3. Proof of Main Result

Lemma 1. *If a centerpoint y has touching distance d_0 (the minimum distance the Voronoi partition), then its touching number cannot exceed τ_n.*

Proof Each touching point y_i of y cannot have touching distance greater thar since $\|y_i-y\|=d_0$. Place a sphere of radius $d_0/2$ about each touching point y_i of Then each sphere will touch the sphere centered at y of radius $d_0/2$; also, the sphe will not overlap since d_0 is the minimum distance of the Voronoi partition. Therefc the touching numder of y cannot exceed τ_n or the kissing number bound would violated.

Corollary 1. *If y has $d_y=d_0$, the minimum distance of the Voronoi partiti then each touching point y_i of y has at most τ_n touching points.*

Proof Since $d_y=d_0$, we have $d_{y_i}=d_0$, so that Lemma 1 applies to the centerpo y_i.

Lemma 2. *If a centerpoint y has touching number $m_y>\tau_n$, then it is necess that* (a) *$d_y>d_0$ and that* (b) *there exist touching points y_1 and y_2 of y with $\|y_1-y_2\|<$* so that

$$d_{y_1}<d_y \quad \text{and} \quad d_{y_2}<d_y.$$

Proof If $d_y=d_0$, then $m_y\leqslant\tau_n$ from Lemma1. If condition (b) does not hc $m_y>\tau_n$ nonoverlapping spheres of radius $d_y/2$ could be placed about each touch point of y, violating the kissing number bound for the sphere centered at y radius $d_y/2$.

Lemma 3. *Given a centerpoint y with touching distance d_y and with separat $c_i\geqslant d_y$ for all touching points y_i. If y is itself the touching point of some centerpoin with $d_z>d_y$, then $m_y\leqslant\tau_n-1$.*

Proof From Lemma 2, $m_y\leqslant\tau_n$. Suppose that $m_y=\tau_n$. Then it is possible to add a new centerpoint w to the partition given by

$$w=\frac{d_z-d_y}{d_z}y+\frac{d_y}{d_z}z,$$

so that $\|w-y\|=d_y$. By the triangle inequality, for any touching point y_i of y,

$$\|w-y_i\|\geqslant\|y_i-z\|-\|w-z\|\geqslant d_z-(d_z-d_y)=d_y,$$

so that w is a touching point of y and the augmented set of touching points of y has separation at least equal to d_y. Then y has τ_n+1 touching points, violating Lemma 2. Therefore $m_y\leqslant\tau_n-1$.

Note that there was room to add an extra touching point for y because of the "forbidden zone" created by z having touching distance $d_z>d_y$.

Lemma 4. *If a centerpoint y has $m_y>\tau_n$, then at least $m_y-\tau_n+1$ of the touching points of y must have separation $c_i<d_y$.*

Proof If $m_y>\tau_n+1$, there exists a touching point y_i of y with $c_i<d_y$ by Lemma 2(b). Remove this touching point, and repeat until there remain exactly τ_n+1 touching points. Then by Lemma 2(b), there are at least two touching points with touching distance less than d_y. After removing these two points, we have τ_n-1 points remaining out of m_y. Thus we have removed $m_y-\tau_n+1$ points with touching distance less than d_y.

Lemma 5. *If a centerpoint y has k touching points $y_1, y_2, \cdots, y_k$ with $k>\tau_n$ and each y_i has its touching points separated from one another by at least d_i, the touching distance of y_i, then the total touching number of y and its touching points satisfies*

$$m_y+\sum_{i=1}^{K}m_i<(k+1)\tau_n-1,$$

where m_i is the touching number of y_i.

Proof Of the k touching points of y at least $k-\tau_n+1$ points have $d_j<d_y$ by Lemma 4. Since the touching points of y_j have a separation of at least d_j, $m_j\leqslant\tau_n-1$ by Lemma 3.

The remaining τ_n-1 or fewer touching points of y have $m_i\leqslant\tau_n$ by Lemma 2. Thus $m_y=k$ and there are at least $k-\tau_n+1$ points with $m_j\leqslant\tau_n-1$ and at most τ_n-1 points with $m_i\leqslant\tau_n$. Hence

$$m_y+\sum_{i=1}^{K}m_i\leqslant K+(K-\tau_n+1)(\tau_n-1)+(\tau_n-1)\tau_n$$
$$=(K+1)\tau_n-1.$$

The next Lemma is a generalization of Lemma 3.

Lemma 6. *Given a centerpoint y with touching distance d_y and with $\|y_i-y_j\|\geqslant d_y$ for any pair y_i, y_j of its touching points, if y is a touching point of points $z_1, z_2, \cdots, z_k$, with $k<\tau_n$ and $d_{zi}>d_y$ for each i, then*

$$m_y\leqslant\tau_n-k.$$

Proof From Lemma 2, $m_y\leqslant\tau_n$, suppose $m_y>\tau_n-k$. Then we show by construction that it is possible to add k additional points w_i, that are touching points of y

with the property that each w_i is a distance at least d_y from any other touching point of y. These new points lie in the forbidden zone created by the points z_i.

Let
$$w_i=\frac{d_{z_i}-d_y}{d_{z_i}}\,y+\frac{d_y}{d_{z_i}}\,z_i$$

It is easily seen that
$$\|w_i-y\|=\|y_i-y\|=d_y,$$
showing that w_i is a touching point of y. From the triangle inequality, for anv touching point y_k of y,
$$\|w_i-y_k\|\geqslant\|y_k-z_i\|-\|w_i-z_i\|\geqslant d_{z_i}-(d_{z_i}-d_y)=d_y,$$
so that each of the augmented touching points w_i is a distance at least d_y from a of the original touching points y_i of y.

We now show that
$$\|w_i-w_j\|>d_y \quad \text{for } i\neq j.$$
Without loss of generality suppose that
$$\|z_i-y\|\leqslant\|z_j-y\|\leqslant\|z_j-z_i\|.$$
Then
$$2(z_i-y)\cdot(z_j-y)=\|z_i-y\|^2+\|z_j-y\|^2-\|z_i-z_j\|^2\leqslant d_{z_i}^2\leqslant d_{z_i}d_{z_j}.$$
But
$$2(w_i-y)\cdot(w_j-y)=\frac{2d_y^2}{d_{z_i}d_{z_j}}(z_i-y)\cdot(z_j-y)$$
by definition of w_i, w_j.

Thus
$$2(w_i-y)\cdot(w_j-y)\leqslant d_y^2$$
and
$$\|w_i-w_j\|^2=\|w_i-y\|^2+\|w_j-y\|^2-2(w_i-y)\cdot(w_j-y)\geqslant d_y^2+d_y^2-d_y^2=d_y^2.$$
Hence by construction we now have a touching number for y that exceeds τ_n, wh contradicts Lemma 2. Therefore the original touching number, m_y, of y must sati $m_y\leqslant\tau_n-k$.

Note that the hypotheses of Lemma 6 cannot be satisfied for $k=\tau_n$, for otherw we would conclude that $m_y=0$ and hence the touching distance d_y of y wouldbe d $\min_i d_{y_i}$. This leads to the following:

Given a Voronoi partition, we define a directed graph associated with partition in the following manner. Let each vertex correspond to a particu centerpoint. A directed edge goes from vertex y to vertex z if z is a touching po of y. (If y is also a touching point of z, then there is another directed edge fror to y). Given the N by N distance matrix for the partition, a unique grapl determined.

Proof of the Theorem Constructa subgraph of the graph of the Voro partition as follows. Pick any vertex y which has more than τ_n outgoing ec (touching number greater than τ_n). Include in the subgraph those vertices y_i which are touching points of y with separation less than d_y. We call these points the descendants of y. By. Lemma 4, the number of descendants is at least $m_y-\tau_n+1$, where m_y is the touching number of y. We continue the construction by adding for each descendant y_i all touching points y_{ij} whose separation is less than d_{y_i}. The graph then includes all edges from each y_i to the newly selected descendants y_{ij}. A vertex z fo the graph J is an endpoint if all touching points of z have separation greater than or equal to touching distance of z. In this case the touching number of z is at most τ_n-1 by Lemma 3.

First we focus our attention on a special case denoted by case A in brief. That is, each centerpoint in the subgraph of the case A is a descendant of the only centerpoint. Since each descendant must have touching distance less than its parent, it follows that descendant cannot be a centerpoint that already exists in the subgraph. Thus the subgraph is a tree all of whose edges are directed toward descendants.

If y is the touching point of some point z and y has separation less than d_z, then the graph can be extended "upwards". In other words there exists a larger tree containing y as an internal vertex. A tree is complete if it cannot be extended upwards or downwards (all of its roots are endpoints).

We now show that the average touching number of a tree in the case A is less than τ_n. Let M denote the number of vertices and E denote the number of endpoints in the tree. Consider a point P_i of the tree that is not an endpoint. The number of descending edges of P_i, l_i, is at least $m_{p_i}-\tau_n+1$.

Hence
$$m_{p_i}\leqslant l_i+\tau_n-1.$$

Summing up all $M-E$ points P_i, except the E endpoints, the total touching number of these points is
$$M_1\leqslant (M-E)\tau_n+M-1-(M-E)$$
since the total number of descending edges in a graph is equal to $M-1$. Since each endpoint has touching number at most τ_n-1, the total touching number of the E endpoints is
$$M_2\leqslant E\tau_n-E.$$
Thus
$$M_1\pm M_2\leqslant M\tau_n-1.$$
So the average touching number of a tree is less than τ_n.

Next we note that any centerpoint q in the graph which is not a member of any tree that can be constructed in the above procedure must have touching number less than τ_n since its touching points have separation greater than d_q by Lemma 2.

Besides the case A to complete the proof, it remains to consider the possibility that a given centerpoint is a member of more than one tree or one subtree.

If z is an internal point or an endpoint of k distinct subtrees of the partition, then z is the touching point of k centerpoints $y_1, y_2, \cdots, y_k$ and z hasseparation less than the minimum of the touching distances of each y_i. The k subtrees are contained in one tree or more. If z is an endpoint, its touching number is at most τ_n-k by

Lemma 6. If z is an internal point, the number of edges descending from z is at least $m_s - \tau_n + k$. If the number of vertices in the subtree of z and its descendants is l, the sum of the touching numbers of these l points is $l\tau_n - k$ by the same reasoning used above. Hence we can consider z equivalent to k separate subtrees starting from z, each subtree having l points. The total touching number of each subtree is $l\tau_n - 1$ so that the average touching number for the trees including the k subtrees is less than τ_n.

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