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FUZZY STONE-ČECH COMPACTIFICATIONS AND THE LARGEST TYCHONOFF COMPACTIFICATIONS**

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Abstract

Using the imbedding theory[6] and the N-compactness of L-fuzzy unit interval[10], the authors establish the Stone-Čech compactification theory of Tychonoff spaces. As well known, the Stone-Čech compactification in general topology is the largest compactification of all the Tychonoff compactifications. But this important property is not true in fuzzy topology. The process of the argument of this negative result is very helpful for establishing a more reasonable Stone-Čech compactification theory[12]. Moreover, as relative results, the metrization theorem of induced spaces and the structure of quasi-Boolean lattice seem to have independent interest.

After proving that each L-fuzzy unit interval is N-compact[10], we can directly ;ablish the theorem of Stone-Čech compactifications of Tychonoff spaces via the ıbedding theory[6]. But as well known, the Stone-Čech compactification has an ıportant property: it is the largest compactification in all the Tychonoff mpactifications. In fact, this property is concerned with a kind of extensions of ıppings on the Stone-Čech compactification, so its investigation is always tractive. Now what is the situation in fuzzy topology? In this paper we will point .t that for remarkably many value fields L and a large kind of fuzzy Tychonoff aces, the corresponding Stone-Čech compactification is not the largest one. The ocess of the argument of this negative result is very helpful for establishing a ore reasonable new type Stone-Čech compactification theory. Moreover, both the etrization theorem of induced spaces and the quasi-Boolean lattice structures which e involved in the investigation of this paper seem to possess independent interests.

In this paper, I denotes a fuzzy lattice, i. e. a completely distributive lattice .th an order reversing involution "′", its largest element and smallest one are noted by 1 and 0 respectively. Let $M(L)$ denote the set of all the union-irreducible ınzero elementes (p is irreducible if $p\leqslant a\vee b\Rightarrow p\leqslant a$ or $p\leqslant b$) of L. (L^X, η) denotes

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an L-fuzzy topological space (fts for short), where $\eta \subset L^X$ is closed for finite union and arbitrary intersection, that is to say, η is the family of all the closed sets, it is called a co-topology on L^X (because it is natural to define the concepts of N-compactness and so on with closed sets, we adopt co-topology instead of topology). A closed set F is called an R-neighborhood of a fuzzy point x_a if $x_a \overline{\in} F$ (i. e., $a \not\leqslant F(x)$). The family of all the R-neighborhoods of fuzzy point x_a is denoted by $\eta(x_a)$. When the concepts of fuzzy points and fuzzy sets are involved, we often call them points, sets directly. Usually, we also identify crisp sets with their characteristic func which take values on $\{0, 1\} \subset L$. For $A \in L^X$, supp $A = \{x \in X: A(x) > 0\}$.

§ 1. Preliminaries

At first, let us briefly recall some concepts and results relative to N-com ness and the imbedding theory.

Definition 1. *Let $a \in L$, $D \subset L$. D is called a minimal set relative to a, if ' a and for each set $A \subset L$ satisfying $\bigvee A \geqslant a$ and for each $d \in D$, there exists $b \in A$ that $d \leqslant b$.*

As shown in [5], for each completely distributive lattice L and each elem of L, there exists a minimal set in L relative to a.

Definition 2. *For each $a \in L$, let $\beta(a)$ denote the union of all the minimal a, let $\beta^*(a) = \beta(a) \cap M(L)$.*

Obviously, $\beta(a)$ is the largest minimal set relative to a. As indicated b following lemma, $\beta^*(a)$ is also a minimal set relative to a.

Lemma 1[18]. *If $a \in L \backslash \{0\}$, then $\bigvee \beta^*(a) = a$. Furthermore, if $a \in M(L)$, according to the order of L, $\beta^*(a)$ is a directed set.*

Definition 3[18,14]. *In (L^X, η), let $A \in L^X$, $\alpha \in M(L)$, $\Phi \subset \eta$. Φ is called a of α-R-neighborhoods of A, if for each $x_\alpha \in A$, there exists a $p \in \Phi \cap \eta(x_\alpha)$. Φ is c family of α^--R-neighborhoods of A, if there exists a $\delta \in \beta^*(\alpha)$ such that Φ is a fa δ-R-neighborhoods of A. A is called N-compact, if for each $\alpha \in M(L)$ and each Φ of α-R-neighborhoods of A, there exists a finite subfamily Φ_0 of Φ which is a of α^--R-neighborhoods of A. (L^X, η) is called N-compact, if the set X is N-com*

Note. The above-mentioned N-compactness preserves the nice properties N-compactness defined in [17] (where the value field L is $I = [0, 1]$); for ex it is closed-hereditary, preserved under continuous mappings; a product of N-c subsets (especially N-compact spaces) is also N-compact, etc.[18].

When the value field is I, it is already shown in [7] that each Tychon (see Definition 6 below) (I^X, η) has a compactification contained in the fuzzy unit cube $I(I)^T$; this result can be extended to the general case that the value field is L.

Definition 4. (L^Y, μ) *is called a compactification of* (L^X, η), *if there exists an* *bedding mapping* c: $(L^X, \eta) \to (L^Y, \mu)$ *such that* $\overline{c(X)}$ *is N-compact and* supp $\overline{c(X)}$ *Y; In the sequel we let* cX, $c\eta$ *denote* Y, μ *respectively. Moreover, if* $\overline{c(X)} = cX$, *we* $(L^{cX}, c\eta)$ *a space-compactification.*

Definition 5. *For each family* $\mathscr{A} = \{f_t : t \in T\}$ *of crisp mappings, where* $f_t : X \to$ *define a crisp mapping*

$$\Delta\mathscr{A} = \mathop{\triangle}_{t\in T} f_t : X \to \prod_{t\in T} Y_t$$

follows: $\forall x \in X$, $(\Delta\mathscr{A})(x) = \{f_t(x)\}_{t\in T}$; *as usual,* $\Delta\mathscr{A}$ *also denotes the fuzzy mapping* *om* L^X *to*

$$\prod_{t\in T} L^{Y_t} = L^{\prod_{t\in T} Y_t}$$

uced by the crisp mapping $\Delta\mathscr{A}$ *(about fuzzy product spaces refer to* [16]). $\Delta\mathscr{A}$ *is* *ed an evaluation mapping.*

For introducing the definition of L-fuzzy Tychonoff space, let us briefly recall ie facts about fuzzy unit intervals[2]: consider all the monotone decreasing ppings from the real line $\mathbb{R}$ to L which takes value $0 \in L$ at the points larger than $\mathbb{R}$, takes value $1 \in L$ at the points smaller than $0 \in \mathbb{R}$. Let $\tilde{I}$ denote all these ppings. For λ, $\mu \in \tilde{I}$, we say that λ is equivalent to μ, if $\forall t \in \mathbb{R}$, the equations $-) = \mu(t-)$, $\lambda(t+) = \mu(t+)$ hold, where

$$\lambda(t-) = \bigwedge_{s<t} \lambda(s), \ \lambda(t+) = \bigvee_{s>t} \lambda(s);$$

equivalence class of λ in $\tilde{I}$ is also denoted by λ. Let $I(L)$ denote all the mappings m the family $\tilde{I}(L)$ of all the equivalence classes in $\tilde{I}$ to the value field L, and uip it with the fuzzy topology generated by the subbase $\{L_t, R_t : t \in \mathbb{R}\}$ as follows:

$$L_t(\lambda) = \lambda(t-)', \ R_t(\lambda) = \lambda(t+), \ \forall \lambda \in \tilde{I}(L).$$

cording to our agreement, $\tilde{I}(L)$ also denotes the largest (crisp) set in $I(L)$. nce we often use $I(L)$ to denote $\tilde{I}(L)$ directly. The Γth power space (Γ is an lex set) of $I(L)$ is denoted by $I(L)^\Gamma$, we call it a fuzzy unit cube. The family all the Fuzzy continuous mappings from (L^X, η) to $I(L)$ is denoted by $\mathscr{F}(X)$.

Definition 6[6,3]. (L^X, η) *is calleda sub-*T_0 *space, if for each pair of points* $x, y \in$ $x \neq Y$, *there exists* $a \in L\backslash\{0\}$ *such that* $x_a \overline{\in} \overline{y_a}$ *or* $y_a \overline{\in} \overline{x_a}$. (L^X, η) *is called a completely* *ular space, if for each open set* U, *there exists a family* $\{W_\alpha\}$ *of fuzzy sets such that* $= \bigcup_\alpha W_\alpha$ *and for each* W_α *there exists a fuzzy continuous mapping* f_α: $(L^X, \eta) \to I(L)$ *h that*

$$W_\alpha \subset f_\alpha^{-1}(L_1') \subset f_\alpha^{-1}(R_0) \subset U.$$

(L^X, η) *is called a fuzzy Tychonoff space, if it is both completely regular and sub-*T_0.

As well known, we have the following definition.

Definition 7[15,2]. (L^X, η) *is called a* T_1 *space, if each fuzzy point is a closedset.*

(L^X, η) is called a normal space, if for each closed set F and each open set U such that $F \subset U$, there exists an open set V such that $F \subset V \subset \overline{V} \subset U$.

From Theorem 1 (Urysohn Lemma) of [2], we know that for a closed set F and an open set U in a normal space (L^X, η) with $F \subset U$, there exists a fuzzy continuous mapping f: $(L^X, \eta) \to I(L)$ such that $F \subset f^{-1}(L_{1'}) \subset f^{-1}(R_0) \subset U$. Therefore, from the relevant definitions we have the following Lemma.

Lemma 2. *If (L^X, η) is both T_1 and normal, then it is also a Tychonoff spa*

By the results of the imbedding theory[6], we know that each fuzzy Tych space (L^X, η) can be imbedded into $I(L)^\Gamma$ with the evaluation mapping $\Delta(\mathscr{F}$ (. Letting $\beta X = \mathrm{supp}\overline{\Delta(\mathscr{F}(X))(X)}$, we get a compactification $(L^{\beta X}, \beta\eta)$ of $(L^X$ where $\beta\eta$ is the relative co-topology of the subspace $L^{\beta X}$ of $I(L)^\Gamma$. The imbed theory above-mentioned takes $I(L)$ as the standard space, and the compactific $(L^{\beta X}, \beta\eta)$ is got with this imbedding theory. We give the following definition.

Definition 8. *The compactification $(L^{\beta X}, \beta\eta)$ above-mentioned is callec Stone-Čech compactification of (L^X, η) relative to the standard space $I(L)$.*

Because we always take $I(L)$ as the standard space in this paper, we call $\beta\eta)$ the Stone-Čech compactification of (L^X, η) for short.

§2. Preorder Relation in Compactifications

As well known, in general topology, the Stone-Čech compactification ofa is the largest element of the family of Tychonoff compactifications of the space is this property still preserved in fuzzy situation? Clearly, we need to introc preorder relation into the family of compactifications of a fuzzy topological at first. Hence a kind of new type subspaces and the definitions of relevant map should be investigated.

Definition 9[11]. *In (L^X, η), let $A \in L^X$, define*

$$\mathscr{P}(A) = \{B \in L^X: B \subset A\}, \quad \eta_A = \{F \cap A: F \in \eta\}.$$

Then η_A is still closed for finte union and arbitrary intersection, $\mathscr{P}(A)$ fo completely distributive lattice with a co-topology structure η_A. We call $(\mathscr{P}(A),$ quasi-subspace of (L^X, η).

Let $(\mathscr{P}(A), \eta_A)$ and $(\mathscr{P}(B), \mu_B)$ be quasi-subspaces of L-fts (L^X, η) and $\mu)$ respectively. Let f: $\mathscr{P}(A) \to \mathscr{P}(B)$ be a mapping, and define its "inverse" $(B) \to \mathscr{P}(A)$ as follows

$$f^\vee(D) = \bigcup\{C \in \mathscr{P}(A): f(C) \subset D\}.$$

We easily verify that the mapping $f^\vee$ is just f^{-1}: $\mathscr{P}(B) \to \mathscr{P}(A)$ if $\mathscr{P}(A)$ and are subspaces[16] of L^X and L^Y respectively.

For each g: $\mathscr{P}(A) \to \mathscr{P}(B)$, if there exists a Fuzzy mapping f: $L^X \to L^Y$ such that 's the restriction of f on $\mathscr{P}(A)$, then let $\tilde{f}$ denote g. In the sequel, if a sign such as $\mathscr{P}(A)\to\mathscr{P}(B)$ appears, it always means that there is a Fuzzy mapping f: L^X—L^Y ch that $\tilde{f}$ is the restriction of f on $\mathscr{P}(A)$.

$\tilde{f}$: $(\mathscr{P}(A), \eta_A)\to(\mathscr{P}(B), \mu_B)$ is said to be $*$-continuous, if $\tilde{f}^\vee(G)\in\eta_A$ for each $\in\mu_B$. Similarly we have the definition of $*$-homeomorphism.

Lemma 3. *Let $\tilde{f}$: $\mathscr{P}(A)\to\mathscr{P}(B)$. Then $\tilde{f}^\vee(D\cap B)=f^{-1}(D)\cap A$ for each $D\in L^Y$. particular, for $D\in\mathscr{P}(B)$, $\tilde{f}^\vee(D)=f^{-1}(D)\cap A$.*

Proof For each C satisfying $f(C)\subset D$, we have $C\subset f^{-1}f(C)\subset f^{-1}(D)$. So

$$\tilde{f}^\vee(D\cap B)=\bigcup\{C\in\mathscr{P}(A): f(C)\subset D\cap B\}\subset\bigcup\{C\in\mathscr{P}(A): f(C)\subset D\}\subset f^{-1}(D).$$

On the other hand,

$$f(f^{-1}(D)\cap A)\subset ff^{-1}(D)\cap f(A)\subset D\cap B.$$

ence $f^{-1}(D)\cap A\subset\tilde{f}^\vee(D\cap B)$. Thus $f^{-1}(D)\cap A=\tilde{f}^\vee(D\cap B)$.

Furthermore we have the following proposition.

Proposition 1[9]. *Let $\tilde{f}$: $\mathscr{P}(A)\to\mathscr{P}(B)$, $\tilde{g}$: $\mathscr{P}(B)\to\mathscr{P}(C)$. i denotes corresponding entity mappings. Then we have the following results*

(i) $\tilde{f}\tilde{g}=fg$, $(\tilde{f}\tilde{g})^\vee=\tilde{g}^\vee\tilde{f}^\vee$;

(ii) $\tilde{f}^\vee\tilde{f}\geqslant i$, $\tilde{f}\tilde{f}^\vee\leqslant i$;

(iii) *$\tilde{f}$ is a one-one correspondence iff $\tilde{f}\tilde{f}^\vee=i$, $\tilde{f}^\vee\tilde{f}=i$.* Quasi-subspaces and $*$-ntinuous mappings possess various properties analogous to that of usual subspaces id continuous mappings. For example, after we define the closure and the R-eighborhood in a quasi-subspace $(\mathscr{P}(A), \eta_A)$ (the closure of $C\in\mathscr{P}(A)$ in it is enoted by $cl_A(C)$) as usual, the following results can be verified without difficulty.

Proposition 2. *$\tilde{f}$: $\mathscr{P}(A)\to\mathscr{P}(B)$ is $*$-continuous iff $f(cl_A(C))\subset cl_B(f(C))$ for ich $C\in\mathscr{P}(A)$.*

Proposition 3. *$\tilde{f}$ preserves arbitrary union, $\tilde{f}^\vee$ preserves arbitrary union and rbitrary intersection.*

Definition 10. *Let $\mathscr{C}(X)$ denote the family of all the compactifications of $(L^X$,). Define a preorder (i. e., a relation which is self-reflexive and transitive, but it need ot to be anti-symmetric) "$\leqslant$" in $\mathscr{C}(X)$ as follows:*

$$(L^{c_1X}, c_1\eta)\leqslant(L^{c_2X}, c_2\eta) \text{ iff there exists a } *\text{-continuous}$$
$$\text{mapping } \tilde{f}: \mathscr{P}(\overline{c_2(X)})\to\mathscr{P}(\overline{c_1(X)}) \text{ such that } fc_2=c_1,$$

here the meanings of c_1 and c_2 are as in Definition 4. We say that $(L^{c_1X}, c_1\eta)$ is quivalent to $(L^{c_2X}, c_2\eta)$ if the mapping f is a $*$-homeomorphism in the above relation.

In [11] we prove that the family $\mathscr{C}(X)$ of compactifications of an L-fts (L^X, η) s always nonempty and each nonempty subfamily of $\mathscr{C}(X)$ has a supremum in $\mathscr{C}(X)$, therefore there exists a largest compactification. In the paper we also prove that there exists a kind of L-fts such that the largest compactifications of them are not unique (i. e., the preorder does not satisfy the anti-symmetric law); but if we limit the class of the compactifications with a certain separation property so called weak T_2, then the largest compactification is unique.

§3. Continuous Mappings on $I(L)^\Gamma$

In this section, we will investigate the value field L and prove that for remarkably many cases, the continuous mappings on $I(L)$ are not enough.

Definition 11. *For a fuzzy lattice L, let $N(L)=\{a\wedge a':\ a\in L\backslash\{0,\ 1\}\}$. called a quasi-Boolean lattice, if $|N(L)|\leqslant 1$.*

It is not difficult to prove that quasi-Boolean lattices include two kinds $N(L)=\phi$ and $N(L)=\{0\}$, at this time, L is a Boolean lattice; (2) $N(L)=\{b$ $b\neq 0$, at this time, $L_1=L\backslash\{0,\ 1\}$ is a Boolean lattice, that is to say, L consists Boolean lattice L_1 plus another largest element and another smallest element.

Proposition 4. *For each fuzzy lattice L, the following conditions are equiv*

(i) L is a quasi-Boolean lattice;

(ii) $a\wedge a'=b\wedge b'$ for each pair of $a,\ b\in L\backslash\{0,\ 1\}$;

(iii) if $a,\ b\in L$ and $a\neq 0$, then $a\geqslant b\wedge b'$.

The proof is obvious.

Definition 12. *For each $a\in L$, let aX denote the fuzzy set which takes co value a on X; $(L^X,\ \eta)$ is called a fully stratified space, if $aX\in\eta$ for each $a\in L$.*

Theorem 1. *In $(L^X,\ \eta)$, if there exist $a,\ b\in L,\ a\neq 0$ such that $a\not\geqslant b\wedge b$ $aX\in\eta$, moreover, suppose that for the closed set A of $I(L)^\Gamma$ we have $a(I(L)^\Gamma)$ ϕ, then there is no fuzzy mapping $f\colon\ I(L)^\Gamma\to(L^X,\ \eta)$ such that the restricti $\mathscr{P}(A)\to(L^X,\ \eta)$ of it is $*$-continuous. Especially when L is not quasi-Boolean and $\eta)$ is fully stratified, there is no continuous mapping $f\colon I(L)^\Gamma\to(L^X,\ \eta)$.*

Proof Suppose that there exists a fuzzy mapping $f\colon\ I(L)^\Gamma\to(L^X,\ \eta)$ suck $\tilde{f}\colon\ \mathscr{P}(A)\to(L^X,\ \eta)$ is $*$-continuous. Let μ denote the co-topology of $I(L)$ Lemma 3 we have

$$G=a(I(L)^\Gamma)\cap A=f^{-1}(aX)\cap A=\tilde{f}^\vee(aX)\in\mu_A$$

and $G\neq\varnothing$. Since A is a closed set of $I(L)^\Gamma$, so is G. Take a crisp point $h\in I($ follows:

$$h(t)=\begin{cases}1,\ t\leqslant 0,\\ b,\ 0<t<1,\\ 0,\ t\geqslant 1.\end{cases}$$

Let z be the crisp point in $I(L)^\Gamma$ such that each coordinate of it is h. Then fro structure of the base of the co-topology of $I(L)^\Gamma$ we know that for each none closed set P in $I(L)^\Gamma$, the relation $P(z)\geqslant b\wedge b'$ always holds. But $G(z)\leqslant a,\ \ a\not\geqslant b\wedge b'$. Hence we have another relation $G(z)\not\geqslant b\wedge b'$ too, this is a contradica tion.

§ 4. Non–Largestness of the Stone–Čech Compactification

The investigations in the preceding section show that the $*$–continuous ppings on $I(L)^{\Gamma}$ and the closed quasi–spaces of it are always insufficient except case that L is quasi–Boolean. Therefore, it can be guessed that the property will troy the largestness (see the defintition of $\leqslant$) of the fuzzy Stone–Čech npactifications which reflects the extensibility of mappings. We have the following ults:

Theorem 2. *For each Tychonoff compactification $(L^{cX}, c\eta)$ of (L^X, η), if there st $a, b \in L$ such that $a \neq 0$, $a \not\geqslant b \wedge b'$ and $a(cX) \in c\eta$, then $(L^{\beta X}, \beta\eta) \not\geqslant (L^{cX}, c\eta)$, refore the Stone–Čech compactification of (L^X, η) is not the largest element of the nily of all the Tychonoff compactifications of (L^X, η).*

Proof Since a subspace of a Tychonoff space is still a Tychonoff space, (L^X, η) ı Tychonoff space and the Stone–Čech compactification of it exists. Since $\overline{\beta(X)} \supset X)$, $\beta(X)$ is a crisp set in $I(L)^{\mathscr{F}(X)}$, the relation $a(I(L)^{\mathscr{F}(X)}) \cap \overline{\beta(X)} \neq \varnothing$ always ds. By Theorem 1, there is no $*$–continuous mapping $\tilde{f}$: $\mathscr{P}(\overline{\beta(X)}) \to (L^{cX}, c\eta)$. nce $(L^{\beta X}, \beta\eta) \not\geqslant (L^{cX}, c\eta)$.

Theorem 3. *If L is not quasi–Boolean, then for each fully stratified Tychonoff npactification $(L^{cX}, c\eta)$ of (L^X, η), $(L^{\beta X}, \beta\eta) \not\geqslant (L^{cX}, c\eta)$.*

Theorem 3 indicates that for giving out a counterexample to show that the ne–Čech compactification is not the largest one we need only to show the existence fully stratified N–compact spaces. We will use the induced space theory to nstruct a large kind of space of this type.

§ 5. Induced Spaces of Pseudo–Metric Spaces

In this section, some properties of induced spaces will be investigated. Chiefly, will prove that the induced space of a pseudo–metric space is also pseudo–trizable.

Definition 13. *For each family $\mathscr{A} \subset L^X$, let $[\mathscr{A}]$ denote the family of all the sp sets in $\mathscr{A}$, let $\mathscr{A}'$ denote $\{A' : A \in \mathscr{A}\}$. $\forall a \in L$, $\forall A \in L^X$, let*

$$A_{[a]} = \{x \in X : A(x) \geqslant a\}, \quad A_{(a)} = \{x \in X : A(x) \not\leqslant a\}.$$

r each crisp set $A \subset X$ and each $a \in L$, let $aA = \bigcup \{x_a : x \in A\}$.

(L^X, η) is called weakly induced or a weakly induced space of $(X, [\eta'])$, if $A_{[a]} \in \eta$ for each $A \in \eta$ and each $a \in L$. (L^X, η) is called induced or the induced space of $(X, [\eta'])$, if (L^X, η) is both weakly induced and fully stratified.

Obviously $(X, [\eta'])$ is a usual topological space. On the other hand, if $(X, \mathscr{T})$ is a usual topological space, then $(X, \mathscr{T})$ determines uniquely an induced space as follows:

Proposition 5[12]. *(L^X, η) is induced iff one of the following conditions is satisfied:*

(i) $A \in \eta \Leftrightarrow \forall a \in L,\ A_{[a]} \in [\eta]$;

(ii) $A \in \eta' \Leftrightarrow \forall a \in L,\ A_{(a)} \in [\eta']$.

Lemma 4. *In (L^X, η), let $\mathscr{B} \subset [\eta']$ be a base of the crisp topological spac $[\eta'])$, $\widetilde{\mathscr{B}} = \{aU: a \in L, U \in \mathscr{B}\}$. Then*

(i) *$\widetilde{\mathscr{B}}$ constitutes a base of a fuzzy (open) topology τ on L^X, and $[\tau] = [\eta']$;*

(ii) *(L^X, τ') is an induced space;*

(iii) *(L^X, η) is an induced space iff $\eta = \tau'$.*

Proof Clearly $[\tau] = [\eta']$. Since $(aU) \cap (bV) = (a \wedge b)(U \cap V)$ for each p $aU, bV \in \widetilde{\mathscr{B}}$, (i) holds. For (ii), clearly $aX \in \tau$ for each $a \in L$, so $aX \in \tau'$ foı $a \in L$, (L^X, τ') is fully stratified. Furthermore, we need to prove that it is w induced: let $U \in \tau$, we can assume

$$U = \bigcup_{\alpha} a_\alpha U_\alpha,$$

where each $a_\alpha U_\alpha \in \widetilde{\mathscr{B}}$, then

$$U_{(a)} = \bigcup_{\alpha} (a_\alpha U_\alpha)_{(a)}$$

for each $a \in L$. But obviously we have $(a_\alpha U_\alpha)_{(a)} \in [\tau]$. Hence $U_{(a)} \in [\tau]$. Noting τ' is a fuzzy co-topology on L^X, from $U'_{[a]} = (U_{(a')})'$ we know that (L^X, τ') is w induced, hence it is induced. For (iii), by (ii) proved above, the sufficie obvious. Let (L^X, η) be an induced space, then by $[\tau] = [\eta']$ (in (i)) and Propo 5 we know $\eta = \tau'$.

Obviously the following proposition is true.

Proposition 6. *An induced space (L^X, η) is a T_1 space iff $(X, [\eta'])$ is space.*

Using Lemma 4 we can prove the following proposition.

Proposition 7. *A weakly induced space (L^X, η) is N-compact iff $(X, [\eta'$ compact.*

Definition 14[1,8,4]. *A family $\{D_r: r > 0\}$ of mappings, where r is a po real number, $D_r: L^X \to L^X$, is called a pseudo-metric on L^X, if it satisfies the follo conditions $\langle A1\rangle$—$\langle A6\rangle$:*

$\langle A1\rangle$ $D_r(\varnothing) = \varnothing$,

$\langle A2\rangle$ $A \subset D_r(A)$,

$\langle A3\rangle$ $D_r(\bigcup_{\alpha} A_\alpha) = \bigcup_{\alpha} D_r(A_\alpha)$,

$\langle A4\rangle$ $D_r \circ D_s \leqslant D_{r+s}$,

$\langle A5\rangle$ $D_r = \bigvee_{s<r} D_s$,

$\langle A6\rangle$ $D_r^{-1} = D_r$, *where* D_r^{-1} *is defined as* $D_r^{-1}(A) = \bigcap\{C: D_r(C') \subset A'\}$.

$\}_r$: $r>0\}$ *satisfies the following condition* $\langle A7\rangle$ *or* $\langle A8\rangle$ *besides* $\langle A1\rangle$—$\langle A6\rangle$:

$\langle A7\rangle$ $\forall x \in X$, $\forall a \in L\backslash\{0\}$, $x_a = \bigcap_{r>0} D_r(x_a)$,

$\langle A8\rangle$ *the fuzzy topology on* L^X *generated by*

$$\{D_r(x_a): x\in X,\ a\in L\backslash\{0\}\} \text{ is sub-}T_0,$$

it is called a I*-type metric or* II*-type metric respectively.*

(L^X, η) is said to be pseudo-metrizable, if there exists a pseudo-metric $\{D_r$: $\}$ on L^X called that the fuzzy (open) topology η' takes $\{D_r(A): A\in L^X, r>0\}$ as a . Correspondently we can give the definition of (L^X, η) being I-type metrizable -type metrizable.

Lemma 5[11]. *Each fuzzy pseudo-metric space is normal. Therefore* T_1 *pseudo- ic space is a Tychonoff space.*

Theorem 4. *If a crisp topological space* $(X, \mathscr{T})$ *is pseudo-metrizable trizable), then the induced space of* $(X, \mathscr{T})$ *is also pseudo-metrizable (both* I*-type izable and* II*-type metrizable).*

Proof Suppose ρ is the pseudo-metric of $(X, \mathscr{T})$, r is an arbitrary positive number. Let

$$B_r: \mathscr{P}(X) \to \mathscr{T},\ B_r(A) \mapsto \{x\in X: \rho(x, A) < r\}.$$

n obviously $\{B_r: r>0\}$ also satisfies $\langle A1\rangle$—$\langle A5\rangle$ of Definition 14. By the Remark ection 5 of [1], $\langle A6\rangle$ is also satisfied. Now for each $r>0$, let D_r: $L^X \to L^X$ be as ows:

$$\forall A\in L^X,\ D_r(A) = \bigcup\{A(x)B_r(x): x\in \text{supp } A\}. \quad (*)$$

en since $\{B_r: r>0\}$ satisfies $\langle A1\rangle$—$\langle A5\rangle$, so does $\{D_r: r>0\}$. Now verify $\langle A6\rangle$. [3] pointed out, for each fuzzy lattice L, if mapping $f: L\to L$ preserves union l it is increasing (i. e., $a\leqslant f(a)$), f^{-1}: $L\to L$ defined as

$$f^{-1}(a) = \bigwedge\{b\in L: f(b')\leqslant a'\}$$

o preserves union and is increasing. In particular, D_r^{-1} also preserves union. nce we need only to verify $\langle A6\rangle$ for each fuzzy point x_a. Now for each fuzzy int x_a we have

$$D_r^{-1}(x_a) = \bigwedge\{C\in L^X: D_r(C')\subset (x_a)'\}, \text{ but}$$

$$\begin{aligned} D_r(C')\subset (x_a)' &\Leftrightarrow (\bigcup\{C'(y)\ B_r(y): y\in \text{supp}C'\})(x)\leqslant a' \\ &\Leftrightarrow (\text{supp}C')\cap B_r(x)\subset\{y\in X: C'(y)\leqslant a'\} \\ &\Leftrightarrow C'\subset (B_r(x))'\cup a'B_r(x) = (aB_r(x))' \\ &\Leftrightarrow D_r(x_a) = aB_r(x)\subset C, \end{aligned}$$

hence $D_r(x_a)\subset D_r^{-1}(x_a)$, $D_r\leqslant D_r^{-1}$. On the other hand, let $D_r(x_a) = C$. Then from the definition of D_r we have $D_r(C')\subset (x_a)'$, hence $D_r^{-1}(x_a)\subset C$, i. e., $D_r\geqslant D_r^{-1}$. Thus we have $D_r = D_r^{-1}$, $\{D_r: r>0\}$ is a fuzzy pseudo-metric.

From [1, Th. 4.8] we know that for the pseudo-metric $\{D_r: r>0\}$ constructed above $\{D_r(A): A \in L^X, r>0\}$ constitutes a base of an L-fuzzy topology on L^X. Let τ denote this topology. Clearly $[\tau] = \mathscr{T}$. Now we need to prove $\tau = \eta'$, where η is the L-fuzzy co-topology generated by Proposition 5 with $(X, \mathscr{T})$, i.e., (L^X, η) is the induced space of $(X, \mathscr{T})$. Clearly $\mathscr{B} = \{B_r(x): x \in X, r>0\}$ is a base of $\mathscr{T}$. Moreover, by the preceding definition $(*)$, obviously $\widetilde{\mathscr{B}} = \{aU: a \in L, U \in \mathscr{B}\}$ constitutes a base of τ. Thusfrom Lemma 4 we get $\tau = \eta'$, that is to say, (L^X, η) is ps metrizable.

If ρ is a metric, then $\{D_r: r>0\}$ satisfies both $\langle A7 \rangle$ and $\langle A8 \rangle$ obviously, so $r>0\}$ is both a I-type metric and a II-type metric. By the proof above we (L^X, η) is both I-type metrizable and II-type metrizable.

§6. Main Results

After making the preparations above, we can give out the main results now Proposition 7, Proposition 6 and Theorem 4, the induced space (L^X, η) of a metrizable campact space $(X, \mathscr{T})$ is T_1 N-compact pseudo-metrizable, so it is stratified Tychonoff N-compact by Lemma 5. Hence it can be looked upon as a stratified Tychonoff compactification of itself. Thus from Theorem 3 we ge ollowing theorem.

Theorem 5. *If L is not quasi-Boolean, $(X, \mathscr{T})$ is a crisp metrizable co space, (L^X, η) is the induced space of $(X, \mathscr{T})$, then (L^X, η) is a compactificati itself, and its Stone-Čech compactification $(L^{\beta X}, \beta\eta) \not\geqslant (L^X, \eta)$. Therefore the Čech compactification of (L^X, η) is not the largest Tychonoff compactificati (L^X, η).*

Because the great majority of fuzzy lattices are not quasi-Boolean and co metric spaces widely exist, there exists a large kind of fuzzy Tychonoff (N-com spaces such that their Stonce-Čech compactifications are not the l compactifications of them.

In the process of constructing the counterexample above, we can see that i research of the Stone-Čech compactifications, induced spaces should be espe regarded. In [12] we especially investigate the induced space $I^*(L)$ on the real interval I and take it as the new standard space. Therefore some neat results re to the imbedding theory and compactifications are given out. About $I^*(L)$ we the following theorem.

Theorem 6. *The Stone-Čech compactification of $I^*(L)$ is not the largest Tychonoff compactification of it.*

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