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A GENERAL PROPERTY OF THE QUADRATIC DIFFERENTIAL SYSTEMS

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Abstract

In this paper, it is proved that the quadratic differential systems with a weak saddle of order 2 or 3 have no closed or singular closed orbit. Then by the results of [3], it follows that the greatest order of the homoclinic loop bifurcation of a quadratic differential system is between 2 and 3 It means a homoclinic loop can be split into at most three limit cycles.

Professor Ye Yanqian conjectures that no quadratic differential system has a homoclinic loop (simply denoted by HLB) through a weak saddle of order 3. Joyal's result (see Theorem 1) makes us confirm this conjecture even more.

In this paper, we prove that a quadratic differential system with a weak saddle of order 2 or 3 has not any closed or singular closed orbits. This conclusion is stronger than what has been expected.

Consider the real quadratic differential systems with a weak saddle

$$\begin{aligned} \dot{x} &= x + a_1x^2 + a_2xy + a_3y^2, \\ \dot{y} &= -y + b_1y^2 + b_2xy + b_3x^2. \end{aligned} \tag{1}$$

Through a polynomial transformation

$$\begin{aligned} x &= u + \sum_{2\le i+j\le r} a_{ij}u^iv^j, \\ y &= v + \sum_{2\le i+j\le r} b_{ij}v^iu^j, \end{aligned} \tag{2}$$

system (1) can become

$$\begin{aligned} \dot{u} &= u + \sum_{i=1}^{k} c_iu^{i+1}v^i + h.o.t., \\ \dot{v} &= -v + \sum_{i=1}^{k} d_iv^{i+1}u^i + h.o.t.. \end{aligned} \tag{3}$$

Denote $R_0=0$, $R_i=k_i(c_i+d_i)$, $i=1, 2, \cdots, k$, where k_i 's are suitable positive constants.

Definition. *If $R_1=R_2=\cdots=R_{j-1}=0$, $R_j\ne 0$, then the origin O is called a weak saddle of order j of the system (1), and R_j the jth saddle value. An HLB through a weak saddle of order j is of order j if $\bar{R}_0=\cdots=\bar{R}_{j-1}=0$, $\bar{R}_j\ne 0$, where $\bar{R}_{2k+1}=R_k$, $\bar{R}_{2k}$ (see [1]).*

For convenience, we first formulate some main results obtained in [1, 2, 3].

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Theorem 1[1]. *If system (1) has an HLB of order k, then any perturbation of (1) has at most k limit cycles and, for any $i \leqslant k$, there exists a perturbation with exactly i limit cycles.*

Theorem 2[2].

$$R_1 = b_1 b_2 - a_1 a_2,$$
$$R_2 = a_2 b_3 (2a_2 - b_1)(a_2 + 2b_1) - a_3 b_2 (2b_2 - a_1)(b_2 + 2a_1),$$
$$R_3 = (a_1 b_1 + a_2 b_2 - 5a_3 b_3)(a_2 b_3 (4b_1^2 - a_2^2) - a_3 b_2 (4a_1^2 - b_2^2)).$$

When $R_1 = R_2 = R_3 = 0$, the system (1) is integrable. Moreover, when the variab and the cofficients in system (1) are complex, iR_1, iR_2, iR_3 are three the first fo values of the singular point O.

In the paper [3], we easily apply the Treorem 2 to prove that the quadra differential system with a weak saddle of order no less than 2 cannot possess we foci, or other weak saddles, or degenerate singular points with two zero eigenvalu

In this paper, we apply the Theorem 2 to prove another general property of quadratic differential systems.

Theorem 3. *The quadratic differential systems with a weak saddle of order 2 3 have not any closed or singular closed orbits.*

Then by the results of [3] about the existence of the HLB of order 2, we ha the following corollary.

Corollary. *The greatest order of the* HLB *of the quadratic differential syste is between* 2 *and* 3.

By Theorem 1, we see that an HLB possessed by a quadratic differential syst can be split into at most three limit cycles.

Before proving Theorem 3, we show four propositions. As we will see, the resu contained in these propositions are more than those in Theorem 3.

Proposition 1. *Suppose* $a \neq b$, $d \neq 2$. *Then the system*

$$\dot{x} = x + x^2 + dxy + by^2,$$
$$\dot{y} = -y - y^2 - dxy - ax^2.$$

has no closed or singular closed orbit.

Proof Under the transformation

$$u = x + y, \quad v = x - y,$$

system (4) becomes

$$\dot{u} = v + \frac{1}{4}(b-a)u^2 + \frac{1}{2}(2-a-b)uv + \frac{1}{4}(b-a)v^2 = P,$$
$$\dot{v} = u + \frac{1}{4}(2+a+b+2d)u^2 + \frac{1}{2}(a-b)uv + \frac{1}{4}(2+a+b-2d)v^2 = Q. \quad (5)$$

Then we consider the comparison system

$$\dot{u}=v+\frac{1}{2}(2-a-b)uv=P_0,$$
$$\dot{v}=u+\frac{1}{4}(2+a+b+2d)u^2+\frac{1}{4}(2+a+b-2d)v^2=Q_0. \tag{7}$$

It is easy to see that the orbits of system (7) are symmetric with respect to the U-axis, and system (7) can be obtained if we substitute $(a+b)1/2$ for a and b in (4) and then use the transfor mation (5). Thus we see that, for system (7), $R_1=R_2=R_3=0$. By Theorem 2, the system is integrable.

The divergence of the system (6) is div $(P, Q)=(2-d)v$, so the closed or singular closed orbit of (6), if it exists, must intersect the U-axis.

If $a+b+2d+2=0$, then the U-axis (cut out the origin) is transversal to the vector field, and the separatrices through saddle 0 are tangent to the lines $u=\pm v$. So the systems (6) and (7) have not any closed or singular closed orbits.

Now we prove the proposition in case $a+b+2d+2<0$. The proof in case $a+b+2d+2>0$ is similar

Without loss of generality, we may assume $b>a$. In fact, if $b<a$, we only need to use a transformation

$$(u, v, t)\mapsto(u, -v, -t).$$

If we notice that: i) both line s $u=\pm v$ are transverses (cut out the origin) when $ab\neq 0$; ii) one of the lines $u=\pm v$ is an integral line and the other consists of two half lines without contact when $ab=0$; iii) the negative U-axis is a transverse, then we see that we only need to prove the proposition in the sector: $u>0$, $-1<u^{-1}v<1$.

Suppose the system (6) has a closed or singular closed orbit T. Then we show that this will lead to a contradiction.

Denote $P^*=P-P_0=\frac{b-a}{4}(u^2+v^2)$, $Q^*=Q-Q_0=\frac{1}{2}(a-b)uv$.

Let $T_1(T_2)$ be the part of T which is situated above (below) the U-axis, $S_1(S_2)$ be the region bounded by $T_1(T_2)$ and the U-axis, A and B (the abscissa $u_A<u_B$) be two intersection points of T with the U-axis, $M(N)$ be the leftest (rightest) point of T, $S_1'(T_1')$ be the symmetric image of $S_1(T_1)$ with respect to the U-axis. If T is a singular closed orbit, then we have $M=A$.

Now we prove

$$S_1'\subset S_2. \tag{8}$$

Suppose the inclusion relation (8) is not true.

Then, i) if A is not the origin, by the relations

$$P|_{A,B}=\frac{1}{4}(b-a)u^2>0,\ Q|_A>0 \text{ and } Q|_B<0,$$

we see that T_1' and T_2 must have at least two intersection points C and D which are different from A and B;

ii) if A is the origin, by the relations

$$vP_0>0,\ Q_0>|Q^*|,\ |P_0|>P^*>0 \text{ when } u>0 \text{ small}$$

and

$$|v|\leqslant u, \tag{9}$$

$$P^*>0,\quad vQ^*<0,$$

we see, when u is small and fixed, the following inequality keeps true

$$\left|\frac{Q}{P}\right|_{T_2}>\left|\frac{Q_0}{P_0}\right|>\left|\frac{Q}{P}\right|_{T_1}.$$

So T_2 is situated below T_1' when u is small enough. Thus T_1' and T_2 also hav least two intersection points C and D which are different from A and B.

It is easy to see that C, D can only be situated on the orbit segment NM.

We use E, F to denote the lowest and the highest points of T respectively.

Consider the symmetric vector field of system (6) with respect to the U-axi

$$\dot{u}=P'$$
$$\dot{v}=Q'$$

where $$P'=-v+\frac{1}{4}(b-a)u^2-\frac{1}{2}(2-a-b)uv+\frac{1}{4}(b-a)v^2,$$

$$Q'=-u-\frac{1}{4}(2+a+b+2d)u^2+\frac{1}{2}(a-b)uv-\frac{1}{4}(2+a+b-2d)v^2.$$

Since T_1' and T_2 have at least four intersection points A, B, C and D, and C are situated on NM, we see that the systems (6) and (11) have at least two tang points I_1, I_2 on the orbit segment BM.

At the point I_i, we must have

$$\frac{Q_0+Q^*}{P_0+P^*}=\frac{Q}{P}=\frac{Q'}{P'}=\frac{-Q_0+Q^*}{-P_0+P^*}.$$

By the properties of the fraction, we get

$$\left.\frac{Q}{P}\right|_{I_i}=\left.\frac{Q^*}{P^*}\right|_{I_i}.$$

From $(P^*)^{-1}Q^*>0$ on T_2, it follows that $P^{-1}Q>0$ at I_i. Thus I_i can only situated on the segmental orbit NE.

Let $k=v/u$. Then

$$\frac{Q^*}{P^*}=-\frac{2uv}{u^2+v^2}=-\frac{2k}{1+k^2}.$$

It is easy to see that $k(1+k^2)^{-1}$ is a strictly increasing function of k when <1. From the convexity of T, it follows that Q/P strictly decreases (from $+\infty$ along the segmental orbit NE, and NE has at most one intersection point w the line $v=ku$. Hence Q^*/P^* strictly increases along NE. Thus there exactly ex one point on NE such that the equality (12) holds.

Now we see T_1' and T_2 cannot have transversal intersection points besides A and B. It means (8) is valid.

Then by
$$0=\int_T Pdv-Qdu=\iint_{S_1\cup S_2}\operatorname{div}(P,\ Q)\,du\,dv$$
$$=\iint_{S_1\cup S_2}(2-d)\,v\,du\,dv\neq 0,$$
the propoisiton follows immediately.

Remark 1. When $a=b$, or $d=0$, or $d=2$, we have $R_1=R_2=R_3=0$, hence the system (4) is integrable. When $d=-1/2$, $ab=1/4$, we also have $R_1=R_2=R_3=0$, i. e., the system (4) is integrable. It is easy to prove that, in the last case, there exists a quadratic curve solution not through the origin
$$4+4x+4y+ax^2+2xy+by^2=0,$$
and a cubic curve solution
$$16+24(x+y)+6((a+1)x^2+(2+a+b)xy+(b+1)y^2)+a(a+1)x^3$$
$$+3(a+1)x^2y+3(b+1)xy^2+b(b+1)y^3=0.$$

Proposition 2. *If $b\neq 2$, then the system*
$$\begin{aligned}\dot{x}&=x+x^2\pm y^2=P,\\ \dot{y}&=-y-ax^2-bxy=Q\end{aligned}\tag{13}$$
has not any closed or singular closed orbits.

Proof.

Since the divergence $\operatorname{div}(P,\ Q)=(2-b)x$, neither closed nor singular closed orbit can be entirely situated in the left half-plane

Then if we notice that we always have $P>0$ on the Y-axis (except the origin) and the coordinate axes are the tangent lines of the separatrices through saddle 0, the proposition follows.

Remark 2. We see $R_2=\pm b(b-2)(2b+1)$ and $R_3=0$ when $b=0$ or $b=2$; $R_3=-75a/8$ when $b=-1/2$. So if $b=-1/2$ and $a=0$, the system (13) is integrable, and has a straight line solution $y=0$ and only two singular points (0, 0), (−1, 0).

Proposition 3. *If $a\neq b$, then the system*
$$\begin{aligned}\dot{x}&=x+xy+by^2,\\ \dot{y}&=-y-xy-ax^2\end{aligned}\tag{14}$$
has not any closed or singular closed orbits.

Proof Through the transformation (5), the system (14) becomes
$$\begin{aligned}\dot{u}&=v+\frac{1}{4}(b-a)u^2-\frac{1}{2}(a+b)uv+\frac{1}{4}(b-a)v^2=P,\\ \dot{v}&=u+\frac{1}{4}(2+a+b)u^2+\frac{1}{2}(a-b)uv+\frac{1}{4}(a+b-2)v^2=Q.\end{aligned}\tag{15}$$
Its divergence is $\operatorname{div}(P,\ Q)=-v$.

Now it is easy to see that the proof of the proposition can be carried on in the same manner as that of the Proposition 1.

Proposition 4. *If $b\neq 0$, then the system*

$$\dot{x}=x+by^2=P,$$
$$\dot{y}=-y-xy-ax^2=Q. \qquad (16)$$

has not any closed or singular closed orbits.

Proof If we notice that: i) $\operatorname{div}(P, Q)=-x$, ii) the Y-axis consists of two half lines without contact, then we can prove the proposition similary to the proof of Proposition 2.

Proof of Theorem 3 Through a scale transformation $x=hu$, $y=kv$, the sys (1) becomes

$$\dot{u}=u+a_1hu^2+a_2kuv+a_3h^{-1}k^2v^2,$$
$$\dot{v}=-v+b_1kv^2+b_2huv+b_3h^2k^{-1}u^2. \qquad ($$

i) If $a_1b_1\neq0$, we take $h=a_1^{-1}$, $k=-b_1^{-1}$. Then $a_1h=-b_1k=1$, and $a_2k=-b_2h$ $R_1=b_1b_2-a_1a_2=0$.

So we may as well assume $a_1=-b_1=1, a_2=-b_2=d$, $a_3=b$ and $b_3=-a$. Now system (1) has the form of (4), and $R_2=d(d-2)(2d+1)(b-a)$.

When $R_2\neq0$, we have $a\neq b$, $d\neq0$, 2 and $-1/2$.

Since $R_3=0$ when $a=b$, or $d=0$, or $d=2$, and $R_3=75(b-a)(4a_3b_3+1)/32$ w $d=-1/2$, we see that we should have $d=-1/2$, $a\neq b$ and $a_3b_3\neq-1/4$ if $R_1=R_2$ $R_3\neq0$.

ii) if $b_1=0$, $a_1\neq0$, then $a_2=0$ by $R_1=0$, and we may assume $a_1=1$ by (17). In this case, the system (1) becomes

$$\dot{x}=x+x^2+a_3y^2,$$
$$\dot{y}=-y+b_2xy+b_3x^2. \qquad ($$

a) When $a_3=0$, we get $R_2=R_3=0$, and the system (18) is integrable.

b) When $a_3\neq0$, we may assume $a_3=1(\text{or}-1)$ when $a_1a_3>0(\text{or}<0)$.

Now $$R_2=\mp b_2(b_2+2)(2b_2-1).$$

If $b_2=0$ or $b_2=-2$, then $R_2=R_3=0$, the system (18) is integrable.

If $b_2=1/2$, then $R_2=0$, $R_3=75b_3/8$.

Thus, when $b_1=0$, $a_1\neq0$, $R_1=0$, $R_2\neq0$ (or $R_1=R_2=0$, $R_3\neq0$), the system can always be ritten in the following form

$$\dot{x}=x+x^2\pm y^2$$
$$\dot{y}=-y-bxy-ax^2, \qquad ($$

where $b\neq0$, 2, $-1/2$(or $b=-1/2$, $a\neq0$).

iii) If $a_1=0$, $b_1\neq0$, we only need to change (x, y, t) into $(y, x, -t)$. Then will return to the case ii).

iv) If $a_1=b_1=0$, then $R_2=2(a_2^3b_3-a_3b_2^3)$.

a) When $a_2b_2\neq0$, we may assume $a_2=-b_2=1$.

Now $R_2=0$ is equivalent to $a_3=-b_3$. Thus $R_2=0$ infers $R_3=0$ and the integrability of the system (1).

b) When $a_2=0$, $b_2\neq 0$, we take $b_2=-1$.

Then $R_2=0$ means $a_3=0$, and it follows that $R_3=0$.

c) When $a_2\neq 0$, $b_2=0$, similar to the case b), $R_2=0$ implies $R_3=0$.

d) When $a_2=b_2=0$, we easily see $R_2=R_3=0$.

Thus, when $a_1=b_1=0$, the system (1) cannot have a weak saddle of order 3 (by 3], the sum of the orders of two weak saddles is 2). And when 0 is a weak saddle f order 2, we must have the case a) or b) (case c)). In the case a), the system (1) ıay be transformed into (14), where $a\neq b$. In the case b), the system (1) may be ewritten into (16), where $b\neq 0$.

From the discussion above, we see, the quadratic differential systems with a eak saddle of order 2 or 3 can always be transformed into system (4) (where $d\neq 0$, $a\neq b$, moreover, $a\cdot b\neq 1/4$ when $d=-1/2$), (19), (14) or (16). Then Theorem 3 ollows immediately form Propositions 1—4.

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