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THE AUTOMORPHISMS OF TWO-DIMENSIONAL LINEAR GROUPS OVER COMMUTATIVE RINGS*

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Abstract

The present paper determines the form of automorphisms of $E_2(R)$ and $GE_2(R)$ over commutative rings provided 2, 3 and 5 are units.

The automorphism problem of linear groups over commutative rings has been lved when $n\geqslant 3$. (Refer to [1, 2]). In case $n=2$ a particular difficulty occurs. From matrix-theoretic approach it does not provide sufficient off-diagonal positions, or eometrically, there is not a sufficient number of distinct lines. The present paper ill discuss and determine the automorphisms of E_2(R) and GE_2(R) under the ypothesis that 2, 3, and 5 are units.

Throughout this paper R will be a commutative ring. Denote by U the multi-licative group of all units in R, by $\max(R)$ the set of all maximal ideals of R. Let $_2(R)$ be the elementary group generated by all elementary matrices $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}$, and let $GE_2(R)$ be the subgroup of $GL_2(R)$ generated by $E_2(R)$ and all iagonal matrices. The order of a matrix $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is the ideal generated by b, c, d $a-d$.

Lemma 1. *Suppose that there exists an element $u\in U$ with $u^4-1\in U$. Then*

1) *for any $x\in U$ the normal subgroup H of $E_2(R)$ generated by $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$ is the hole $E_2(R)$, and*

2) *the normal subgroup of $E_2(R)$ generated by $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ is also $E_2(R)$.*

Hence the images of these matrices under every automorphism of $E_2(R)$ have the me property. In particular, if 2, 3, and $5\in U$, then 1) and 2) hold.

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Proof 1) Since

$$H \ni \begin{pmatrix} 1 & 0 \\ -x^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 2 & x \\ -x^{-1} & 0 \end{pmatrix}$$

we have
$$H \ni \left[\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \begin{pmatrix} 2 & x \\ -x^{-1} & 0 \end{pmatrix}\right] = \begin{pmatrix} u^2 & 2x(u^2-1) \\ 0 & u^{-2} \end{pmatrix}.$$

For any $a \in R$, take $b = a(1-u^4)^{-1}$. Then H contains

$$\left[\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} u^2 & 2x(u^2-1) \\ 0 & u^{-2} \end{pmatrix}\right] = \begin{pmatrix} 1 & b(1-u^4) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

Furthermore, $H \ni \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}$. Thus, $H = E_2(R)$.

2) Use the fact $\left[\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] = \begin{pmatrix} u^2 & 0 \\ 0 & u^{-2} \end{pmatrix}$, and then proceed as in tl proof of part 1).

Lemma 2. *Assume that* 2, 3, *and* $5 \in U$. *Let* Λ *be an automorphism of* $E_2(R$ *Then for any* $M \in \max(R)$ *there is* $g_M \in GL_2(R_M)$ *such that*

$$g_M\left[\Lambda\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right]g_M^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in E_2(R_M),$$

where R_M *is the localization of* R *with respect to* M.

Proof Let $\Lambda\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. For $M \in \max(R)$ we may always assume th: $y \notin M$. Indeed, if $z \notin M$, then we may transform z into the (1, 2)-position l conjugation. If $y, z \in M$, then by Lemma 1, the order of $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ is R, hence $x-w$ M. The (1, 2)-entry of the conjugate of $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is $w-x+y-z \notin M$.

Since $y \notin M$, y is invertible in R_M. Take $g = \begin{pmatrix} y^{-1} & 0 \\ y^{-1}x & 1 \end{pmatrix}$. Then

$$g\left[\Lambda\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right]g^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & v \end{pmatrix} \in E_2(R_M).$$

Any conjugate of $\begin{pmatrix} 0 & 1 \\ -1 & v \end{pmatrix}$ which commutes with $\begin{pmatrix} 0 & 1 \\ -1 & v \end{pmatrix}$ is of the form $a+$ $\begin{pmatrix} 0 & 1 \\ -1 & v \end{pmatrix}$ with determinant $a^2+abv+b^2=1$ and trace $2a+bv=v$. It follows that

$$2a = (1-b)v, \quad (1-b^2)(v^2-4) = 0.$$

Claim that $v^2-4 \in MR_M$. Otherwise, v^2-4 is invertible in R_M, hence $1-b^2=$ Thus, $b = \pm 1$ since R_M is a local ring and 2 is a unit. Moreover, $a=0$ if $b=1$, ar $a=v$ if $b=-1$. Therefore, $\begin{pmatrix} 0 & 1 \\ -1 & v \end{pmatrix}$ and $\begin{pmatrix} v & -1 \\ 1 & 0 \end{pmatrix}$ are the only conjugates of $\begin{pmatrix} 0 & 1 \\ -1 & v \end{pmatrix}$

which commute with $\begin{pmatrix} 0 & 1 \\ -1 & v \end{pmatrix}$. On the other hand, $g\left[A\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right]g^{-1}$, $g\left[A\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\right]g^{-1}$ and $g\left[A\begin{pmatrix} 1 & 16 \\ 0 & 1 \end{pmatrix}\right]g^{-1}$ are conjugates of $\begin{pmatrix} 0 & 1 \\ -1 & v \end{pmatrix}$ which commut ewith $\begin{pmatrix} 0 & 1 \\ -1 & v \end{pmatrix}$. By Lemma 1, the normal subgroup of $E_2(R)$ generated by $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ is $E_2(R)$, so the order of $A\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ is R. It follows that the order of $g\left[A\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}A\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}\right]g^{-1}$ is R_M. Hence $g\left[A\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\right]g^{-1}\neq g\left[A\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right]g^{-1}$. Similarly,

$$g\left[A\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\right]g^{-1}\neq g\left[A\begin{pmatrix} 1 & 16 \\ 0 & 1 \end{pmatrix}\right]g^{-1}\neq g\left[A\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right]g^{-1}.$$

This shows that there are at least three conjugates of $\begin{pmatrix} 0 & 1 \\ -1 & v \end{pmatrix}$ which commute with $\begin{pmatrix} 0 & 1 \\ -1 & v \end{pmatrix}$. This is a contradiction. Therefore, we must have $v^2-4\in MR_M$, hence $v\equiv \pm 2 \mod MR_M$. Now

$$g\left[A\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\right]g^{-1}=\begin{pmatrix} 0 & 1 \\ -1 & v \end{pmatrix}^4=a+b\begin{pmatrix} 0 & 1 \\ -1 & v \end{pmatrix},$$

where $a=1-v^2$, $b=v(v^2-2)$. Thus, $b^2-1\equiv 15 \mod MR_M$, i. e., b^2-1 is invertible. It follows from $(1-b^2)(v^2-4)=0$ that $v^2=4$. Further, we derive that $v=2$ from the equality $2a=(1-b)v$.

Set $g_M=\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}g$. Then $g_M\left[A\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right]g_M^{-1}=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Lemma 3. *Assume that 2, 3, and $5\in U$. Let A be an automorphism of $E_2(R)$. Then for any $M\in \max(R)$ there is $g_M\in GL_2(R_M)$ such that*

$$g_M\left[A\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right]g_M^{-1}=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$g_M\left[A\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]g_M^{1}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proof Let $A\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By Lemma 1, the order of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is R. We see that $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2=-1$ belongs to the center of $E_2(R)$, so does $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2$. Hence, $b(a+d)=c(a+d)=(a-d)(a+d)=0$. But $(b, c, a-d)=R$, so $a+d=0$, i. e., $d=-a$. Obviously, this property is invariant under conjugation.

By Lemma 2, there is $g_M\in GL_2(R_M)$ such that

$$g_M\left[A\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right]g_M^{-1}=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Write
$$g_M\left[\Lambda\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right]g_M^{-1}=\begin{pmatrix}x & y\\ z & -x\end{pmatrix}.$$

Claim that z is invertible in R_M. Otherwise, $z\in MR_M$. Since R_M/MR_M is naturally isomorphic to R/M, if we identify R_M/MR_M with R/M, then the following diagram is commutative:

$$\begin{array}{ccccccc}
E_2(R) & \xrightarrow{\Lambda} & E_2(R) & \longrightarrow & E_2(R_M) & \xrightarrow{\mathrm{Int}g_M} & E_2(R_M)\\
 & & & \searrow & & & \downarrow\\
 & & & & & & E_2(R_M/MR_M)\\
 & & & & & & \|\\
 & & & & E_2(R/M) & \xrightarrow{\mathrm{Int}g'_M} & E_2(R/M)
\end{array}$$

where g'_M is induced by g_M under the natural homomorphism $R_M\to R_M/MR_M=R/1$ and the unmarked arrows in the above diagram are natural group homomorphism Denote $\varphi=\mathrm{Int}\,g'_M\circ\Lambda$. Then φ is surjective. Since the above diagram is commutati $\varphi\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $\varphi\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}=\begin{pmatrix}x & y\\ 0 & -x\end{pmatrix}$. Since $\begin{pmatrix}1 & r\\ 0 & 1\end{pmatrix}$ commutes with $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ assume $\varphi\begin{pmatrix}1 & r\\ 0 & 1\end{pmatrix}=\alpha\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}$. Every matrix in $E_2(R)$ can be expressed as a produ $\prod_i\left[\begin{pmatrix}1 & r_i\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right]$, and its image under φ is $\prod_i\left[\alpha_i\begin{pmatrix}1 & \beta_i\\ 0 & 1\end{pmatrix}\begin{pmatrix}x & y\\ 0 & -x\end{pmatrix}\right]$, whose (2, -entry is always zero. This is contrary to the surjectivity of φ. Therefore, z must invertible in R_M. We now have

$$\begin{pmatrix}-z_1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}x & y\\ z & -x\end{pmatrix}\begin{pmatrix}-z & x\\ 0 & 1\end{pmatrix}^{-1}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$
$$\begin{pmatrix}-z & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}-z & x\\ 0 & 1\end{pmatrix}^{-1}=\begin{pmatrix}1 & -z\\ 0 & 1\end{pmatrix}.$$

Using the equality $\left[\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right]^3=1$, we obtain $\left[\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & -z\\ 0 & 1\end{pmatrix}\right]^3=1$, whi implies that $z=-1$. Replacing $\begin{pmatrix}-z & x\\ 0 & 1\end{pmatrix}g_M$ by g_M, we complete the proof of t lemma.

Set $\tilde{R}=\prod_{M\in\max(R)}R_M$. Then we have the natural embedding $R\hookrightarrow\tilde{R}$.

Let $g=(g_M)_{M\in\max(R)}\in GL_2(\tilde{R})$. Since the following diagram is commutative:

$$\begin{array}{ccccccc}
E_1(R) & \xrightarrow{\Lambda} & E_2(R) & \hookrightarrow & E_2(\tilde{R}) & \xrightarrow{\mathrm{Int}g} & E_2(\tilde{R})\\
 & & & \searrow & \downarrow & & \downarrow\\
 & & & & E_2(R_M) & \xrightarrow{\mathrm{Int}g_M} & E_2(R_M)
\end{array}$$

where the unmarked arrows are natural group homomorphisms, by Lemma 3, we have

$$g\left[\Lambda\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\right]g^{-1}=\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix},$$

$$g\left[\Lambda\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\right]g^{-1}=\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}.$$

Theorem 1. *Suppose that 2, 3, and $5\in U$. Let Λ be an arbitrary automorphism of $E_2(R)$. Then there exist $g\in GL_2(\tilde{R})$ and a bijection β: $R\to R$ satisfying*

$$\beta(1)=1,\ \beta(x+y)=\beta(x)+\beta(y) \text{ for all } x,\ y\in R,$$

such that

$$g\left[\Lambda\begin{pmatrix}1 & x\\0 & 1\end{pmatrix}\right]g^{-1}=\begin{pmatrix}1 & \beta(x)\\0 & 1\end{pmatrix} \text{ for all } x\in R,$$

$$g\left[\Lambda\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\right]g^{-1}=\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}.$$

Proof The matrix $g=(g_M)_{M\in\max(R)}\in GL_2(\tilde{R})$ has been given by Lemma 3. Write $g=\begin{pmatrix}a & b\\c & d\end{pmatrix}$. Since $g^{-1}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}g=\Lambda\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$ and $g^{-1}\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}g=\Lambda\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$ belong to $E_2(R)$, it is easy to see that $\frac{a^2}{\Delta}$, $\frac{b^2}{\Delta}$, $\frac{c^2}{\Delta}$, $\frac{d^2}{\Delta}$, $\frac{ab}{\Delta}$, and $\frac{cd}{\Delta}$ are elements in R, where $\Delta=ad-bc$. It follows that, for any $\begin{pmatrix}x & y\\z & w\end{pmatrix}\in E_2(R)$, the (1, 2) and (2, 1)-entries of $g\begin{pmatrix}x & y\\z & w\end{pmatrix}g^{-1}$ are elements in R.

Since $\begin{pmatrix}1 & x\\0 & 1\end{pmatrix}$ commutes with $\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$, we may assume that $g\left[\Lambda\begin{pmatrix}1 & x\\0 & 1\end{pmatrix}\right]g^{-1}=$ $\begin{pmatrix}\alpha(x) & \beta(x)\\0 & \alpha(x)\end{pmatrix}$ where $\alpha(x)\in\tilde{R}$, $\beta(x)\in R$. Clearly, $\alpha(x+y)=\alpha(x)\alpha(y)$ and $\alpha(x)^2=1$. Hence $\alpha(x)=\alpha\left(\frac{x}{2}+\frac{x}{2}\right)=\alpha\left(\frac{x}{2}\right)^2=1$ for any $x\in R$, i. e.,

$$g\left[\Lambda\begin{pmatrix}1 & x\\0 & 1\end{pmatrix}\right]g^{-1}=\begin{pmatrix}1 & \beta(x)\\0 & 1\end{pmatrix}.$$

This shows that Int g induces an automorphism of $E_2(R)$. Evidently, β: $R\to R$ is a bijection, and $\beta(1)=1$, $\beta(x+y)=\beta(x)+\beta(y)$.

Remark 1. In fact, β satisfies a stronger condition. For $x_1, \cdots, x_n\in R$ define

$$P_{-1}=0,$$
$$P_0=1,$$
$$P_1(x_1)=x_1,$$
$$P_2(x_1, x_2)=P_1(x_1)x_2-P_0,$$
$$P_n(x_1, \cdots, x_n)=P_{n-1}(x_1, \cdots, x_{n-1})x_n-P_{n-2}(x_1, \cdots, x_{n-2}).$$

It can be proved by induction that

$$\begin{pmatrix}x_1 & 1\\-1 & 0\end{pmatrix}\cdots\begin{pmatrix}x_n & 1\\-1 & 0\end{pmatrix}=\begin{pmatrix}P_n(x_1, \cdots, x_n) & P_{n-1}(x_1, \cdots, x_{n-1})\\-P_{n-1}(x_2, \cdots x_n) & -P_{n-2}(x_2, \cdots, x_{n-1})\end{pmatrix}.$$

(See [3].) Note that $\begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and

$$g\left[A\begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix}\right]g^{-1} = \begin{pmatrix} 1 & \beta(-x) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\beta(x) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \beta(x) & 1 \\ -1 & 0 \end{pmatrix}.$$

Therefore, β satisfies the following condition (*):

$$\begin{pmatrix} P_n(x_1, \cdots, x_n) & P_{n-1}(x_1, \cdots, x_{n-1}) \\ -P_{n-1}(x_2, \cdots, x_n) & -P_{n-2}(x_2, \cdots, x_{n-1}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

if and only if

$$\begin{pmatrix} P_n(\beta(x_1), \cdots, \beta(x_n)) & P_{n-1}(\beta(x_1), \cdots, \beta(x_{n-1})) \\ -P_{n-1}(\beta(x_2), \cdots, \beta(x_n)) & -P_{n-2}(\beta(x_2), \cdots, \beta(x_{n-1})) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since every relation in $E_2(R)$ can be expressed as $\prod_{i=1}^{n} \begin{pmatrix} x_i & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ some $x_1, \cdots, x_n \in R$, by Theorem 1 and the above remark, we derive the follow corollary.

Corollary. *Suppose that 2, 3, and $5 \in U$. Let A be an automorphism of $E_2($. Then there exist $g \in GL_2(\tilde{R})$ and a bijection β: $R \to R$ satisfying the condition (*) s that*

$$A\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = g^{-1}\begin{pmatrix} 1 & \beta(x) \\ 0 & 1 \end{pmatrix} g \quad \textit{for all } x \in R,$$

$$A\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = g^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g.$$

Conversely, if there is a bijection β: $R \to R$ satisfying (*), then we may define automorphism $\tilde{\beta}$ of $E_2(R)$ such that

$$\tilde{\beta}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \beta(x) \\ 0 & 1 \end{pmatrix} \quad \textit{for all } x \in R,$$

$$\tilde{\beta}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Hence, every automorphism A of $E_2(R)$ can be expressed as $A = \mathrm{Int}\, g^{-1} \circ \tilde{\beta}$.

Lemma 4. *If there is $u \in U$ with $u^2 - 1 \in U$, then*

$$[GE_2(R), GE_2(R)] = E_2(R)].$$

Proof Since $[E_2(R), E_2(R)] \subseteq [GE_2(R), GE_2(R)] \subseteq GE_2(R) \cap SL_2(R$ $E_2(R)$, it suffices to show that $E_2(R) \subseteq [E_2(R), E_2(R)$.

For any $x \in R$, take $a = x(u^2-1)^{-1}$. Then

$$[E_2(R), E_2(R)] \ni \left[\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & a(u^2-1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Similarly, $[E_2(R), E_2(R)]$ contains $\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}$ for any $x\in R$. This proves the lemma.

Theorem 2. *Suppose that* 2, 3, *and* $5\in U$. *Let* Λ *be an arbitrary automorphism of* $GE_2(R)$. *Then there exist* $g\in GL_2(\tilde{R})$, *a bijection* β: $R\to R$ *satisfying the condition* $(*)$, *and a homomorphism* γ: $U\to U$, *such that*

$$g\left[\Lambda\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right]g^{-1}=\begin{pmatrix}1 & \beta(x)\\ 0 & 1\end{pmatrix}\quad \textit{for all } x\in R,$$

$$g\left[\Lambda\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right]g^{-1}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

$$g\left[\Lambda\begin{pmatrix}v & 0\\ 0 & 1\end{pmatrix}\right]g^{-1}=\gamma(v)\begin{pmatrix}\beta(v) & 0\\ 0 & 1\end{pmatrix}\quad \textit{for all } v\in U.$$

Proof By Lemma 4, $[GE_2(R), GE_2(R)]=E_2(R)$, Λ induces an automorphism of $E_2(R)$. By Theorem 1 and its corollary, there exist $g\in GL_2(\tilde{R})$ and a bijection β: $R\to R$ satisfying $(*)$ such that

$$g\left[\Lambda\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right]g^{-1}=\begin{pmatrix}1 & \beta(x)\\ 0 & 1\end{pmatrix},$$

$$g\left[\Lambda\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right]g^{-1}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

Let $g\left[\Lambda\begin{pmatrix}v & 0\\ 0 & 1\end{pmatrix}\right]g^{-1}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$. Using the relations $\begin{pmatrix}v & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix}\begin{pmatrix}v & 0\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}v & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ v & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}v & 0\\ 0 & 1\end{pmatrix}$, we obtain $b=c=0$, $a=d\beta(v)$. Thus,

$$g\left[\Lambda\begin{pmatrix}v & 0\\ 0 & 1\end{pmatrix}\right]g^{-1}=\gamma(v)\begin{pmatrix}\beta(v) & 0\\ 0 & 1\end{pmatrix}.$$

It is easily seen that γ: $U\to U$ is a group homomorphism.

Remark 2. For a bijection β: $R\to R$ satisfying $(*)$, the automorphism $\tilde{\beta}$ of $E_2(R)$ can be extended to an automorphism of $GE_2(R)$ if we define

$$\tilde{\beta}\begin{pmatrix}v & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}\beta(v) & 0\\ 0 & 1\end{pmatrix}.$$

Therefore, when 2, 3, and $5\in U$, every automorphism Λ of $GE_2(R)$ can be expressed as $\Lambda=\Gamma\circ \operatorname{Int} g^{-1}\circ \tilde{\beta}$, where Γ is a radial automorphism of $GE_2(R)$.

Remark 3. If R is an integral domain, by using continued fractions, the condition $(*)$ may be simplified[4].

Remark 4. If R is universal for GE_2, i. e., the following relations

$$\begin{pmatrix}x & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}y & 1\\ -1 & 0\end{pmatrix}=-\begin{pmatrix}x+y & 1\\ -1 & 0\end{pmatrix},$$

$$\begin{pmatrix}u & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}u^{-1} & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}u & 1\\ -1 & 0\end{pmatrix}=-\begin{pmatrix}u & 0\\ 0 & u^{-1}\end{pmatrix},$$

$$\begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}=\begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}\begin{pmatrix} v^{-1}xu & 1 \\ -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\begin{pmatrix} u' & 0 \\ 0 & v' \end{pmatrix}=\begin{pmatrix} uu' & 0 \\ 0 & vv' \end{pmatrix}$$

form a complete set of defining relations of $GE_2(R)$ (refer to [3] or [5]), then the condition (*) can be simply expressed as

$$\beta(x+y)=\beta(x)+\beta(y) \quad \text{for all } x,\ y\in R,$$

$$\beta(ux)=\beta(u)\beta(x) \quad \text{for all } u\in U,\ x\in R.$$

The Reiner's automorphism[6] is just induced by such a bijection β.

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