Chin. Ann. of Math.
10B (1) 1989

KAEHLER SUBMANIFOLDS IN A LOCALLY SYMMETRIC BOCHNER-KAEHLER MANIFOLD

GUO XIAOYING (郭孝英)* SHEN YIBING (沈一兵)*

Abstract.

This paper gives some sufficient conditions for a compact Kaehler submanifold M^n in locally symmetric Bochner-Kaehler manifold $\widetilde{M}^{n+p}$ to be totally geodesic. The conditior are given by inequalities which are established between. the sectional curvature (resp. hol morphic sectional curvature) of M^n and the Ricci curvature of $\widetilde{M}^{n+p}$. In particular, simila results in the case where $\widetilde{M}^{n+p}$ is a complex projective space are contained.

§ 0. Introduction

A locally symmetric Bochner-Kaehler manifold means a Kaehler man with parallel Riemannian curvature tensor and vanishing Bochner curva tensor. Complex space forms are special locally symmetric Bochner-Ka manifolds. In this paper, we will prove the following results.

Theorem 1. *Let M^n be a compact Kaehler submanifold of complex dimensio 2 in a locally symmetric Bochner-Kaehler manifold $\widetilde{M}^{n+p}$ of complex dimension Let $Q_{\max}=\max_{x\in M^n}\{\widetilde{Ric}(\widetilde{M})_x\}$ and $Q_{\min}=\min_{x\in M^n}\{\widetilde{Ric}(\widetilde{M})_x\}$, where $\widetilde{Ric}(\widetilde{M})_x$ denotes the curvature of $\widetilde{M}^{n+p}$ at the point x. If the sectional curvature K_M of M^n satisfies*

$$K_M>\frac{1}{2(n+p+2)}\left[Q_{\max}-\frac{n+p}{2(n+p+1)}Q_{\min}\right], \qquad (1)$$

then M^n must be totally geodesic in $\widetilde{M}^{n+p}$.

Theorem 2. *Let M^n and $\widetilde{M}^{n+p}$ be the same as in Theorem 1. If the holomo sectional curvature H_M of M^n satisfies*

$$H_M>\frac{2}{n+p+2}\left[Q_{\max}-\frac{n+p}{2(n+p+1)}Q_{\min}\right], \qquad (2)$$

then M^n must be totally geodesic in $\widetilde{M}^{n+p}$.

As is well known, a complex projective space CP^{n+p} endowed with the St Fubini metric of constant holomorphic sectional curvature 1 is a locally symmetric

Manuscript received July 17, 1986.

* Department of Mathematics, Hangzhou University, Hangzhou, Zhejiang, China.

ohner–Kaehler manifold with constant Ricci curvature $\frac{1}{2}(n+p+1)$[2]. Therefore, have from Theorems 1 and 2 the following corollaries immediately.

Corollary1.[5] *Let M^n be a compact Kaehler submanifold of complex dimension ≥2 in CP^{n+p} with constant holomorphic sectional curvature 1. If the sectional curvature M^n is larger than 1/8, then M^n is totally geodesic in CP^{n+p}.*

Corollary 2[4]. *Let M^n and CP^{n+p} be the same as in Corollary 1. If the holomor·ic sectional curvature of M^n is larger than 1/2, then M^n is totally geodesic in CP^{n+p}.*

Note that the proof of the theorem in [5] is based on the result of [4]. Here we ll give an entirely self-contained proof of Theorem 1, which differs from that in]. By the way, we will use the moving frame method andthe notation of [3] and].

§1. Fundamental Formulas

Let M^n be a Kaehler submanifold of complex dimension n in a Bochner–Kaehler ınifold $\widetilde{M}^{n+p}$ of complex dimension $n+p$, and J (resp. $\widetilde{J}$) the complex structure M^n(resp. $\widetilde{M}^{n+p}$). We choose a local field of orthonormal frames $e_1, \cdots, e_{n+p}, e_{1^*}=$ $_1, \cdots, e_{(n+p)^*}=\widetilde{J}e_{n+p}$ in $\widetilde{M}^{n+p}$ in such a way that, restricted to M^n, $e_1, \cdots e_n, e_{1^*}, \cdots, e_{n^*}$ e tangent to M^n. (*) With respect to the frame field of $\widetilde{M}^{n+p}$ chosen above, let $^A\}$ be the field of dualframes. Then, restricted to M^n, we have (cf. [3])

$$\omega^\lambda=0,\ \omega_j^\lambda=\sum_j h_{ij}^\lambda\omega^j,\ h_{ij}^\lambda=h_{ji}^\lambda, \tag{1.1}$$

$$d\omega^i=-\sum_j\omega_j^i\wedge\omega^j,$$

$$d\omega_j^i=-\sum_k\omega_k^i\wedge\omega_j^k+\frac{1}{2}\sum_{k,l}R_{ijkl}\omega^k\wedge\omega^l, \tag{1.2}$$

$$R_{ijkl}=\widetilde{R}_{ijkl}+\sum_\lambda(h_{ik}^\lambda h_{jl}^\lambda-h_{il}^\lambda h_{jk}^\lambda), \tag{1.3}$$

$$R_{\lambda\mu ij}=\widetilde{R}_{\lambda\mu ij}+\sum_k(h_{ik}^\lambda h_{jk}^\mu-h_{jk}^\lambda p_{ik}^\mu), \tag{1.4}$$

here

$$\widetilde{R}_{ABCD}=\delta_{BD}L_{AC}-\delta_{BC}L_{AD}+\delta_{AC}L_{BD}-\delta_{AD}L_{BC}+\widetilde{J}_{BD}M_{AC}-\widetilde{J}_{BC}M_{AD}$$
$$+\widetilde{J}_{AC}M_{BD}-\widetilde{J}_{AD}M_{BC}+2\widetilde{J}_{CD}M_{AB}-2\widetilde{J}_{AB}M_{CD}, \tag{1.5}$$

$$L_{AB}=\frac{1}{2(n+p+2)}[\widetilde{R}_{AB}-\delta_{AB}\widetilde{\rho}/4(n+p+1)], \tag{1.6}$$

$$\widetilde{R}_{AB}=\sum_C\widetilde{R}_{CACB},\ \widetilde{\rho}=\sum_A\widetilde{R}_{AA},$$

$$M_{AB}=-\sum_C\widetilde{J}_{BC}L_{AC}, \tag{1.7}$$

(*) We use the following convention on the range of indices unless otherwise stated:

$A, B, C, \cdots=1, \cdots, n+p, 1^*, \cdots, (n+p)^*$;

$a, b, c, \cdots=1, \cdots n$; $i, j, k, \cdots=1, \cdots, n, 1^*, \cdots, n^*$;

$\alpha, \beta, \cdots=n+1, \cdots, n+p$; $\lambda, \mu, \cdots=n+1, \cdots, n+p, (n+1)^*, \cdots, (n+p)^*$.

$$(\tilde{J}_{AB})=\left[\begin{array}{cc|c} 0 & -I_n & \\ I_n & 0 & 0 \\ \hline & & \begin{array}{cc} 0 & -I_p \\ I_p & 0 \end{array} \\ 0 & & \end{array}\right]. \tag{1.8}$$

Moreover, we have

$$L_{ab}=L_{a^*b^*},\quad L_{\alpha\beta}=L_{\alpha^*\beta^*} \tag{1.9}$$

and

$$M_{ab}=L_{ab^*},\quad M_{a^*b}=-M_{ab^*}=L_{ab},\quad M_{a^*b^*}=-L_{a^*b},$$
$$M_{\alpha\beta}=L_{\alpha\beta^*},\quad M_{\alpha^*\beta}=-M_{\alpha\beta^*}=L_{\alpha\beta},\quad M_{\alpha^*\beta^*}=-L_{\alpha^*\beta}. \quad ($$

The second fundamental form $\sigma(X, Y)$ of M^n in $\tilde{M}^{n+p}$ is

$$\sigma(X, Y)=\sum_\lambda h^\lambda_{ij}\omega^i(X)\omega^j(Y)e_\lambda, \quad ($$

which satisfies[3]

$$\sigma(JX, Y)=\sigma(X, JY)=\tilde{J}\sigma(X, Y), \quad ($$

$$h^\lambda_{a^*b^*}=-h^\lambda_{ab},\quad h^\lambda_{a^*b}=h^\lambda_{ab^*},$$
$$h^\alpha_{ab^*}=-h^\alpha_{ab},\quad h^{\alpha^*}_{a^*b}=h^\alpha_{ab}. \quad ($$

Let h^λ_{ijk} and h^λ_{ijkl} be the first and second covariant derivatives of Va Waerden-Bortolotti of h^λ_{ij}. Suppose that $\tilde{M}^{n+p}$ is locally symmetric. We have[1]

$$h^\lambda_{ijk}-h^\lambda_{ikj}=-\tilde{R}_{\lambda ijk}, \quad ($$

$$\begin{aligned} h^\lambda_{ijkl}-h^\lambda_{ikjl}&=-\tilde{R}_{\lambda ijkl}\\ &=-\sum_\mu \tilde{R}_{\lambda\mu jk}h^\mu_{il}-\sum_\mu \tilde{R}_{\lambda i\mu k}h^\mu_{jl}-\sum_\mu \tilde{R}_{\lambda ij\mu}h^\mu_{kl}+\sum_m \tilde{R}_{mijk}h^\lambda_{ml},\end{aligned}$$

$$h^\lambda_{ijkl}-h^\lambda_{ijlk}=\sum_m h^\lambda_{im}R_{mjkl}+\sum_m h^\lambda_{mj}R_{mikl}+\sum_\mu h^\mu_{ij}R_{\mu\lambda kl}. \quad ($$

Let $H_M(X)$ denote the holomorphic sectional curvature of M^n determined unit vector X tangent to M. From (1.3) and (1.5) it follows that[3]

$$H_M(X)=8L(X, X)-2\|\sigma(X, X)\|^2,$$

where $L(X, X)=\sum_{i,j} L_{ij}\omega^i(X)\omega^j(X)$.

§2. Maximum Principles

Let M^n be a compact Kaehler submanifold in a locally symmetric Bo Kaehler manifold $\tilde{M}^{n+p}$. Let $UM=\bigcup_{x\in M^n} U_x(M)$ and $U_x(M)=\{X\in T_x(M)\mid \|X$ so that $UM\to M$ is the unit tangent bundle over M^n. We define a function f: R by $f(X)=\|\sigma(X, X\|^2$ for $X\in UM$. On putting $X=\sum_i \xi^i e_i$, we have

$$f(X)=\|\sigma(X, X)\|^2=\sum_\lambda\Big(\sum_{i,j} h^\lambda_{ij}\xi^i\xi^j\Big)^2.$$

Since UM is compact, f attains the maximum at a unit vector in UM. Suppo t this vector is $X=\sum\xi^i e_i\in U_x(M)$ for a point $x\in M$. For any $Y=\sum_i \eta^i e_i\in U_x(M)$, let $\gamma_Y(t)$ be a geodesic in M^n determined by the initial conditions $\gamma_Y(0)=x$, $\gamma'_Y(0)=Y$.

By parallel translating X along $\gamma_Y(t)$, we obtain a vector field $\tilde{X}_Y(t)=\sum\xi^i(t)e_i$ with $\tilde{X}_Y(0)=X$. Put $f_Y(t)=f(\tilde{X}_Y(t))$. From the maximum condition we get

$$0=\frac{d}{dt}f_Y(t)\Big|_{t=0}=2\sum_\lambda\Big(\sum_{i,j}h^\lambda_{ij}\xi^i\xi^j\Big)\Big(\sum_{i,j,k}h^\lambda_{ijk}\xi^i\xi^j\eta^k\Big), \tag{2.2}$$

$$0\geq\frac{d^2}{dt^2}f_Y(t)\Big|_{t=0}=2\sum_\lambda\Big(\sum_{i,j,k}h^\lambda_{ijk}\xi^i\xi^j\eta^k\Big)^2+2\sum_\lambda\Big(\sum_{i,j}h^\lambda_{ij}\xi^i\xi^j\Big)\Big(\sum_{i,j,k,l}h^\lambda_{ijkl}\xi^i\xi^j\eta^k\eta^l\Big). \tag{2.3}$$

Hence, we have the following lemma.

Lemma 1. *For arbitrary unit vector* $Y=\sum_i\eta^ie_i\in U_x(M)$, (2.2) *and* (2.3) *hold.*

Now assume that $Y=\sum\eta^ie_i\in U_x(M)$ and $\langle X, Y\rangle=0$. Let $\beta(t)$ be a curve on the here $U_x(M)$ such that $\beta(0)=X$ and $\beta'(0)=Y$. Then, we have the following mma.

Lemma 2. *At the maximum point* $X\in U_x(M)$ of f,

$$0=\sum_\lambda\Big(\sum_{i,j}h^\lambda_{ij}\xi^i\xi^j\Big)\Big(\sum_{i,j}h^\lambda_{ij}\xi^i\eta^j\Big), \tag{2.4}$$

$$0\geq2\sum_\lambda\Big(\sum_{i,j}h^\lambda_{ij}\xi^i\eta^j\Big)^2+\sum_\lambda\Big(\sum_{i,j}h^\lambda_{ij}\xi^i\xi^j\Big)\Big(\sum_{i,j}h^\lambda_{ij}\eta^i\eta^j\Big)-f(X) \tag{2.5}$$

r *any* $Y\in U_x(M)$ *with* $\langle Y, X\rangle=0$.

Proof Suppose that $\beta(t)=\sum_i\beta^i(t)e_i$ with $\sum_i(\beta_i(t))^2=1$.

As X is a critical point of f we have

$$0=\frac{d}{dt}f(\beta(t))\Big|_{t=0}=2\sum_\lambda\Big(\sum_{i,j}h^\lambda_{ij}\beta^i(0)\beta^j(0)\Big)\Big(2\sum_{i,j}h^\lambda_{ij}\beta^i(0)\beta'^j(0)\Big),$$

ich is just (2.4) by the initial condition of $\beta(t)$.

Moreover, we have at $t=0$

$$0\geq\frac{d^2}{dt^2}f(\beta(t))\Big|_{t=0}=8\sum_\lambda\Big(\sum h^\lambda_{ij}\xi^i\eta^j\Big)^2+4\sum_\lambda\Big(\sum_{i,j}h^\lambda_{ij}\xi^i\xi^j\Big)\Big(\sum_{i,j}h^\lambda_{ij}\eta^i\eta^j+\sum_{i,j}h^\lambda_{ij}\xi^i\beta''^j(0)\Big). \tag{2.6}$$

ce $\sum_i(\beta^i(t))^2=1$, we have $\sum_i\beta^i(t)\beta'^i(t)=0$ and $\sum_i\beta^i(t)\beta''^i(t)=-1$, which im- es that

$$\beta''^i(0)=-\xi^i+\zeta^i \tag{2.7}$$

h $\sum_i\xi^i\zeta^i=0$.

Substituting (2.7) into (2.6) and using (2.4), one obtains (2.5) immediately. ıs, Lemma 2 is proved.

§3. The Proof of Theorem 1

Let $X\in U_x(M)$ be a maximum point of the function f defined by (2.1). It follows from (1.12) and (2.1) that $JX\in U_x(M)$ is also one. So we can choose a local field of frames such that $e_1=X$ and $e_{1^*}=JX$ at x. With respect to such a frame field, Let Y be e_2 and e_{2^*}, respectively, in Lemma 1. Then we have from (2.3)

$$\sum_\lambda (h^\lambda_{112})^2+\sum_\lambda h^\lambda_{11}h^\lambda_{1122}\leqslant 0 \quad\text{and}\quad \sum_\lambda (h^\lambda_{112^*})^2+\sum_\lambda h^\lambda_{11}h^\tau_{112^*2^*}\leqslant 0$$

which imply that

$$\sum_\lambda h^\lambda_{11}(h^\lambda_{1122}+h^\lambda_{112^*2^*})\leqslant 0. \tag{3.1}$$

Moreover, by Lemma 2, we have

$$\sum_\lambda h^\lambda_{11}h^\lambda_{1j}=0 \quad (j\neq 1),$$
$$\sum_\lambda h^\lambda_{1^*1^*}h^\lambda_{1^*j}=0 \quad (j\neq 1^*)$$ (

and

$$2\sum_\lambda (h^\lambda_{1j})^2+\sum_\lambda h^\lambda_{11}h^\lambda_{jj}\leqslant f(X) \quad (j\neq 1),$$
$$2\sum_\lambda (h^\lambda_{1^*j})^2+\sum_\lambda h^\lambda_{1^*1^*}h^\lambda_{jj}\leqslant f(X) \quad (j\neq 1^*),$$ (

where

$$f(X)=\|\sigma(e_1,e_1)\|^2=\sum_\lambda (h^\lambda_{11})^2.$$ (

From (1.13), (1.14) and (1.15) we have

$$h^\lambda_{112^*2^*}=-h^\lambda_{1122}+\sum_i h^\lambda_{1^*i}R_{i122^*}+\sum_i h^\lambda_{i1}R_{i1^*22^*}+\sum_\mu h^\mu_{1^*1}R_{\mu\lambda22^*}$$
$$+\tilde{R}_{\lambda1212}-\tilde{R}_{\lambda1^*12^*2}-\tilde{R}_{\lambda1^*212^*}-\tilde{R}_{\lambda1112^*2^*},$$

which together with (3.1) yields

$$0\geqslant\underbrace{\sum_{\lambda,i}h^\lambda_{11}(h^\lambda_{1^*i}R_{i122^*}+h^\lambda_{i1}R_{i1^*22^*})}_{\text{(I)}}+\underbrace{\sum_{\lambda,\mu}h^\lambda_{11}h^\mu_{11^*}R_{\mu\lambda22^*}}_{\text{(II)}}+\underbrace{\sum_\lambda h^\lambda_{11}\tilde{R}_{\lambda1212}}_{\text{(III)}}$$
$$-\underbrace{\sum_\lambda h^\lambda_{11}\tilde{R}_{\lambda1^*12^*2}}_{\text{(IV)}}-\underbrace{\sum_\lambda h^\lambda_{11}\tilde{R}_{\lambda1^*212^*}}_{\text{(V)}}-\underbrace{\sum_\lambda h^\lambda_{11}\tilde{R}_{\lambda112^*2^*}}_{\text{(VI)}}.$$ (

Since $\qquad R=_{11^*22^*}=-R_{122^*1^*}-R_{12^*1^*2}=R_{1212}+R_{1^*21^*2}=K_{12}+K_{1^*2},$

where K_{ij} denotes the sectional curvature of M^n at x for the plane spanned by e e_j, it follows from (1.13), (3.2) and (3.4) that

$$\text{(I)}=2f(X)\,(K_{12}+K_{1^*2}).$$ (

By means of (3.4), (1.4) and (1.5)–(1.13), a direct computation shows

$$\text{(II)}=-2\sum_{\lambda,\mu}h^\lambda_{11}h^\mu_{11}L_{\lambda\mu}-2\,L_{22}f(X)-2\,(\sum_{\lambda,i}h^\lambda_{11}h^\lambda_{2i})^2.$$ (

Using (1.14) and (1.5)—(1.13), we can derive that

$$\text{(III)}=\sum_{\lambda,\mu}h^\lambda_{11}h^\mu_{22}L_{\lambda\mu}-\sum_{\lambda,i}h^\lambda_{11}h^\lambda_{2i}L_{i2},$$
$$\text{(IV)}=\sum_{\lambda,i}h^\lambda_{11}h^\lambda_{2i}L_{i2},$$
$$\text{(V)}=\sum_{\lambda,i}h^\lambda_{11}h^\lambda_{2^*i}L_{i2^*},$$
$$\text{(VI)}=-\sum_{\lambda,\mu}h^\lambda_{11}h^\mu_{22}L_{\lambda\mu}-\sum_{\lambda,i}h^\lambda_{11}h^\lambda_{2^*i}L_{i2^*}.$$ (

Substituting (3.6), (3.7) and (3.8) into (3.5), we obtain

$$0\geqslant 2f(X)(K_{12}+K_{1^*2})-2\sum_{\lambda,\mu}h^\lambda_{11}h^\mu_{11}L_{\lambda\mu}-2f(X)L_{22}-2(\sum_{\lambda,i}h^\lambda_{11}h^\lambda_{2i})^2 \tag{3.9}$$

the point x.

Now, consider the symmetric matrix $S=(S_{ij})$ where $S_{ij}=\sum_\lambda h^\lambda_{11}h^\lambda_{ij}$. (3.2) together ...th (1.13) shows that e_1 as well as $e_{1^*}=Je_1$ is an eigenvector of S. Since $n\geqslant 2$, we ...n always choose $\{e_a, e_{a^*}=Je_a\}$ $(a\neq 1)$ in such a way that S is diagonal at x. With ...spect to the frame field chosen above, (3.9) becomes

$$0\geqslant 2f(X)(K_{12}+K_{1^*2})-2\sum_{\lambda,\mu}h^\lambda_{11}h^\mu_{11}L_{\lambda\mu}-2f(X)L_{22}-2\Big(\sum_\lambda h^\lambda_{11}h^\lambda_{22}\Big)^2. \tag{3.10}$$

Noting that $\widetilde{R}_{1212}=\widetilde{R}_{1^*21^*2}=L_{11}+L_{22}$, by (1.5), it follows from (1.3) and (1.13) ...at

$$2\sum_\lambda h^\lambda_{11}h^\lambda_{22}=\sum_\lambda(h^\lambda_{11}h^\lambda_{22}-h^\lambda_{1^*1^*}h^\lambda_{22})=R_{1212}-R_{1^*21^*2}=K_{12}-K_{1^*2}. \tag{3.11}$$

Substituting (3.11) into (3.10), we get

$$0\geqslant AK_{12}+BK_{1^*2}-2\sum_{\lambda,\mu}h^\lambda_{11}h^\mu_{11}L_{\lambda\mu}-2f(X)L_{22}, \tag{3.12}$$

...ere

$$\begin{aligned}A&=2f(X)-(K_{12}-K_{1^*2})/2,\\ B&=2f(X)-(K_{1^*2}-K_{12})/2.\end{aligned} \tag{3.13}$$

On the other hand, from (3.11) and (3.3) we have

$$(K_{12}-K_{1^*2})/2=\sum_\lambda h^\lambda_{11}h^\lambda_{22}\leqslant 2\sum_\lambda(h^\lambda_{12})^2+\sum_\lambda h^\lambda_{11}h^\lambda_{22}\leqslant f(X). \tag{3.14}$$

By similar arguments we also have

$$\frac{1}{2}(K_{1^*2}-K_{12})=\sum_\lambda h^\lambda_{1^*1^*}h^\lambda_{22}\leqslant f(X). \tag{3.15}$$

...nce, one sees from (3.13)—(3.15) that $A\geqslant 0$ and $B\geqslant 0$. Then, it follows from ...12) and (3.13) that

$$0\geqslant (A+B)K_{\min}-2\sum_{\lambda,\mu}h^\lambda_{11}h^\mu_{11}L_{\lambda\mu}-2f(X)L_{22},$$

...,

$$0\geqslant 2f(X)K_{\min}-\sum_{\lambda,\mu}h^\lambda_{11}h^\mu_{11}L_{\lambda\mu}-f(X)L_{22}, \tag{3.16}$$

...ere $K_{\min}$ is the minimum of the sectional curvature of M^n at x.

Under the assumption of Theorem 1, we have from (1.6) and (3.4)

$$\sum_{\lambda,\mu}h^\lambda_{11}h^\mu_{11}L_{\lambda\mu}+f(X)L_{22}\leqslant 2f(X)\frac{1}{2(n+p+2)}\Big[Q_{\max}-\frac{n+p}{2(n+p+1)}Q_{\min}\Big], \tag{3.17}$$

...ch together with (3.16) yields

$$0\geqslant f(X)\Big\{K_{\min}-\frac{1}{2(n+p+2)}\Big[Q_{\max}-\frac{n+p}{2(n+p+1)}Q_{\min}\Big]\Big\}. \tag{3.18}$$

...ollows from (0.1) and (3.18) that $f(X)=0$, which implies, by (2.1), that $\sigma=0$. ...nce, M is totally geodesic, and Theorem 1 is proved.

§ 4. The Proof of Theorem 2

By similar arguments as in § 3, we take $Y=e_1$ and e_{1^*}, respectively, in Lemma 1. Thus, we get

$$\sum_{\lambda} h^{\lambda}_{11}(h^{\lambda}_{1111}+h^{\lambda}_{11 1^* 1^*})\leqslant 0. \tag{4.1}$$

Since

$$h^{\lambda}_{11 1^* 1^*}= -h^{\lambda}_{1111}+\sum_{i} h^{\lambda}_{1^* i}R_{i11 1^*}+\sum_{i} h^{\lambda}_{1i}R_{i1^* 11^*}+\sum_{\mu} h^{\mu}_{11^*}R_{\mu\lambda 11^*}-\widetilde{R}_{\lambda 111^* 1^*}-\widetilde{R}_{\lambda 1^* 11^* 1},$$

(4.1) becomes

$$0\geqslant\underbrace{\sum_{\lambda,i} h^{\lambda}_{11}(h^{\lambda}_{1^* i}R_{i11 1^*}+h^{\lambda}_{1i}R_{i1^* 11^*})}_{(\mathrm{I})}+\underbrace{\sum_{\lambda,\mu} h^{\lambda}_{11}h^{\mu}_{11^*}R_{\mu\lambda 11^*}}_{(\mathrm{II})}+\underbrace{\sum_{\lambda} h^{\lambda}_{11}(\widetilde{R}_{\lambda 111^* 1^*}+\widetilde{R}_{\lambda 1^* 11^* 1})}_{(\mathrm{III})}. \tag{4.2}$$

It is seen easily from (1.13) that

$$f(X)=\sum_{\lambda}(h^{\lambda}_{11})^2=\sum_{\alpha}(h^{\alpha}_{11})^2+\sum_{\alpha^*}(h^{\alpha^*}_{11})^2$$
$$=\sum_{\alpha^*}(h^{\alpha^*}_{11^*})^2+\sum_{\alpha}(h^{\alpha}_{11^*})^2=\sum_{\lambda}(h^{\lambda}_{11^*})^2. \quad (\text{4.3})$$

Using (1.5)—(1.14) and (4.3), a similar computation as in (3.6)—(3.8) gives to

$$(\mathrm{I})=16f(X)L_{11}-4f^2(X),$$
$$(\mathrm{II})=-2\sum_{\lambda,\mu} h^{\lambda}_{11}h^{\mu}_{11}L_{\lambda\mu}-2f(X)L_{11}-2f^2(X),$$
$$(\mathrm{III})=-8\sum_{\lambda,\mu} h^{\lambda}_{11}h^{\mu}_{11}L_{\lambda\mu}+8f(X)L_{11}.$$

Substituting these into (4.2), we obtain

$$0\geqslant-10\sum_{\lambda,\mu} h^{\lambda}_{11}h^{\mu}_{11}L_{\lambda\mu}-f(X)\,[6f(X)-22L_{11}]. \quad (\text{4.4})$$

By virtue of (1.16) and $X=e_1$ we have

$$f(X)=4L_{11}-H_M(X)/2. \quad (\text{4.5})$$

Substituting (4.5) into (4.4) and using (1.6), we can get

$$0\geqslant-10\sum_{\lambda,\mu} h^{\lambda}_{11}h^{\mu}_{11}L_{\lambda\mu}-f(X)\,[2L_{11}-3H_M(X)]$$
$$=3f(X)\left[\frac{\tilde{\rho}}{2(n+p+1)(n+p+2)}+H_M(X)\right]$$
$$-\frac{1}{n+p+2}\Big[5\sum_{\lambda,\mu} h^{\lambda}_{11}h^{\mu}_{11}\widetilde{R}_{\lambda\mu}+f(X)\widetilde{R}_{11}\Big]$$
$$\geqslant 3f(X)\left\{H_M(X)-\frac{2}{n+p+2}\left[Q_{\max}-\frac{n+p}{2(n+p+1)}Q_{\min}\right]\right\}. \quad (\text{4.6})$$

From the hypothesis (0.2) and (4.6) it follows that $f(X)=0$, so that M is totally geodesic and we have proved Theorem 2.

References

[1] Chern, S. S., Do Carmo, M. & Kobayashi, S., Minimal submanifolds of a sphere with second fundamental form of constant length, *Func. Anal. Rel. Fie.*, (1970), 59—75.

[2] Houh, C. S., Totally real submanifolds in a Bochner-Kaehler manifold, *Tensor (N. S.)*, **32** (1978) —296.

[3] Ogiue, K., Differential geometry of Kaehler submanifolds, *Adv. in Math.*, **13** (1974), 73—114.

[4] Ros, A., Positively curved Kaehler submanifolds, *Proc. A. M. S.*, **93** (1985), 329—331.

[5] Ros, A., & Verstraelen, L., On a conjecture of K. Ogiue, *J. Diff. Geom.*, **19** (1984), 561—566.

[6] Shen Yibing, Totally real minimal submanifolds in a locally symmetric Bochner-Kaehler manifold *J. Hangzhou Univ.*, **12** (1985), 432—440.