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ON EULER CHARACTERISTIC OF MODULES**

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Abstract

This paper gives a characteristic property of the Euler characteristic for IBN rings. The following results: are proved. (1) If R is a commutative ring, M, N are two stable free R-modules, then $\chi(M\otimes N)=\chi(M)\chi(N)$, where χ denotes the Euler characteristic. (2) If f: $K_0(R)\to\mathbb{Z}$ is a ring isomorphism, where $K_0(R)$ denotes the Grothendieck group of R, $K_0(R)$ is a ring when R is commutative, then $f([M])=\chi(M)$ and $\chi(M\otimes N)=\chi(M)\chi(N)$ when M, N are finitely generated projective R-modules, where the isomorphism class $[M]$ is a generator of $K_0(R)$. In addition, some applications of the results above are also obtained.

§1. Introduction

Let M be a left module over a ring R. A finite free resolution of M is any exact sequence

$$0\to F_n\to F_{n-1}\to\cdots\to F_0\to M\to 0, \tag{1}$$

where each F_j is a finitely generated free R-module. If M has a finite free resolution (1), denote $M\in$FFR, and if R has the invariant basis number property, denote $R\in$IBN, the Euler characteristic of M is defined to be the number

$$\chi(M)=\sum_{j=0}^{n}(-1)\ \text{rank}\ F_j.$$

The Euler characteristic of M is independent of the choice of the finite free resolution (see[1]). If, R is a commutative ring, then $R\in$IBN and $\chi(M)\geqslant 0$ [1, Theorem 192]. But $\chi(M)\geqslant 0$ need not hold when R is a non-commutative IBN ring (see[1], p. 145).

It is well known (see[2], p. 255) that assume $R\in$IBN and

$$0\to M'\to M\to M''\to 0 \tag{2}$$

is an exact sequence of left R-modules, two of which have an Euler characteristic, then the third module has also an Euler characteristic, and

$$\chi(M)=\chi(M')+\chi(M'').$$

One of the main purpose in this paper is to give a characteristic property of

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the Euler characteristic. By this result, we give some other properties of the Euler characteristic. In addition, we give some relations between $\chi(P\otimes Q)$ and $\chi(P)$, $\chi(Q)$ for certain classes of R-modules.

All rings in the paper are supposed to be associative with unit, and all modules in the paper are supposed to be left unitary modules.

§ 2. Main Results

We shall begin with the following definition.

Definition. *Let $R\in IBN$, φ: $FFR\to\mathbb{Z}$ be a mapping which satisfies the follo condition:*

In any exact sequence (2), if two of M', M, M'' have FFR, then the third mo has also FFR, and

$$\varphi(M)=\varphi(M')+\varphi(M'').$$

In this case, we denote $\varphi\in EC$.

Now we prove the following result.

Theorem 1. *If $R\in IBN$, then φ is the Euler characteristic $\Leftrightarrow\varphi\in EC$ and $\varphi = 1$.*

Proof "$\Rightarrow$" is clear.

$\Leftarrow$: If F is a free R-module and rank $F=j$, then $F\simeq R^j$, Consider the e sequence of R-modules

$$0\to R\to R^j\to R^{j-1}\to 0,$$

then $$\varphi(R^j)=\varphi(R^{j-1})+\varphi(R)=\varphi(R^{j-1})+1.$$

By induction, we have

$$\varphi(F)=\varphi(R^j)=j-1+1=j=\text{rank } F.$$

Take any $N\in$ FFR, and let

$$0\to F_n\xrightarrow{\partial_n}F_{n-1}\xrightarrow{\partial_{n-1}}\cdots\to F_0\xrightarrow{\partial_0}N\to 0$$

be a finite free resolution of N, denote $K_j=\ker\partial_j$, $j=0, 1, \cdots, n$. Note that

$$0\to K_0\to F\to N\to 0$$

is an exact sequence of left R-modules, and $N, F_0\in$ FFR. Thus

$$\varphi(F_0)=\varphi(K_0)+\varphi(N).$$

Similarly, by the following exact sequences

$$0\to K_{j+1}\to F_{j+1}\to K_j\to 0,\ j=0, 1, \cdots, n-2,$$

noting that $F_n=\operatorname{Im}\partial_n=\ker\partial_{n-1}=K_{n-1}$, we have

$$\varphi(F_{j+1})=\varphi(K_{j+1})+\varphi(K_j),\quad j=0, 1, \cdots, n-3,$$
$$\varphi(F_{n-1})=\varphi(F_n)+\varphi(K_{n-2}).$$

Hence
$$\varphi(N)=\varphi(F_0)-\varphi(K_0)$$
$$=\varphi(F_0)-\varphi(F_1)+\varphi(K_1)=\cdots$$

$$= \sum_{j=0}^{n} (-1)^j \varphi(F_j) = \sum_{j=0}^{n} (-1)^j \operatorname{rank} F_j$$
$$= \chi(N), \ \forall N \in \text{FFR}.$$

Thus φ is the Euler characteristic.

Using Theorem 1, we can obtain the following corollaries.

Corollary 1. *If $R \in IBN$, N is a submodule of R-module M, and two of M, N, M/N have FFR, then the third module has also FFR, and*

$$\chi(M/N) = \chi(M) - \chi(N).$$

Hence $\chi(N) \leqslant \chi(M)$ when R is commutative.

Proof Noting that

$$0 \to N \to M \to M/N \to 0$$

is an exact sequence, by Theorem 1, we have

$$\chi(M/N) = \chi(M) - \chi(N).$$

If R is also commutative, then $\chi(M/N) \geqslant 0$. So $\chi(N) \leqslant \chi(M)$.

Corollary 2. *If R is a commutative ring, N is a submodule of R-module M, and two of M, N, M/N have FFR, then*

$$\operatorname{Ann}_R M \neq 0 \Leftrightarrow \operatorname{Ann}_R N \neq 0 \ \text{and} \ \operatorname{Ann}_R M/N \neq 0,$$

where $\operatorname{Ann}_R X$ is the annihilator of R-module X.

Proof By [3, p. 115, Theorem 12], if R is commutative and $X \in \text{FFR}$, then $\chi(X) \geqslant 0$, and $\chi(X) = 0 \Leftrightarrow \operatorname{Ann}_R X \neq 0$. It follows from Corollary 1, that this corollary holds.

It is well known that any submodule F_1 of a free module F_0 over PID is free, and $\operatorname{rank} F_1 \leqslant \operatorname{rank} F_0$. Now we prove a weaker result for commutative rings.

Corollary 3. *If R is a commutative ring, F_0 is a free R-module and F_1 is a free submodule of F_0, then*

$$\operatorname{rank} F_1 \leqslant \operatorname{rank} F_0,$$

and $\operatorname{rank} F_1 < \operatorname{rank} F_0 \Leftrightarrow \operatorname{Ann}_R F_0/F_1 = 0$.

Proof If $\operatorname{rank} F_0 = \infty$, it is clear that $\operatorname{rank} F_1 \leqslant \operatorname{rank} F_0$. If $\operatorname{rank} F_0 = n < \infty$, consider the exact sequence

$$0 \to F_1 \to F_0 \to F_0/F_1 \to 0,$$

then

$$\chi(F_0) = \chi(F_1) + \chi(F_0/F_1).$$

But R is commutative, so $\chi(F_0/F_1) \geqslant 0$. Thus

$$\operatorname{rank} F_1 = \chi(F_1) \leqslant \chi(F_0) = \operatorname{rank} F_0,$$

and $\operatorname{rank} F_1 < \operatorname{rank} F_0 \Leftrightarrow \chi(F_0/F_1) > 0$, i. e., $\operatorname{Ann}_R F_0/F_1 = 0$, by [3, p. 115, Theorem 12].

Corollary 4. *If $R \in IBN$, $M_j \in FFR$, $j = 1, \cdots, n$, then*

$$\chi\Big(\bigoplus_{j=1}^{n} M_j\Big) = \sum_{j=1}^{n} \chi(M_j).$$

Proof Consider the exact sequence

$$0 \to M_1 \to M_1 \oplus M_2 \to M_2 \to 0,$$

then we have

$$\chi(M_1 \oplus M_2) = \chi(M_1) + \chi(M_2).$$

By induction, we can obtain

$$\chi\Big(\bigoplus_{j=1}^{n} M_j\Big) = \sum_{j=1}^{n} \chi(M_j).$$

Now we give some results on tensor products of R-modules.

Theorem 2. *If R is a commutative ring, M, N are stable free R-modules, de $M, N \in SF_R\mathfrak{M}$, then*

$$\chi(M \otimes N) = \chi(M)\chi(N).$$

Proof Note that $SF_R\mathfrak{M} \subseteq$ FFR, so $\chi(M)$, $\chi(N)$ exist. Since M, $N \in SF_R\mathfrak{M}$ e., there exist r, s, p, q such that (see [4])

$$M \oplus R^r \simeq R^s,$$
$$N \oplus R^p \simeq R^q,$$

we have

$$(M\otimes N) \oplus (M \otimes R^p) \oplus (R^r \otimes N) \oplus R^{pr} \simeq (M\otimes N) \oplus M^p \oplus N^r \oplus R^{pr} \simeq R^{qs},$$

and

$$0 \to M\otimes N \to R^{qs} \to M^p \oplus N^r \oplus R^{pr} \to 0$$

is an exact sequence of R-modules. By Corollary 4, $\chi(M^p \oplus N^r \oplus R^{pr})$ and $\chi($ exist. Hence $\chi(M\otimes N)$ also exists, by Theorem 1. It follows from Corollary 4 tl

$$\chi(M\otimes N) = \chi(R^{qs}) - p\chi(M) - r\chi(N) - \chi(R^{pr}).$$

But $\chi(M) = s - r$, $\chi(N) = q - p$, by (3) and (4). Thus

$$\begin{aligned}\chi(M\otimes N) &= qs - pr - p(s-r) - r(q-p) \\ &= (s-r)(q-p) = \chi(M)\chi(N).\end{aligned}$$

This completes the proof of the theorem.

Now we give a relation between the Grothendieck group $K_0(R)$ (see [4]) an Euler characteristic. Note that for the commutative local rings, the Bezout ri the PID rings, the polynomial rings over PID rings in n indeterminates, the fo power series rings over fields, their the Grothendieck groups $K_0(R)$ are rings, $K_0(R) \simeq \mathbb{Z}$ (see [4], [5]). In addition, in [6], we gave the following result:

Lemma[6]. *If R is a commutative ring, f. $K_0(R) \to \mathbb{Z}$ is a ring isomorpl then the finitely generated projective R-modules are stable free R-modules.*

Using the result above, we can obtain the following result.

Theorem 3. *If R is a commutative ring, f: $K_0(R) \to \mathbb{Z}$ is a ring isomorpl and M, N are finitely generated projective R-modules, then*

$$\chi(M\otimes N) = \chi(M)\chi(N)$$

and

$$f([M]) = \chi(M),$$

where $[M]$ denotes the generator in $K_0(R)$ for the isomorphism class of M (see [4]).

Proof Since f: $K_0(R) \to \mathbb{Z}$ is a ring isomorphism, we see that finitely generated projective R-modules M, $N\in SF_R\mathfrak{M}$, by Lemma. It follows from Theorem 2 that

$$\chi(M\otimes N)=\chi(M)\chi(N).$$

In addition, by the proof of Theorem 2, if

$$M\oplus R^r\simeq R^s,$$

then $f(M)=s-r$. But $f([M])=s-r$. Hence

$$f([M])=\chi(M).$$

By Theorem 3, we can prove the following result.

Theorem 4. *If R is a commutative ring, $f: K_0(R)\to\mathbb{Z}$ is a ring isomorphism, then*

(1) *For any finitely generated projective R-module M, $f([M])\geqslant 0$, and $f([M])=0\Leftrightarrow \mathrm{Ann}_R M\neq 0$,*

(2) *If* $$0\to M_n\to M_{n-1}\to\cdots\to M_1\to M_0\to 0$$ *is an exact sequence of R-modules, and n modules of $M_n, M_{n-1}, \cdots, M_1, M_0$ are finitely generated projective R-modules, then*

$$\sum_{2|j,0\leqslant j\leqslant n}\chi(M_j)=\sum_{2\nmid j,0\leqslant j\leqslant n}\chi(M_j),$$

and hence $$\sum_{2|j,0\leqslant j\leqslant n}f([M_j])=\sum_{2\nmid j,0\leqslant j\leqslant n}f([M_j]).$$

Proof (1) By [1, Theorem 192], $\chi(M)\geqslant 0$. By [3, p. 115, Theorem 12], $\chi(M)=0\Leftrightarrow \mathrm{Ann}_R M\neq 0$. Hence (1) holds, by Theorem 3.

(2) Let $$0\to M_n\xrightarrow{d_n}M_{n-1}\xrightarrow{d_{n-1}}\cdots\to M_1\xrightarrow{d_1}M_0\to 0$$ and $K_j=\ker d_j$, $j=1, \cdots, n$. Note that $SF_R\mathfrak{M}\subseteq FFR$, and each finitely generated projective R-module is a stable free R-module, by Lemma, Hence has FFR.

If $M_1, \cdots, M_n$ are finitely generated projective R-modules, then $M_j\in$ FFR, $j=1, \cdots, n$. Thus by Theorem 1

$$0\to M_n\to M_{n-1}\to K_{n-2}\to 0 \text{ is exact}\Rightarrow K_{n-2}\in \text{FFR},$$

$$0\to K_{n-2}\to M_{n-2}\to K_{n-3}\to 0 \text{ is exact}\Rightarrow K_{n-3}\in \text{FFR},$$

....

$$0\to K_2\to M_2\to K_1\to 0 \text{ is exact}\Rightarrow K_1\in \text{FFR},$$

$$0\to K_1\to M_1\to M_0\to 0 \text{ is exact}\Rightarrow M_0\in \text{FFR}.$$

Similarly, if $M_0, \cdots, M_{n-1}$ are finitely generated projective R-modules, then $M_n\in$ FFR.

Now we assume that M_j, $j=0, \cdots, i-1, i+1, \cdots, n$, are finitely generated projective R-modules. Similarly, we see that $K_i, K_{i-1}\in$ FFR. Hence

$$0\to K_i\to M_i\to K_{i-1}\to 0 \text{ is exact}\Rightarrow M_i\in \text{FFR}.$$

From the proof above, by Theorem 1, we have

$$\chi(M_1)=\chi(K_1)+\chi(M_0),$$

$$\chi(M_j)=\chi(K_j)+\chi(K_{j-1}),\quad j=2, \cdots, n-2.$$

$$\chi(M_{n-1})=\chi(M_n)+\chi(K_{n-2}),$$

hence
$$\sum_{2|j,\, 0\leqslant j\leqslant n} \chi(M_j) = \sum_{2\nmid j,\, 0\leqslant j\leqslant n} \chi(M_j).$$

Thus, by Theorem 3, we have
$$\sum_{2|j,\, 0\leqslant j\leqslant n} f([M_j]) = \sum_{2\nmid j,\, 0\leqslant j\leqslant n} f([M_j]),$$
which completes the proof.

By the proof above, we obtain immediately the following corollary.

Corollary 5[2]. *If $R \in IBN$ and*
$$0 \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 \to 0$$
is an exact sequence of R–modules, n modules of $M_n, \cdots, M_0$ have FFR, then $M_j \in$ FF $j=0, 1, \cdots, n$, *and*
$$\sum_{2\nmid j,\, 0\leqslant j\leqslant n} \chi(M_j) = \sum_{2\nmid j,\, 0\leqslant j\leqslant n} \chi(M_j).$$

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