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ON THE GERM-MARKOV PROPERTY OF THE GENERALIZED N-PARAMETER ORNSTE IN-UHLENBECK PROCESS

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Abstract

A generalized N-parameter Ornstein-Uhlenbeck process (GOUP_N) is defined as

$$X(t)=[f(t)]^{-1}\int_a^t f(s)\eta(ds),$$

$t\in R_a^N$, where $a=(0,\cdots,0)$ or $(-\infty,\cdots,-\infty)$, correspondently $R_a^N=R_+^N$ or R^N, and $\eta(ds)$ is the standard Gaussian orthogonal random measure and f is an infinitely differentiable and locally quadratically integrable positive function. In this paper it is proved that the GOUP_N has the so called germ-Markov property with respect to any bounded domain, and two examples are given which say that for spherical and some pyramid-like domains, the minimal splitting σ-algebras for the "interior" and the "outer" information σ-algebras are strictly "larger" than the boundary information σ-algebras.

§1. Introduction

Prof. Wang Zikun[1] defined the 2-parameter Ornstein-Uhlenbeck proce and studied its Markov and strong Markov properties. According to [3] and [4 the OUP_2 has the germ-Markov property with respect to any open sets in R_+^2. Th paper is to study the germ-Markov property of the generalized N-parameter OUP.

For $t\in R^N$, we always think of its ith component as t_i, $i=1,\cdots,N$. For $s,t\in R^N$ $s\leqslant t$ means $s_i\leqslant t_i$, $i=1,\cdots,N$, $s<t$ means $s_i<t_i$, $i=1,\cdots,N$; and $[s,t]=\{u\in R^N$ $s\leqslant u\leqslant t\}$. Write $R_b^N=\{t\in R^N,\ t\geqslant b\}$. Suppose $a=(0,\cdots,0)\in R^N$ or $=(-\infty,\cdots,-\infty)$, and $\eta(dt)$ is the standard Gaussian orthogonal random measure on R ("standard" means $\mathrm{E}\,\eta(dt)=0$ and $\mathrm{E}|\eta(dt)|^2=dt$). Let

$$X(t)=\frac{1}{g(t)}\int_a^t f(s)\eta(ds),\ t\in R_a^N, \tag{1.1}$$

where f is a quadratically integrable real-valued function on $[a,t]$, $t\in R_a^N$ and g i a positive real-valued Borel measurable function in R_a^N. We call the rando process X a generalized N-parameter Ornstein-Uhlenbeck process (briefly GOUP_N).

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When $a=(0, \cdots, 0)$ and $f(s)=f_1(s_1)\cdots f_N(s_N)$, X is the extended OUP_N in [5] which proves that the extended OUP_2 has the germ-Markov property with respect to any domains with piecewise-smooth boundaries. If

$$f(s)=\exp(\alpha s), \quad \alpha s=\sum_{i=1}^{N}\alpha_i s_i,$$

$\alpha_i>0$, and $g(s)=f(s)$, X is the OUP_N, in particular, if $a=(-\infty, \cdots, -\infty)$, X is stationary, and if $N=2$, $a=(0, 0)$, X is the OUP_2 in [2]. If $f=g=1$ and $a=(0, \cdots, 0)$, X is the N-parameter Wiener process. With the help of the Markovian theory of generalized random functions in [6], we prove in this paper that the $GOUP_N$ X, under some conditions on g and f, has the germ-Markov property with respect to any open sets which are bounded or have bounded complements in R_a^N.

For the germ-Markov theory of multiparameter stochastic process, to make clear the relationship between the minimal splitting σ-algebra with respect to an open set and the information σ-algebra in its boundary arouses much interest. As concerns Brownian Sheet, [6] says that for the triangle domain surrounded by the axes and a straight line with a negative slope, the later is strictly "less than" the former. In section 4 of this paper it is shown that for any sphere in R_a^N and some pyramid-like domains, the $GOUP_N$ X also has this prorerty.

§2. Preliminaries

For convenience we will display several concepts (ref. [6]). Let $(\Omega, \mathscr{F}, P)$ be our basic probability space. σ-Subalgebras $\mathscr{A}_1$, $\mathscr{B}$, $\mathscr{A}_2$ of $\mathscr{F}$ are called a Markovian system if $\mathscr{B}$ is splitting for $\mathscr{A}_1$ and $\mathscr{A}_2$. The definitions of splitting and minimal splitting algebras are refered to [6, § 2.1].

Suppose D is an open set in R^N and $Y(t)$, $t\in D$ a quadra-tically integrable real-valued random process. For open $S\subset D$, let

$$H(S)=\vee_{t\in S}Y(t), \; H_+(\Gamma)=\cap\{H(S), \text{ open } S\supset\Gamma\}, \; \Gamma\subset D.$$

$H(S)$ is the closed linear subspace generated by $Y(t)$, $t\in S$ in $L_2(\Omega, \mathscr{F}, P)$. Let

$$\mathscr{A}(S)=\sigma\{\eta, \eta\in H(S)\}, \; \mathscr{A}_+(\Gamma)=\cap\{\mathscr{A}(S), \text{ open } S\supset\Gamma\}, \; \Gamma\subset D.$$

$H(S)$, $S\subset D$ and $\mathscr{A}(S)$, $S\subset D$ are all called random fields.

Denote by $C_0^\infty(D)$ the space of infinitely differentiable functions $u=u(t)$, $t\in D$, with compact support Supp $u\subset D$, which is endowed a suitable topology (ref. [6, p. 11]). Let $\xi=(u, \xi)$, $u\in C_0^\infty(D)$ be a generalized random function. For open $S\subset D$, let $H(S)=\vee_{\text{supp} u\subset S}(u, \xi)$. Also we define $H_+(\Gamma)$, $\mathscr{A}(S)$ and $\mathscr{A}_+(\Gamma)$, $\Gamma\subset D$ as before. $H(S)$, $S\subset D$ and $\mathscr{A}(S)$, $S\subset D$ are also called random fields generated by ξ.

Suppose $Y=Y(t)$, $t\in D$ is a quadratically integrabl real-valued random process on D. Let

$$\xi = (u, \xi) = \int_D u(t) Y(t)\, dt,\ u \in C_0^\infty(D).$$

Lemma 1. *If Y is weakly continuous, i. e., $t \to EY^2(t)$ is continuous, Y and ξ generate the same random feld $H(S)$, $S \subset D$.*

The Proof of Lemma 1 is easy. We omit it.

Let $\mathscr{A}(S)$, $S \subset D$, corresponding to $H(S)$, $S \subset D$, be a random field generated by a random function Y (or a generalized random function ξ). Y (or ξ) is said to have germ-Markov property with respect to an open set $S \subset D$ if $\mathscr{A}(S)$, $\mathscr{A}_+(\partial$ $\mathscr{A}(D \backslash \bar{S})$ is a Markov system. If Y (or ξ) is Gaussian, for any closed subspace I $H(D)$ we have

$$E\{\eta \mid \sigma(H)\} = P(H)\eta, \quad \sigma(H) \equiv \sigma\{\eta,\ \eta \in H\},$$

where $P(H)$ is the orthogonal projection operator on H, and for closed subspace and H_2 that $\sigma(H)$ splits $\sigma(H_1)$ and $\sigma(H_2)$ is equivalent to

$$\eta_1 - P(H)\eta_1 \perp \eta_2 - P(H)\eta_2,\ \eta_i \in H_i. \tag{2}$$

Hence if (2.1) is true, we also call H_1, H, H_2 a Markov system. Also we have concepts of splitting and minimal splitting spaces.

§ 3. Germ-Markov Property

Recall (1.1). Write $H(S) = \vee_{t \in S} X(t)$, open set $S \subset R^N$.

Theorem 3. 1. *Suppose f is an infinitely differentiable positive real-val function on R_a^N with*

$$\int_a^t f^2(s)\, ds < \infty$$

and $f^{-1} \equiv 1/f$ bounded in $[t, \infty)$, $t \in R_a^N$, and suppose $g = f$. Then the $GOUP_N$ X has germ-Markov property with respect to any open sets which are bounded or have boun complemenets in R_a^N. Furthermore, $H_+(\partial S)$ is the minimal splitting space of $H(S)$ $H(R_a^N \backslash \bar{S})$ and consists of variables having the following form

$$\eta = \int_{R_a^N} u(t)\, \eta(dt),$$

where $u \in L_2(R_a^N) \equiv L_2(R_a^N, \mathscr{B}_a^N, dt)$ is a generalized solution of the differential equa

$$L_0^* u(t) = 0,\ t \in D \backslash \partial S, \tag{3}$$

where for $u \in C_0^\infty(R_a^N)$,

$$L_0^* u(t) = (-1)^N f(t) \frac{\partial^N}{\partial t_1 \cdots \partial t_N} [f^{-1}(t) u(t)]. \tag{3}$$

Remark. L_0^* is the adjoint operator of the continuous linear operator $C_0^\infty(R_N^a) \to L_2(R_a^N)$ defined by

$$L_0 u(t) = f^{-1}(t) \frac{\partial^N}{\partial t_1 \cdots \partial t_N} [f(t) u(t)],\ u \in C_0^\infty(R_a^N). \tag{3.3}$$

Proof Define a continuous linear operator J_0: $C_0^\infty(R_a^N)\to L_2(R_a^N)$ by

$$J_0u(t)=f(t)\int_t^\infty u(s)f^{-1}(s)\,ds,\ t\in R_a^N.$$

Let

$$Ju=\int_a^\infty J_0u(s)\eta(ds),\ u\in C_0^\infty(R_a^N).$$

Then

$$\begin{aligned}Ju&=\int_a^\infty f(s)\left[\int_s^\infty u(t)f^{-1}(t)\,dt\right]\eta(ds)\\&=\int_a^\infty u(t)f^{-1}(t)\left[\int_a^t f(s)\eta(ds)\right]dt\\&=\int_a^\infty u(t)X(t)\,dt.\end{aligned}$$

So by Lemma 1, the random field generated by the generalized random function Ju, $u\in C_0^\infty(R_a^N)$ is the same as that by the GOUP$_N$ $X=X(t)$, $t\in R_a^N$. According to the theory of Markov transformations of white noise in [6, § 3.5], to prove the theorem, we need only to verify:

(a) $J_0L_0^*u=L_0^*J_0u=u$, $u\in C_0^\infty(R_a^N)$;

(b) L_0u, $u\in C_0^\infty(R_a^N)$ is dense in $L_2(R_a^N)$;

(c) For any $w\in C_0^\infty(R_a^N)$ there exists a constant $C(w)$ depending only on w, such that

$$\|L_0(w\cdot u)\|\leqslant C(w)\|L_0u\|,\ u\in C_0^\infty(R_a^N),\tag{3.4}$$

where $(w\cdot u)(t)=w(t)\cdot u(t)$, $t\in R_a^N$.

Claim (a).

$$\begin{aligned}J_0L_0^*u(t)&=f(t)\int_t^\infty f^{-1}(s)L_0^*u(s)\,ds\\&=f(t)\int_t^\infty(-1)^N\frac{\partial^N}{\partial s_1\cdots\partial s_N}[f^{-1}(s)u(s)]\,ds=u(t).\end{aligned}$$

Similarly $L_0^*J_0u(t)=u(t)$.

Claim (b). At first we check that $\mathscr{H}\equiv\{(fL_0)u,\ u\in C_0^\infty(R_a^N)\}$ is dense in $L_2(R_a^N)$, where

$$(fL_0)u(t)=f(t)L_0u(t)=\frac{\partial^N}{\partial t_1\cdots\partial t_N}[f(t)u(t)].$$

Because f is infinitely differentiable and positive, we need only to check that

$$\left\{\frac{\partial^N u}{\partial t_1\cdots\partial t_N},\ u\in C_0^\infty(R_a^N)\right\}$$

is dense in $L_2(R_a^N)$. It is not difficult to see that this set contains

$$C_{00}^\infty(R_a^N)\equiv\left\{u:\ u=u_1\cdots u_N,\ u_i\in C_0^\infty(R_{a_i}^1)\text{ and}\int u_i(t_i)\,d_i=0,\ 1\leqslant i\leqslant N\right\}.\tag{3.5}$$

Because the set of functions in $L_2(R_a^N)$ with compact supports is dense in $L_2(R_a^N)$, by the following act we fcan affirm that $C_{00}^\infty(R_a^N)$ is dense in $L_2(R_a^N)$. Therefore $\mathscr{H}$

is dense in $L_2(R_a^N)$.

FACT. Suppose $g_0 \in L_2(R_a^N)$ with support Supp $g_0 \subset [S, T]$, S, $T \in R_a^N$, $S<T$. Then there exist $g_n \in C_{00}^\infty(R_S^N)$, $n \geqslant 1$, such that $g_n \xrightarrow{L_2} g_0$ as $n \to \infty$. (As for the proof, one only needs to appraximate functions of the form

$$g_0 = \prod_{i=1}^{N} I_{[s_i, t_i]}$$

with functions in $C_{00}^\infty(R_a^N)$. So the problem can be turned into one dimensional ca

Suppose $g \in L_2(R_a^N)$. Given $\varepsilon > 0$, there exist S, $T \in R_a^N$, $S<T$ such that

$$\int_{R_a^N/[S,T]} g^2(t)\,dt < \varepsilon. \qquad ($$

Let $g_1(t) = f(t) g(t) I_{[S,T]}$, I is the indicator function of $[S, T]$. Write

$$M = \max\{f^{-2}(t),\ t \geqslant S\}.$$

It follows from what we have obtained and the above fact that there exist h_n (Supp $h_n \subset [S, \infty)$, $n=1, 2, \cdots$, such that for sufficiently large n

$$\int_{R_a^N} [h_n(t) - g_1(t)]^2\,dt$$
$$= \int_{[S,T]} [h_n(t) - g_1(t)]^2\,dt + \int_{R_a^N [S,T]} h_n^2(t)\,dt < \varepsilon/(1+M). \qquad ($$

Hence from (3.6)—(3.7), we have

$$\int_{R_a^N} [f^{-1}(t) h_n(t) - g(t)]^2 dt$$
$$\leqslant \int_{[S,T]} [f^{-1}(t) h_n(t) - g(t)]^2\,dt + 2\left\{\int_{R_a^N/[S,T]} g^2(t)\,dt\right.$$
$$\left. + M \int_{[S,\infty)\setminus[S,T]} h_n^2(t)\,dt < 4\varepsilon. \right.$$

Therefore $f^{-1}\mathscr{H} = \{L_0 u,\ u \in C_0^\infty(R_a^N)$ is dense in $L_2(R_a^N)\}$.

Claim (c). Fix $w \in C_0^\infty(R_a^N)$. Suppose Supp $w \subset [b, T]$, b, $T \in R_a^N$, $b<T$. By (we have

$$L_0(u \cdot w)(t) = f^{-1}(t) \frac{\partial^N}{\partial t_1 \cdots \partial t_N} [w(t) f(t) u(t)]$$
$$= f^{-1}(t) \sum \frac{\partial^k w(t)}{\partial t_{i_1} \cdots \partial t_{i_k}} \cdot \frac{\partial^{N-k}[f(t) u(t)]}{\partial t_{j_1} \cdots \partial t_{j_{N-k}}}.$$

Write

$$h_{j_1 \cdots j_{N-k}}(t) = \frac{\partial^{N-k}}{\partial t_{j_1} \cdots \partial t_{j_{N-k}}} [f(t) u(t)].$$

Then by (3.8),

$$\|L_0(u \cdot w)\|^2 = \int_{[b,T]} [L_0(u(t) w(t))]^2 dt$$
$$\leqslant \sum{}' C_{i_1 \cdots i_k}(w) C_{i'_1 \cdots i'_k}(w) \int_{[b,T]} h_{j_1 \cdots j_{N-k}}(t) h_{j'_1 \cdots j'_{N-k}}(t)\,dt, \qquad (3.9)$$

where

$$G_{m_1\cdots m_k}(w)=\max\left\{\left|f^{-1}(t)\frac{\partial^k w(t)}{\partial t_{m_1}\cdots\partial t_{m_k}}\right|^2,\ t\in[b,\ T]\right\}.$$

we have proved that for any $h_{j_1\cdots j_{N-k}}$ there exists constant $C(w)$ such that

$$\int_{[b,T]} h^2_{j_1\cdots j_{N-k}}(t)\,dt\leqslant C(w)\,\|L_0u\|^2,\ u\in C_0^\infty(R_a^N),\tag{3.10}$$

ien by Cauchy-Schwarz inequality, so does every integral in (3.9), and thus (3.4) justified. We are going to prove (3.10) for $h_{k+1,\cdots,N}$. Other cases can be treated milarly. It follows from (3.3) that

$$\frac{\partial^N[f(t)u(t)]}{\partial t_{k+1}\cdots\partial t_N}=\int_{a_1}^{t_1}\cdots\int_{a_k}^{t_k}(fL_0u)(s_1,\ \cdots,\ s_k,\ t_{k+1},\ \cdots,\ t_N)\,ds_1\cdots ds_k.$$

) by Cauchy-Schwarz inequality we have

$$\begin{aligned}\int_{[b,T]} h^2_{k+1,\cdots,N}(t)&\leqslant C_1(w)\int_{[b,T]}\left\{\int_{a_1}^{t_1}\cdots\int_{a_k}^{t_k}[L_0u(s_1,\ \cdots,\ s_k,\ t_{k+1},\ \cdots,\right.\\ &\qquad \left. t_N)]^2\,ds_1\cdots ds_k\right\}dt_1\cdots dt_N\\ &\leqslant C_1(w)\prod_{i=1}^{k}(T_i-b_i)\,\|L_0u\|^2,\end{aligned}$$

here $C_1(w)=\max\left\{\int_{a_1}^{t_1}\cdots\int_{a_k}^{t_k}f^2(s_1,\ \cdots,\ s_k,\ t_{k+1},\ \cdots,\ t_N)\,ds_1\cdots ds_k,\ (t_1,\ \cdots,\ t_N)\in[b,\ T]\right\}.$

§4. Discussion for Minimal Splitting Spaces

Suppose open set $S\subset R_a^N$ is bounded or has a bound complement in R_a^N. Theorem 1 says that for the GOUP$_N$ X, the minimal splitting space of $H_+(S)$ and $H_+(R_a^N\backslash$ is $H_+(\partial S)$. Naturally one would ask what is the relationship between $H_+(\partial S)$ id the boundary space $H(\partial S)=\vee_{t\in\partial S}X(t)$, and whether $H(\partial S)$ is the minimal litting space of $H(S)$ and $H(R_a^N\backslash\bar{S})$. Generally speaking, the answer is No. We ill give two examples in this section.

Example 1 Suppose $N=2$. Let f be a function on R_a^N satisfying the condition ven in Theorem 3.1 and having the form $f(s_1,\ s_2)=f_1(s_1)f_2(s_2)$, $(s_1,\ s_2)\in R_a^N$. Let $\subset R_a^N$ be the circular domain with center $Z=(z_1,\ z_2)$ and radius r. Then there ists $u\in L_2(R_a^2)$ satisfying (3.1), such that for variable

$$\zeta=\int_{R_a^2}u(t)\,\eta(dt),$$

e have

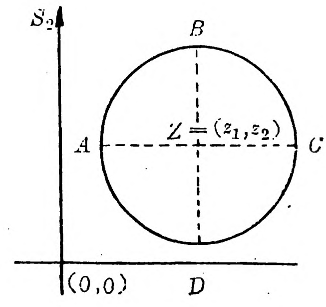
(i) $\zeta\in H_+(\partial S)$;

(ii) $E\zeta^2\neq0$ and $\zeta\notin H(\partial S)$, and so the minimal splitting space of $H_+(\bar{S})$ and $_+(R_a^N/\bar{S})$ is strictly "larger than" the boundary space $H(\partial S)$;

(iii) $H(\partial S)$ is not even a splitting space of $H(S)$ and $H(R_a^N\backslash\bar{S})$.

Proof We only have to prove (i)—(iii) for the case $a=(0,\ 0)$. The other case can be treated similarly. We may assume $r=1$. For any $u_1, u_2 \in C^1([0, \infty))$, let

$$u(t)=\begin{cases} f(t)\,[u_1(t_1)-u_2(t_2)], & t=(t_1, t_2)\in S, \\ 0, & t\in R_+^2\backslash S. \end{cases} \qquad (4.1)$$



Then u is a generalized solutin of (3.1) for $N=2$, and so

$$\zeta=\int u(t)\eta(dt)\in H_+(\partial S)$$

by Theorem 3.1, (i) is verified. Next we shall choose suitable u_i such that ζ is independent of $H(\partial S)$, i. e.,

$$g(t)\equiv E\zeta X(t)=f^{-1}(t)\int_0^t f(s)u(s)\,ds=0,\quad t\in\partial S. \qquad ($$

Our task is to find conditions on u_i, $i=1, 2$, which make the third equality in (true.

See the figure. For $t\in\widehat{AD}$, (4.2) is always true.

(a) For $t=(t_1, t_2)\in\widehat{AB}$, Set $\beta(s)=\sqrt{1-(z_1-s)^2}$. If $g(t)=0$, then

$$\int_{z_1-1}^{t_1}\int_{z_2-\beta(s_1)}^{z_2+\beta(s_1)} f_1^2(s_1)f_2^2(s_2)\,[u_1(s_1)-u_2(s_2)]\,ds_1ds_2=0. \qquad ($$

Differentiate with respect to t_1 to obtain for $z_1-1\leqslant t_1\leqslant z_1$,

$$u_1(t_1)=\int_{z_2-\beta(t_1)}^{z_2+\beta(t_1)} f_2^2(s_2)\,u_2(s_2)\,ds_2\Big/\int_{z_2-\beta(t_1)}^{z_2+\beta(t_1)} f_2^2(s_2)\,ds_2. \qquad ($$

(b) For $t\in\widehat{CD}$. If $g(t)=0$, we have

$$\int_{2z_1-t_1}^{t_1} ds_1\int_{z_2-\beta(s_1)}^{z_2-\beta(t_1)} f_1^2(s_1)f_2^2(s_2)\,[u_1(s_1)-u_2(s_2)]\,ds_1ds_2=0. \qquad ($$

Differentiate (4.5) and use $\beta(t_1)=\beta(2z_1-t_1)$ to obtain

$$\int_{2z_1-t_1}^{t_1} f_1^2(s_1)\,u_1(s_1)\,ds_1=\int_{2z_1-t_1}^{t_1} f_1^2(s_1)\,ds_1 u_2[Z_2-\beta(t_1)]. \qquad ($$

Differentiating (4.6) and making some arrangement, we get

$$\begin{aligned} &f_1^2(t_1)\,u_1(t_1)-f_1^2(2z_1-t_1)\,u_1(2z_1-t_1) \\ &=[f_1^2(t_1)-f_1^2(2z_1-t_1)]\,u_2[z_2-\beta(t_1)]. \end{aligned} \qquad ($$

Because $2z_1-t_1\leqslant z_1$, from (4.7) and (4.4), we have

$$\begin{aligned} u_1(t_1)=f_1^{-2}(t_1)\{&f_1^2(2z_1-t_1)\,u_1(2z_1-t_1)+[f_1^2(t_1)-f_1^2(2z_1-t_1)] \\ &\cdot u_2[z_2-\beta(t_1)]\},\quad z_1\leqslant t_1\leqslant z_1+1. \end{aligned} \qquad ($$

(c) For $t\in\widehat{BC}$ let $g(t)=0$. Then

$$\int_{z_1-1}^{2z_1-t_1}\int_{z_2-\beta(s_1)}^{z_2+\beta(s_1)}+\int_{2z_1-t_1}^{t_1}\int_{z_2-\beta(s_1)}^{z_2+\beta(t_1)} f^2(s)\,[u_1(s_1)-u_2(s_2)]=0. \qquad ($$

Because $2z_1-t_1\leqslant z_1$, by (4.3) we know that the first integral in (4.9) equals 0. By

$$[z_2-\beta(s_1),\ z_2+\beta(t_1)]=[z_2-\beta(s_1),\ z_2-\beta(t_1)]\cup[z_2-\beta(t_1),\ z_2+\beta(t_1)]$$

and by (4.9) and (4.5), we can obtain

$$\int_{2z_1-t_1}^{t_1}\int_{z_2-\beta(t_1)}^{z_2+\beta(t_1)} f_1^2(s_1)f_2^2(s_2)[u_1(s_1)-u_2(s_2)]ds_1ds_2=0. \tag{4.10}$$

follows from (4.10) and (4.6) that

$$u_2[z_2-\beta(t_1)]\int_{z_2-\beta(t_1)}^{z_2+\beta(t_1)} f_2^2(s_2)\,ds_2=\int_{z_2-\beta(t_1)}^{z_2+\beta(t_1)} f_2^2(s_2)\,u_2(s_2)\,ds_2.$$

ut $t_2=\beta(t_1)$ and differentiate with respect to t_2 to obtain

$$u_2'(z_2-t_2)\int_{z_2-t_2}^{z_2+t_2} f_2^2(s_2)\,ds_2=f_2^2(z_2+t_2)[u_2(z_2-t_2)-u_2(z_2+t_2)]. \tag{4.11}$$

he equations (4.11), (4.4) and (4.8) are just the conditions which $u_i(t_i)$ should tisfy. The existence of solutions of these equations is clear. For instance, we can oose $u_2(t_2)$ as a solution of the following system of equations:

$$\begin{cases} u_2(z_2-t_2)-u_2(z_2+t_2)=\int_{z_2-t_2}^{z_2+t_2} f_2^2(s_2)\,ds_2, \\ u_2'(z_2-t_2)=f_2^2(z_2+t_2), \end{cases} \quad 0\leqslant t_2\leqslant 1,$$

nd then determine u_1 by (4.4) and (4.8).

Choose u_i, $i=1, 2$, satisfying equations (4.4), (4.8) and (4.11). Reversing the rocedure from (a)—(c), we can see that u defined by (4.1) satisfies the equation 4.2). So ζ is independent of $H(\partial S)$. But clearly

$$E\xi^2=\int u^2(t)\,dt\neq 0,$$

ence $\zeta\notin H_+(\partial S)$, and (ii) is verified. To check (iii), let S_α be the circular domain ith center $Z=(z, z_2)$ and radius α. For chosen $u_i(t_i)$ let

$$u_\alpha(t)=\begin{cases} f(t)\{u_1[z_1+\alpha(t_1-z_1)]-u_2[z_2+\alpha(t_2-z_2)], & t\in S_\alpha, \\ 0, & t\in R_+^2\backslash S_\alpha. \end{cases}$$

$$\zeta_\alpha=\int_{R_+^2} v_\alpha(t)\eta\,(dt).$$

t can be seen as in the beginning of this proof that $\zeta_\alpha\in H_+(\partial S_\alpha)$. So when $\alpha>1$ and $S_\alpha\subset R_+^2$, $\zeta_\alpha\in H(R_+^2\backslash\bar{S})$; when $\alpha<1$, $\zeta_\alpha\in H(S)$. But

$$E(\zeta_\alpha-\zeta)^2=\int[v_\alpha(t)-u(t)]^2dt\to 0 \text{ as } \alpha\to 1.$$

o $\zeta=\lim\limits_{\alpha\to 1}\zeta_\alpha\in H(S)\cap H(R_+^2\backslash\bar{S})$. If $H(\partial S)$ splits $H(S)$ and $H(R_+^2\backslash\bar{S})$, then

$$E\{\zeta\cdot\zeta|H(\partial S)\}=E\{\zeta|H(\partial S)\}E\{\zeta|H(\partial S)\}. \tag{4.12}$$

But we have proved that ζ is independent of $H(\partial S)$, so $E\{\zeta|H(\partial S)\}=0$. Hence by (4.12), $E\zeta^2=E\{\zeta^2|H(\partial S)\}=0$. This is a contradiction and we are done.

Remark. The results in Example 1 can be extended to the case $N>2$ without ubstantial difficulty.

Suppose $b\in R_+^N$. Denote by $[\![b,h]\!]$ any simply-connected closed domain surrounded by all hyperplanes which pass b and are parallel to coordinate planes and by a hypersurface h which is a continuous function over a domain

$$G\subset R_b^{N-1}=\{(t_1, \cdots, t_{N-1}),\ t_i\geqslant b_i,\ i=1, \cdots, N-1\}.$$

Example 2 Let f be a function on R_a^N which satisfies the condition in Theorem 3.1 and has the form $f(t_1, \cdots, t_N) = f_1(t_1, \cdots, t_{N-1}) f_2(t_N)$. Let S be the interior of a bounded domain of the form $[\![b, h]\!]$ contained in R_b^N. Suppose h has continuous partial derivatives $\frac{\partial^{N-1}h(t)}{\partial t_1 \cdots \partial t_{N-1}}$, $t \in G$, and for any $t \in S$, $[b, t] \subset S$ (e. g. it $N=2$, h is a nonincreasing differentiable curve with finite length). Let $u_2(t_N)$, $t_N \geqslant a_N$ be a real-valued function which is differentiable and not a constant. Put

$$u_1(t_1, \cdots, t_{N-1}) = \frac{1}{f^2(t_1, \cdots, t_{N-1})} \frac{\partial^{N-1}}{\partial t_1 \cdots \partial t_{N-1}} \int_{b_1}^{t_1} \cdots \int_{b_{N-1}}^{t_{N-1}} \int_{b_N}^{h}$$

$$f^2(s_1, \cdots, s_N) u_2(s_N) ds_1 \cdots ds_N \Big/ \int_{b_N}^{h} f_2^2(s_N) ds_N,$$

$$u(t) = \begin{cases} f(t)[u_1(t_1, \cdots, t_{N-1}) - u_2(t_N)], & t = (t_1, \cdots, t_N) \in S \\ 0, & t \in R_a^N \setminus S, \end{cases}$$

$$\zeta = \int_{R_a^N} u(t) \eta(dt).$$

Then we have the same results as (i)—(ii) in Example 1. Furthermore if for fixed $(t_2, \cdots, t_N)$, $h(\cdot, t_2, \cdots, t_N)$ is a strictly decreasing function, we also hav result (iii).

Thh proof can be finished similarly as we have done in Example 1, and much easier.

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