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ON THE TAYLOR'S JOINT SPECTRUM OF 2n-TUPLE (L_A, R_B)

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Abstract

Let $A=(A_1, \cdots, A_n)$ and $B=(B_1, \cdots, B_n)$ be double commuting n-tuples of operator: on Hilbert space H and let L_{A_i} and R_{B_j} denote the left and right multiplications induced b A_i and B_j, respectively. The following results are proven: $Sp(L_A, R_B)=Sp(A)\times Sp(B)$; $Sp_e(L_A, R_B)=Sp_e(A)\times Sp(B)\cup Sp(A)\times Sp_e(B)$.

In the operator system theory, people take more and more interest in elemen operators. Let $A=(A_1, \cdots, A_n)$ and $B=(B_1, \cdots, B_n)$ be commuting n-tuple operators on a Hilbert space H. Corresponding to A and B, there is an elem-en operator Δ on $B(H)$ given by $\Delta(X)=\sum_{i=1}^{n} A_i X B_i$. Let L_{A_i} and R_{B_j} denote the left right multiplications induced by A_i and B_j, respectively. Then $\Delta=\sum_{i=1}^{n} L_{A_i}R_{B_i}$ $(L_{A_1}, \cdots, L_{A_n}, R_{B_1}, \cdots R_{B_n})$ is commutative. It has been proved that, for the elen tary operator Δ, the following is true:

$$\sigma(\Delta)=Sp(A)\circ Sp(B),$$
$$\sigma_e(\Delta)=Sp(A)\circ Sp_e(B)\cup Sp_e(A)\circ Sp(B).$$

Obviously, the above results can be concluded from the fovowing conjecture:

$$Sp(L_A, R_B)=Lp(A)\times Sp(B),$$
$$Sp_e(L_A, R_B)=Sp(A)\times Sp_e(B)\cup Sp_e(A)\times Sp(B).$$

Many people have been striving to prove the conjectrue. The main aim of paper is trying to solve it. We shall show that if $A=(A_1, \cdots, A_n)$ and $B=(B_1,$ $B_n)$ are doubly commuting n-tuples of operators on a Hilbert space H, then the expression of the Taylor's joint spectrum of $2n$-tuple (L_A, R_B) is true. Let $A=($ $\cdots, A_n)$ be a commuting n-tuple of operators on a Hilbert space H. We define σ_* $=\{(\lambda_1, \cdots, \lambda_n) \mid$ there exists a sequence $\{f_m\}$ in H with $\|f_m\|=1$ such that $(A$ $\lambda_{i_k})f_m\to 0$ as $m\to\infty$ for $k=1, \cdots, j$ and $(A_{l_t}-\lambda_{l_t})^* f_m\to 0$ as $m\to\infty$ for all $l_t\in\{1,$ $n\}\setminus\{i_1, \cdots, i_j\}\}$. From [2], we know that if $A=(A_1, \cdots, A_n)$ is doubly commutat then $Sp(A)=\sigma_*(A)$.

Theorem 1. *Let $A=(A_1, \cdots, A_n)$ and $B=(B_1, \cdots, B_n)$ be two commuting n-*

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les of operators on a Hilbert space H. Then we have

$$\sigma_*(A)\times\sigma_*(B)\subset Sp(L_A, R_B)\subset Sp(A)\times Sp(B).$$

Proof The inclusion on the right hand is obvious. We only need to prove the clusion on the left hand. Next, we will divide the proof into three steps.

1° If there exist two elements f and g of H with $\|f\|=1$ and $\|g\|=1$ such that

$$A_{i_1}f=0,\ \cdots,\ A_{i_j}=0,\ A^*_{k_1}f=0,\ \cdots,\ A^*_{k_{j'}}f=0$$

d

$$B_{l_1}g=0,\ \cdots,\ B_{l_s}g=0,\ B^*_{t_1}g=0,\ \cdots,\ B^*_{t_{s'}}g=0,$$

ere $\{i_1, \cdots, i_j\}=\{1, \cdots, n\}\setminus\{k_1, \cdots, i_{j'}\}$ and $\{l_1, \cdots, l_s\}=\{1, \cdots, n\}\setminus\{t_1, \cdots, t_{s'}\}$, en the koszul complex associated with (L_A, R_B) is not exact at the $\binom{2n}{j+s'}$th stage $(H)^{\binom{2n}{s+j'}}$. In fact, if we let $X=(\cdot, g)f$ and $\xi=e_{i_1}\wedge\cdots\wedge e_{i_j}\wedge\sigma_{t_1}\wedge\cdots\wedge\sigma_{t_{s'}}$ and denote $X\otimes\xi\in B(H)^{\binom{2n}{s+j'}}$; then we have

$$d_{j+s'}Z_m\to 0 \text{ as } m\to\infty.$$

Next, we will prove that if Ran $(d_{j+s'+1})$ is closd, then

$$\varliminf_{m\to\infty}\rho(Z_m, \mathrm{Ran}(d_{j+s'+1}))\geqslant 1, \tag{1}$$

ere $\rho(Z_m, \mathrm{Ran}(d_{j+s'+1}))$ expresses the distance between Z_m and Ran $(d_{j+s'+1})$. If t, we can assume

$$\lim_{m\to\infty}\rho(Z_m, \mathrm{Ran}(d_{j+s'+1}))=l<1. \tag{2}$$

en there exists a sequence $\{U_m\}$ in $B(H)^{\binom{2n}{j+s'+1}}$,

$$U_m=Y^{(m)}_1e_{k_1}\wedge\xi+\cdots+Y^{(m)}_{j'}e_{k_{j'}}\wedge\xi+V^{(m)}_1\sigma_1\wedge\xi+\cdots+V^{(n)}_s\sigma_{l_s}\wedge\xi+U'_m,$$

here U'_m does not contain the factor ξ, such that

$$\rho(Z_m, d_{j+s'+1}U_m)\leqslant\|Z_m-d_{j+s'1}u_m\|\leqslant\rho(Z_m, \mathrm{Ran}(d_{j+s'+1}))+m^{-1}.$$

rom (2), we get $\lim_{m\to\infty}\|Z_m-d_{j+s'+1}u_m\|=l$. And we can choose u_m such that $\|u_m\|\leqslant$ $\gamma(d_{j+s'+1})^{-1}\|d_{j+s'+1}\ u_m\|\leqslant 5\gamma(d_{j+s'+1})^{-1}$ if m is large enough (since $\mathrm{Ran}(d_{j+s'+1})$ is osed and $\|Z_m\|=1$), so $\|Y^{(m)}_j\|\leqslant 5r(d_{j+s'+1})^{-1}$ and $\|V^{(m)}_j\|\leqslant 5r(d_{j+s'+1})^{-1}$. But, on the her hand, we have

$$\begin{aligned}\varliminf\|Z_m-d_{j+s'+1}u_m\|&\geqslant\varliminf_{m\to\infty}\|X_m-A_{k_1}Y^{(m)}_1-\cdots-A_{kj'}Y^{(m)}_{j'}-V^{(m)}_1B_{l_1}-\cdots-V^{(m)}_sB_{l_s}\|\\&\geqslant\varliminf_{m\to\infty}|((X_m-A_{k_1}Y^{(m)}_1-\cdots-A_{F_{j'}}Y^{(m)}_{j'}V^{(m)}_1B_{1l}-\cdots-V^{(m)}_sB_{l_s})g_m,f_m)|\\&=1,\end{aligned}$$

en we have

$$d_{j+s'}Z=0.$$

Next, we will prove $Z\overline{\in}\mathrm{Ran}(d_{j+s'+1})$ (therefore the complex is not exact at $B(H)^{\binom{2n}{j+s'}}$).

Assume not, i. e., there exists an element u in $B(H)^{\binom{2n}{j+s'}}$,

$$u=Y_1e_{k_1}\wedge\xi+\cdots+Y_{j'}e_{k_{j'}}\wedge\xi+V_1\sigma_{l_1}\wedge\xi+\cdots+V_s\sigma_{l_s}\wedge\xi+u',$$

where u' does not contain the factor ξ, such that

$$Z = d_{j+s'+1}u.$$

In this case, it must be

$$A_{k_1}Y_1 + \cdots + A_{k_{j'}}Y_{j'} + V_1B_{l_1} + \cdots + V_sB_{l_s} = X.$$

On the other hand, we have $(X_g, f) = 1$ but

$$((A_{k_1}Y_1 + \cdots + A_{k_{j'}}Y_{j'} + V_1B_{l_1} + \cdots + V_sB_{l_s})g, f) = 0.$$

This contradiction proves the above equality.

2° If there exist two sequences $\{f_m\}$ and $|g_m\}$ in H with $\|f_m\| = \|g_m\| =$ at least, one is uncompact such that

$$A_{i_1}f_m \to 0, \cdots, A_{i_j}f_m \to 0, A^*_{k_1}f_m \to 0, \cdots, A^*_{k_{j'}}f_m \to 0$$

and $$B_{l_1}g_m \to 0, \cdots, B_{l_s}g_m \to 0, B^*_{t_1}g_m \to 0, \cdots, B^*_{t_{s'}}g_m \to 0,$$

then $\mathrm{Ran}(dj+s')$ is not closed if the complex is exact at $B(H)^{\binom{2n}{j+s'}}$.

Let us prove it. If let $X_m = (\cdot, g_m)f_m$, then $\|X_m\| = 1$ and $\{X_m\}$ is noncoı

Let $Z_m = X_m \otimes \xi \in B(H)^{\binom{2n}{j+s'}}$ This contradicts (2).

From (1), we can see that if $\mathrm{Ran}(d_{j+s'+1}) = \mathrm{Ker}(d_{j+s'})$ and $s+j'>0$, then

$$\varliminf_{m\to\infty} \rho(Z_m, \mathrm{Ker}(d_{j+s'})) > 1.$$

Hence $\mathrm{Ran}(d_{j+s'})$ is not closed for $\{Z_m\}$ is noncompact. If $s'+j=0$, then Raı $B(H)$. (that is, d_1 is not onto).

3° If both $\{f_m\}$ and $\{g_m\}$ in case 2° are compact, then we can find f anc H which satisfy the hypothesis of case 1°. Therefore we can change the cas case 1°

From 1°, 2° and 3° we get the inclusion

$$\sigma_*(A) \times \sigma_*(B) \subset Sp(L_A, \ \ ).$$

Coroloary 2. *For commuting n-tuple $A = (A_1, \cdots, A_n)$, the following is t*

$$\sigma_*(A) \subset Sp(A).$$

Corollary 3. *Let $A = (A_1, \cdots, A_n)$ and $B = (B_1, \cdots, B_n)$ be two doubly com n-tuples of operators on H. Then we have*

$$Sp(L_A, R_B) = Sp(A) \times Sp(B).$$

Corollary 4. *If A and B are two operators in $B(H)$, then $Sp(L_A, R_B) = \sigma$ $\sigma(B)$.*

To study the essential spectrum $Sp_e(L_A, R_B)$, we, at first, introduce the spe $\sigma_{*e}(A)$. Let $A = (A_1, \cdots, A_n) \in B(H)^n$ de a commuting n-tuple on H. We $\sigma_{*e}(A) = \{(\lambda_1, \cdots, \lambda_n) \mid$ there exists an uncompact sequence $\{f_m\}$ in H with $\|f$ for all m such that $(A_{i_k} - \lambda_{i_k})f_m \to 0$ for $k=1, \cdots, j$ and $(A_{l_t} - \lambda_{l_t})^*f_m \to 0$ for al $\{1, \cdots, n\} \setminus \{i_1, \cdots, i_j\}\}$.

For the needs in the sequel, we give a lemma.

Lemma 5. *Let A be a bounded linear opeoator from a Banach space E into another* *ach space F. If there exist an uncompact sequence $\{f_m\}\subset E$ with $\|f_m\|=1$ and a te dimensional subspace $N\subset \text{Ker } A$, $\text{Ker } A=N+M$, such that $\rho(f_m, M)\geq 1$ and $\to 0$, then the range of A is not closed.*

The proof of the lemma can be easily got by the properties of finite dimensional space. Bo means of this lemma and the same discussion as in the proof of Theorem ve can obtain the following analogous result.

Theorem 6. *Let* $A=(A_1, \cdots, A_n)$ and $B=(B_1, \cdots, B_n)$ *be commuting n-tuples* H. *Then we have*

$$\sigma_{*e}(A)\times\sigma_*(B)\cup\sigma_*(A)\times\sigma_{*e}(B)\subset Sp_e(L_A, R_B).$$

Proof This proof is similar to that of Theorem 1. First, let either

$$\bigcap_{u=1}^{j}\text{Ker}A_{i_u}\cap\bigcap_{r=1}^{'j}\text{Ker}A^*_{k_r}$$

$$\bigcap_{u=1}^{s}\text{Ker}B_{l_u}\cap\bigcap_{r=1}^{s'}\text{Ker}B^*_{t_r}$$

infinitedimensional and the other be not $\{0\}$. Then, at $B(H)^{\binom{2n}{j+s'}}$, $\text{Ker }(d_{j+s'})/$ $\text{n}(d_{j+s'+1})$ is infinte dimensional. The other case, $\text{Ker}(d_{j+s'})/\text{Ran}(d_{j+s'+1})$ is finite iensional, and then $\text{Ran}(d_{j+s'+1})$ is closed. Consequently, by Lemma 5 and the quality (1) in the proof of Theorem 1, we can conclude that $\text{Ran}(d_{j+s'})$ is not ;ed.

Corollary 7. *For doubly commuting n-tuples* $A=(A_1, \cdots A_n)$ *and* $B=(B_1, \cdots,$, we have the inclusion

$$Sp(A)\times Sp_e(B)\cup Sp_e(A)\times Sp(B)\subset Sp_e(L_A, R_B).$$

For a commuting n-tuple $A=(A_1, \cdots, A_n)$ on a Banach space E, we can define $(A)=\{(\lambda_1, \cdots, \lambda_n)\,|\,$there exist $\{f_m\}\subset E$ and $\{f^*_m\}\subset E^*$ satisfying $\|f_m\|=\|f^*_m\|=1$ d $f^*_m(f_m)\geq\delta$ for some constant $\delta>0$ such that $(A_{i_k}-\lambda_{i_k})f_m\to 0$ for $k=1, \cdots, j$ and $(A_{l_t}-\lambda_{l_t})^*f^*_m\to 0$ for all $l_t\in\{1,\cdots, n\}\setminus\{i_1, \cdots, i_j\}\}$. Then we have the follow result.

Theorem 8. *Let* $A=(A_1, \cdots, A_n)$ *and* $B=(B_1, \cdots, B_n)$ *be commuting n-tuples* E. *Then* $\sigma_*(A)\times\sigma_*(B)\subset Sp(L_A, R_B)$.

Proof If there exist $\{f_m\}$, $\{f^*_m\}$, $\{g_m\}$, $\{g^*_m\}$ and $\delta>0$ satisfying $\|f_m\|=\|f^*_m\|=$ $_m\|=\|g^*_m\|=1$, $f^*_m(f_m)\geq\delta$ and $g^*_m(g_m)\geq\delta$

such that

$$A_{i_1}f_m\to 0, \cdots, A_{i_j}f_m\to 0, A^*_{k_1}f^*_m\to 0, \cdots, A^*_{k_{j'}}f^*_m\to 0$$

d

$$B_{l_1}g_m\to 0, \cdots, B_{l_s}g_m\to 0, B^*_{t_1}g^*_m\to 0, \cdots, B^*_{t_{s'}}g^*_m\to 0,$$

en either the complex associated with (L_A, R_B) is not exact at $B(E)^{\binom{2n}{j+s'}}$ or Ran $_{j+s'})$ is not closed. In fact, if let $X_m=g^*_m(\cdot)f_m\in B(H)$, $\xi=e_{i_1}\wedge\cdots\wedge e_{i_j}\wedge\sigma_{t_1}\wedge\cdots\wedge\sigma_{t'_s}$ and $Z_m=X_m\otimes\xi\in B(E)^{\binom{2n}{j+s'}}$, then $d_{j+s'}Z_m\to 0$.

By the same discussion as in Theorem 1, and notice that $f^*(X_m, g_m)\geq\delta^2$ it follows that if $\text{Ran}(d_{j+s'+1})$ is closed, then

$$\varlimsup_{m\to\infty} \rho(Z_m, \text{ Ran } (d_{j+s+1})) \geqslant \delta^2.$$

If $\text{Ran}(d_{j+s'+1}) = \text{Ker}(d_{j+s'})$, then $R(d_{j+s'})$ is surely not closed. Now let us prove it. Assume it is false, that is, $\text{Ran}(d_{j+s'})$ is closed. By Open mapping Theorem, there must be $U_m \in \text{Ker}(d_{j+s'})$ such that

$$\|Z_m - U_m\| \to 0, \quad \text{as} \quad m\to\infty. \tag{3}$$

But, on the other hand, we have

$$\varlimsup_{m\to\infty} \|Z_m - U_m\| \geqslant \varlimsup_{m\to\infty} \rho(Z_m, \text{ Ker}(d_{j+s'}))$$
$$= \varlimsup_{m\to\infty} \rho(Z_m, \text{ Ran}(d_{j+s'+1})) < \delta^2.$$

This contradicts (3), so the result follows.

For commuting n-tuple $A = (A_1, \cdots, A_n)$ on E, we can also define $\sigma_{*e}(A) =$ $\cdots, \lambda_n) \mid$ there exist uncompact sequences $\{f_m\} \subset E$ and $\{f_m^*\} \subset E^*$ satisfying $\|$ $\|f_m^*\| = 1$ and $f_m^*(f_m) \geqslant \delta$ for some constant δ such that $(A_{i_k} - \lambda_{i_k}) f_m \to 0$ for $k = 1$ and $(A_{l_t} - \lambda_{l_t})^* f_m^* \to 0$ for all $l_t \in \{1, \cdots, n\} \setminus \{i_1, \cdots, i_j\}\}$. Then the following th is obvious.

Theorem 9. *let $A = (A_1, \cdots, A_n)$ and $B = (B_1, \cdots, B_n)$ be commuting n-tu Banach space E. Then we have*

$$\sigma_*(A) \times \sigma_{*e}(B) \cup \sigma_{*e}(A) \times \sigma_*(B) \subset Sp_e(L_A, R_B).$$

The proof is similar to that of Theorem 6.

References

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