*Chin. Ann. of Math.*
**10B (1) 1989**

# ADJACENT COEFFICIENTS OF UNIVALENT FUNCTIONS

Hu Ke (胡 克)*


## Abstract

Let the function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in S$. It is obtained that

$$-2.793<|a_{n+1}|-|a_n|<3.26,$$

which is an improvement of the result in [1] or [2].


1. Let $S=\{f(z)=z+\sum_{n=2}^{\infty}a_nz^n|f(z)$ is analytic and univalent in the unit disk $<1\}$. Denote $d_n=|a_{n+1}|-|a_n|$. In 1963 Hayman first proved[3] that $|d_n|<A$, for ıe absolute constant $A$. Several years later I. M. Milin[4] ound a simpler proof ıch also provided a better numerical bound for $d_n$. And then Ilina[5] refined the ument and got $|d_n|<4.26$, Grinspan modified Milin's proof and obtained $d_n\in$ 2.97, 3.61). Further Ye[1] obtained $d_n\in(-2.945, 3.394)$. In this paper we have following theorem.

**Theorem.** *For each function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ in the class $S$, the following ɣualities*

$$-2.793<|a_{n+1}|-|a_n|<3.26,\ n=1,\ 2,\ \cdots,\tag{1}$$

Given $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in S$ $(|z|<1)$, it is obvious that $g(t)=1/f(z)\in\Sigma$ for $t=$ . Fixed $\rho\in(0,1)$, let $\zeta$ be such a point on the circle $|z|=\rho$ that $|f(\zeta)|$ is maum on the circle. Let

$$\varphi(z)=\log\frac{1/z-1/\zeta}{g\left(\frac{1}{z}\right)-g\left(\frac{1}{\zeta}\right)}=\sum_{n=1}^{\infty}\alpha_n(\zeta)z_n,\tag{2}$$

$$\Psi(z)=e^{\varphi(z)}=\sum_{n=0}^{\infty}\beta_n(\zeta)z^n,\tag{3}$$

$$B_n=\left\{\frac{f(z)}{f(\zeta)}\Psi(z)\right\}_n,\tag{4}$$

$$X^2=\sum_{k=1}^{n}k^2|\alpha_k|^2/n.$$

Manuscript received July 7, 1986.

* Department of Mathematics, Jiangxi Normal College, Nanchang, Jiangxi, China.

We have the following lemma.

**Lemma 1** [2].

$$\sqrt{\sum_{k=1}^{n-1}|\beta_k|^2}\leqslant\frac{\sqrt{n}}{\sqrt{(1-\rho^2)(n+1/2)}}\exp\frac{1}{2}(1-x^2-c),\tag{5}$$

$$|\beta_n|\leqslant\frac{x}{\sqrt{(1-\rho^2)(n+1/2)}}\exp\frac{1}{2}(1-x^2-c),\tag{6}$$

*where c is the Euler constant.*

**Lemma 2.**

$$|B_n|\leqslant\frac{x+\rho^{-n}}{\sqrt{(1-\rho^2)(n+1/2)}}\exp\frac{1}{2}(1-x^2-c).$$

*Proof* Let $x_0=0,\ x_k=k\rho^{n-k}\ (k=1,2,\cdots,n),\ y_k=n-x_k$. Then

$$\begin{aligned}B_n&=\frac{1}{f(\zeta)}\sum_{k=0}^{n-1}\beta_k a_{n-k}\\&=\frac{1}{nf(\zeta)}\left\{\sum_{k=1}^{n-1}x_k\beta_k a_{n-k}+\sum_{k=1}^{n-1}y_{n-k}\beta_{n-k}a_k\right\}\\&=\frac{1}{nf(\zeta)}\{I_1+I_2\}.\end{aligned}$$

By $k\beta_k=\sum_{l=1}^{k}l\alpha_l\beta_{k-l}\ \max_{|z|=\rho}|f(z)|=|f(\zeta)|$ and Schwarz's inequality we have

$$\begin{aligned}\frac{1}{n|f(\zeta)|}|I_1|&=\frac{1}{n|f(\zeta)|}\frac{1}{2\pi}\left|\int_0^{2\pi}z^{-n}\sum_{k=1}^{n}k\beta_k z^k\sum_{k=1}^{n-1}a_k\rho^k\bar z^k d\theta\right|\ (z=e^{i\theta})\\&=\frac{1}{n|f(\zeta)|}\frac{1}{2\pi}\left|\int_0^{2\pi}z^{-n}\sum_{k=1}^{n}k\alpha_k z^k\sum_{k=0}^{n-1}\beta_k z^k\sum_{k=1}^{\infty}a_k\rho^k\bar z^k d\theta\right|\\&\leqslant\frac{1}{n}\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=1}^{n}k\alpha_k z^k\right|\left|\sum_{k=0}^{n-1}\beta_k z^k\right|d\theta\\&\leqslant\frac{1}{2\pi n}\sqrt{\int_0^{2\pi}\left|\sum_{k=1}^{n}k\alpha_k z^k\right|^2d\theta\int_0^{2\pi}\left|\sum_{k=0}^{n-1}\beta_k z^k\right|^2d\theta}\\&=\frac{1}{n}\sqrt{\sum_{k=1}^{n}k^2|\alpha_k|^2\sum_{k=0}^{n-1}|\beta_k|^2}.\ [1]\end{aligned}$$

It follows from (5) that

$$\frac{1}{n|f(\zeta)|}|I_1|\leqslant\frac{x}{\sqrt{(1-\rho^2)(n+1/2)}}\exp\frac{1}{2}(1-x^2-c).$$

To estimate the second term of (8) let $\rho=e^{-\lambda/n}$, $\lambda\in(0,1)$, $n=2,3\cdots$. We see $y_{n-k}\rho^{n-k}\leqslant\sqrt{kn}$, $1\leqslant k\leqslant n$. And it is known that $\sum_{k=1}^{\infty}k|a_k|^2\rho^{2k}\leqslant(\operatorname{Max}_{|z|=\rho}|f(z)|)^2$. from Cauchy inequality we deduce[1]

$$\begin{aligned}I_2&=\left|\sum_{k=1}^{n}y_{n-k}\beta_{n-k}a_k\right|\leqslant\rho^{-n}\sqrt{n}\sqrt{\sum_{k=0}^{n-1}|\beta_k|^2\sum_{k=1}^{n}k|a_k|^2\rho^{2k}}\\&\leqslant\rho^{-n}\sqrt{n}\,|f(\zeta)|\sqrt{\sum_{k=0}^{n}|\beta_k|^2}.\end{aligned}$$

(11) becomes, from (5)

$$\frac{1}{n|f(\zeta)|}|I_2|\leqslant\rho^{-n}\frac{1}{\sqrt{(1-\rho^2)(n+1/2)}}\exp\frac{1}{2}(1-x^2-c).\tag{12}$$

quality (7) follows from (10), (12) and (8).

**Lemma 3.** *If* $|a_{n+1}|>0$, *and* $\rho^2=e^{-\frac{1}{n+1}}$, *then*

$$|B_n|\leqslant\frac{n\sqrt{(n+)}\cdot\rho^{-n-1}}{\sqrt{3}\ |a_{n+1}|^2(1-\rho^2)^{1/2}}\exp\frac{1}{2}(1-x^2-c). \tag{13}$$

*Proof* Application of L. de Branges theorem[6] gives

$$\sum_{k=1}^{n}|a_k|^2\leqslant\sum_{k=1}^{n}k^2=\frac{1}{3}n(n+1)\left(n+\frac{1}{2}\right). \tag{14}$$

observe that if $\rho^2=e^{-\frac{1}{n+1}}$, then $(k+n+1)\rho^{2k}\geqslant(k+n+1)\left(1-\frac{k}{n+1}\right)$ It follows m Fitz Gerald theorem[7] that

$$|a_{n+1}|^4\rho^{2(n+1)}\leqslant\left\{\sum_{k=1}^{n+1}k|a_k|^2+\sum_{k=n+2}^{2n+1}(2(n+1)-k|a_k|^2\right\}\rho^{2(n+1)}$$

$$\leqslant\sum_{k=1}^{\infty}k|a_k|^2\rho^{2k}\leqslant|f(\zeta)|^2. \tag{15}$$

us we obtain

$$|B_n|=\frac{1}{|f(\zeta)|}\left|\sum_{k=1}^{n}a_k\beta_{n-k}\right|\leqslant\frac{1}{|f(\zeta)|}\sqrt{\sum_{k=1}^{n}|a_k|^2\sum_{k=1}^{n}|\beta_k|^2}$$

$$\leqslant\frac{n\sqrt{(n+1)(n+1/2)}}{\sqrt{3}\,|a_{n+1}|^2\rho^{n+1}}\sqrt{\sum_{k=1}^{n}|\beta_k|2/n}. \tag{16}$$

bstitution of (5) into (16) completes the proof of (13).

2. Proof. of Theorem.

Given a function $f\in S$, consider the tollowing identity

$$\left(\frac{1}{z}-\frac{1}{\zeta}\right)f(z)=\left\{1-\frac{f(z)}{f(\zeta)}\right\}\frac{1/z-1/\zeta}{g(1/z)-g(1/\zeta)}$$

$$=\psi(z)-\frac{f(z)}{f(\zeta)}\psi(z).\quad\left(f(z)=\frac{1}{g(1/z)}\right)$$

$$a_{n+1}-\zeta^{-1}a_n=\{\psi(z)\}_n-\left\{\frac{f(z)}{f(\zeta)}\psi(z)\right\}_n=\beta_n-B_n,$$

ere $\max\limits_{|z|=\rho}|f(z)|=|f(\zeta)|$, $|\zeta|=\rho$.

By (6) and (7) we get

$$||a_{n+1}|-\rho^{-1}|a_n||$$

$$\leqslant|a_{n+1}-\zeta^{-1}a_n|\leqslant|\beta_n|+|B_n|$$

$$\leqslant\frac{2x+\rho^{-n}}{\sqrt{(1-\rho^2)\left(n+\frac{1}{2}\right)}}\exp\frac{1}{2}(1-x^2-c)$$

$$\leqslant\frac{1}{\sqrt{(1-\rho^2)\left(n+\frac{1}{2}\right)}}\cdot\frac{\rho^{-n}+\sqrt{\rho^{-2n}+16}}{2}\exp\frac{1}{2}\left\{1-c-\left(\frac{\sqrt{\rho^{-2n}+16}-\rho^{-n}}{4}\right)^2\right\}$$

$$=G(\rho). \tag{17}$$

Setting $\rho=\exp\left(-\frac{1}{1.3n}\right)$ and performing some simple computation we conclude that

$$||a_{n+1}|-\rho^{-1}|a_n||\leqslant G\exp\left(-\frac{1}{1.3n}\right)<2.79375.$$

Thus $$d_n=|a_{n+1}|-|a_n|>|a_{n+1}|-\rho^{-1}|a_n|>-2.79375.$$

The lower bound of $d_n$ is shown.

To establish the upper bound we consider two cases.

(i) $|a_{n+1}|\leqslant 0.6685(n+1)$. Take $\rho=\exp\left(-\frac{1}{1.5(n+1)}\right)$ and use (17) to deduce

$$\begin{aligned}|a_{n+1}|-|a_n|&=(1-\rho)|a_{n+1}|+\rho|a_{n+1}|-|a_n|\\&\leqslant\left(1-\exp\left(-\frac{1}{1.5(n+1)}\right)\right)|a_{n+1}|\\&\quad+\exp\left(-\frac{1}{1.5(n+1)}\right)G\left(\exp\left(-\frac{1}{1.5(n+1)}\right)\right)\\&<3.26.\end{aligned}$$

(ii) $|a_{n+1}|\geqslant 0.6685(n+1)$. Use of (6) and (13) gives

$$\begin{aligned}||a_{n+1}|-\rho^{-1}|a_n||&\leqslant|\beta_n|+|B_n|\\&\leqslant\frac{1}{\sqrt{(1-\rho^2)(n+1/2)}}\left\{x+\frac{n\sqrt{(n+1)(n+1/2)}}{\sqrt{3}|a_{n+1}|^2\rho^{n+1}}\right\}\\&\quad\times\exp\frac{1}{2}(1-x^2-c)\\&=G_1(\rho,\ |a_{n+1}|).\end{aligned}$$

Setting $\rho^2=\exp\left(-\frac{1}{n+1}\right)$, $|a_{n+1}|=y(n+1)$ $(0.6685\leqslant y\leqslant 1)$ |, we conclude fro[m]

that

$$\begin{aligned}|a_{n+1}|-|a_n|&=\rho|a_{n+1}|-a_n|+(1-\rho)|a_{n+1}|\\&\leqslant\rho G_1(\rho,\ |a_{n+1}|)+(1-\rho)y(n+1)\\&\leqslant\frac{1}{2}y+\left\{x+\frac{\sqrt{e}}{\sqrt{3}y^2}\right\}\exp\frac{1}{2}(1-x^2-c)\\&\leqslant\frac{1}{2}y+\frac{1}{2}\left\{\frac{\sqrt{e}}{\sqrt{3}y^2}+\sqrt{\frac{e}{3y^4}+4}\right\}\\&\quad\times\exp\frac{1}{2}\left\{1-c-\frac{1}{4}\left(\sqrt{\frac{e}{3y^4}+4}-\frac{\sqrt{e}}{\sqrt{3}y^2}\right)^2\right\}\\&=G_1(y)\leqslant G_2(0.6685)\leqslant 3.26.\end{aligned}$$

The proof of Therem is complete.

## References

[1] Ye, Z. Q., On successive coefficients of univalent functiosn, *J. of Jiangxi Normal Univ.* (Science 1(1985), 24—33.

[2] Grinspan, A. Z., Improved bound for the difference of moduli of adjacent coefficients of u[n] functions, in Some Questions in the Modern Theory of Functions (Sib. Inst. Mat.: NovosibirsL, [illegible]), 41—45 (in Russian).

[3] Hayman, W. K., On successive coefficients of univalent functions, *J. London Math. Soc.*, **38** (1963), 228—243

Milin, I. M., Adjacent coefficients of univalent functions, *Dokl. Akad. Nauk SSSR*, **180** (1968), 1294—1297 (in Russian).

Ilina, L. P., On the relative growth of adjacent coefficients of univalent functions, *Mat. Zametki*, **4** (1968), 715—722 (in Russian).

L. de Branges, A proof of the Bieberbach conjecture, Steklov Math. Inst. Lomi preprint E-f-84, Leningrad, 1984, 1—21.

FitzGerald, C. H., Quadratic inequalities and coefficient estimates for schlicht functions, *Arch. Rational Mech. Anal.*, **46** (1972), 356—368.